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REGULARITY OF DYNAMICAL GREEN'S FUNCTIONS

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ABSTRACT. For meromorphic maps of complex manifolds, ergodic theory and pluripotential theory are closely related. In nice enough situations, dynamically defined Green's functions give rise to invariant currents which intersect to yield measures of maximal entropy. 'Nice enough' is often a condition on the regularity of the Green's function. In this paper we look at a variety of regularity properties that have been considered for dynamical Green's functions. We simplify and extend some known results and prove several others which are new. We also give some examples indicating the limits of what one can hope to achieve in complex dynamics by relying solely on the regularity of a dynamical Green's function.

INTRODUCTION

A holomorphic, or more generally, meromorphic self-map $f : X \to X$ of a compact complex manifold X induces actions $f^*, f_* : H^*(X, \mathbb{R}) \to H^*(X, \mathbb{R})$ on the real cohomology groups of X. It is conjectured that when these actions are suitably well-behaved, then the topological entropy $h_{top}(f)$ of f should be $\log \rho(f^*)$, where $\rho(\cdot)$ denotes the spectral radius. This conjecture has motivated a great deal of research in the past fifteen years, and it has been verified in some important cases (see [Gr], [Y], [Sm], [FS 2], [Du], [G 3]). It is known, for instance, that the inequality $h_{top}(f) \leq \log \rho(f)$ always holds [DS 1].

The main strategy for proving the reverse inequality has been to look for an invariant measure whose metric entropy is maximal, i.e. equal to $\log \rho(f)$. However, rather than try to realize the measure directly from the dynamics of f, it often seems more promising to use the dynamics to construct invariant positive closed currents and then try to obtain the measure as an intersection of these currents. The drawback is that in passing from currents to measures, one must somehow make sense of what is essentially a product of distributions. For positive closed currents, this is usually done by resorting to 'potentials' for the currents and integrating by parts. Success depends on having potentials that are substantially more regular than the currents themselves. The purpose of this paper is to better understand the regularity properties of potentials associated to dynamically-defined positive closed $(1,1)$-currents. Such potentials will be functions, the *dynamical Green's functions* in the title of the paper.

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In the first section we describe the best possible situation: *holomorphic* maps. We present a simple proof, due to Dinh and Sibony (see [DS 2], Theorem 3.7.1; also [DS 3], Proposition 2.4), of the fact that a dynamical Green's function associated to a holomorphic map must be Hölder continuous with Hölder exponent controlled by what we call the *topological Lyapunov exponent*

$$\chi_{top}(f) := \lim_{n\to+\infty} \frac{1}{n} \log \sup_{x\in\mathbb{P}^k} ||D_x f^n||$$

of the map. A straightforward example shows that this estimate is sharp.

In the remaining sections of the paper, we turn to the more general class of meromorphic self-maps, a principal goal being to see what remains of Hölder continuity for the Green's function once one leaves the holomorphic setting.

Our first result, proven in Section 2, is a general existence theorem for dynamical Green's functions of meromorphic maps in any dimension. If $f : X \to X$ is a meromorphic map of a compact Kähler manifold, then we say that f is 1-*stable* if the induced action f^* on $H^{1,1}_{\mathbb{R}}(X)$ satisfies $(f^n)^* = (f^*)^n$ for all $n \in \mathbb{N}$. Given a class $\eta \in H^{1,1}_{\mathbb{R}}(X)$ satisfying $f^*\eta = \lambda\eta$ and a smooth form ω representing η, one can try to construct an invariant current representing η as follows. For each $n \in \mathbb{N}$ we have an L^1 function $g_n^\omega : X \to \mathbb{R} \cup \{\pm\infty\}$ satisfying

$$\lambda^{-n} f^{n*}\omega - \omega = dd^c g_n^\omega.$$

If g_n^ω converges in L^1 to some function g^ω (the dynamical Green's function), then the current $T_\eta := \omega + dd^c g^\omega$ automatically satisfies $f^*T = \lambda T$.

Theorem A. *Suppose that $f : X \to X$ is a* 1*-stable meromorphic map of a compact Kähler surface and that the induced action f^* has a unique simple eigenvalue λ of largest modulus with eigenspace generated by a nef class η. Then for any smooth form ω representing η, it can be arranged that the sequence (g_n^ω) is decreasing and L^1 convergent. The closed current $T := \omega + dd^c g^\omega$ is positive and independent of ω.*

Our proof follows Sibony [S] who considered the case $X = \mathbb{P}^k$ and Guedj [G 1, G 2] who considered general X. The novelty here is that we do not assume that the smooth representative ω can be chosen to be positive. Hence it requires some new ideas to establish that the sequence approximating g^ω is decreasing and to show that the invariant current T is positive. We remark that in dimension two, the theorem applies to nearly all reasonable meromorphic maps (see Corollary 2.7). After Section 2, we restrict attention to maps of complex *surfaces.*

When the map f in Theorem A is not holomorphic, the Green's function g_ω will not be continuous. It will typically have a logarithmic pole at each point of indeterminacy for f and its iterates. We let $\mathcal{I}_f$ denote the closure of the set of all such points. Though $\mathcal{I}_f$ can be all of X, as the first example in Section 6 shows, there are many situations where the complement of $\mathcal{I}_f$ is large, and one can then hope for continuity of g^+ in $X - \mathcal{I}_f$. In Section 3, we validate this hope in some interesting special cases. Indeed, we give a unified approach to proving something analogous to, but weaker than, Hölder continuity for g^+ for some large classes of birational surface maps (Theorem 3.1 and the comment following its proof) and of polynomial maps of $\mathbb{C}^2$ (Theorem 3.4). We point out concerning Theorem 3.4 that in the important case where the first dynamical degree λ exceeds the topological degree, we know of no examples where the hypothesis of the theorem fails.

In Section 4, we consider a still weaker regularity condition for birational surface maps. We state a quantitative recurrence property for points of indeterminacy that turns out to be equivalent to the condition that the derivative dg^+ of the Green's function be in L^2. A similar, slightly stronger L^2 condition has been used with much success in [BD] and [Du] to produce measures of maximal entropy for birational maps. With our version, the construction of the measure still succeeds, but its fine dynamical properties remain unclear; in particular, we do not know if $\log ||Df||$ is integrable with respect to the measure, a property that is important for applying many of the theorems and techniques from smooth ergodic theory.

Continuing with birational surface maps in Section 5, we consider what is perhaps the weakest relevant regularity condition of all: g^+ is integrable with respect to (the trace measure of) T^-, the invariant current associated to f^{-1}. This condition guarantees that $\mu = T^+ \wedge T^-$ is a well-defined probability measure. Indeed with no further assumption on f (i.e. on g^+), we prove the following.

Theorem B. *The measure μ_f is f-invariant and mixing, and it does not charge any compact complex curve.*

The proof that μ_f does not charge curves is distinctly indirect, depending on, among other things, a characterization (Proposition 5.4) of the L^2 condition used in [BD].

We present several telling examples throughout the paper, and Section 6 is devoted to two of these. The first shows that indeterminacy orbits of a meromorphic map can be dense. That is, $\mathcal{I}_f = X$. The second builds on an example due to Favre [F] and demonstrates that one can have g^+ integrable with respect to T^- without necessarily having that dg^+ is in L^2.

1. Holomorphic maps

Let $f : \mathbb{P}^k \to \mathbb{P}^k$ be a holomorphic endomorphism of the complex projective space $\mathbb{P}^k$. In homogeneous coordinates, $f = [P_0 : \ldots : P_k]$ where the P_j's are homogeneous polynomials of the same degree λ with no common zero outside the origin. We shall always assume $\lambda \geq 2$.

Let ω denote the Fubini-Study Kähler form on $\mathbb{P}^k$. Then $f^*\omega$ is a well-defined smooth closed $(1,1)$-form on $\mathbb{P}^k$ which is cohomologous to $\lambda\omega$. Thus it follows from the dd^c-lemma (see [GH], p. 149) that

$$\frac{1}{\lambda} f^*\omega = \omega + dd^c\gamma,$$

where $\gamma \in \mathcal{C}^\infty(\mathbb{P}^k)$ is uniquely determined up to an additive constant. Here $d = \partial + \overline{\partial}$ and $d^c = \frac{1}{2i\pi}(\overline{\partial} - \partial)$. Pulling back the previous equation by f^n yields

$$\frac{1}{\lambda^n}(f^n)^*\omega = \omega + dd^c g_n, \text{ where } g_n = \sum_{j=0}^{n-1} \frac{1}{\lambda^j} \gamma \circ f^j.$$

The sequence of positive closed $(1,1)$-forms $\lambda^{-n}(f^n)^*\omega$ converges weakly to the so-called Green current

$$T_f = \omega + dd^c g_f, \text{ where } g_f := \sum_{j \geq 0} \frac{1}{\lambda^j} \gamma \circ f^j.$$

This is a dynamically interesting current. It was constructed by H. Brolin [Bro] (polynomial case) and M. Lyubich [Ly] (rational case) when $k = 1$, and by Hubbard-Papadopol [HP] and Fornaess-Sibony [FS 2] in higher dimensions. We refer the reader to [S] for its basic properties. Our aim here is to give a very simple proof of the fact that the (dynamical) Green function g_f is Hölder continuous. To this end we introduce the topological Lyapunov exponent of f,

$$\chi_{top}(f) := \lim_{n\to+\infty} \frac{1}{n} \log \sup_{x\in\mathbb{P}^k} ||D_x f^n||.$$

That the limit exists follows from the submultiplicativity of the sequence $(\sup_{x\in\mathbb{P}^k} ||D_x f^n||)$. The definition clearly does not depend on the choice of the norm $||\cdot||$. Also, the supremum in the definition can be considered only on the Julia set J_f of f. Let us recall that the *Fatou set* $\mathcal{F}_f$ of f is the largest open subset of $\mathbb{P}^k$ on which the sequence of iterates (f^n) forms a normal family. The *Julia set* J_f is the complement of the Fatou set.

The next result, together with its proof, is essentially Theorem 3.7.1 in [DS 2] (see Proposition 2.4 in [DS 3] for a more general statement). We include it here for the convenience of the reader, because many of the results in the following sections may be viewed as attempts to see what remains when one passes from holomorphic to more badly behaved meromorphic maps.

Theorem 1.1. *The Green function g_f is Hölder continuous of exponent $\alpha > 0$, for every $\alpha < \log\lambda/\chi_{top}(f)$.*

Proof. Set $M = \sup_{x\in\mathbb{P}^k} ||D_x f||$. A straightforward induction yields, for all $x, y \in \mathbb{P}^k$ and all $j \in \mathbb{N}$,

$$d(f^j x, f^j y) \le M^j d(x, y).$$

Here d denotes the distance associated to the Fubini-Study metric on $\mathbb{P}^k$. Since γ is smooth, it is in particular Hölder-continuous of exponent $\alpha > 0$, for any $\alpha \le 1$. We fix $\alpha < \log\lambda/M$ and estimate

$$|g_f(x) - g_f(y)| \le \sum_{j\ge 0} \frac{1}{\lambda^j} |\gamma \circ f^j(x) - \gamma \circ f^j(y)| \le C_\alpha d(x, y)^\alpha,$$

where $C_\alpha = \sum_{j\ge 0} \lambda^{-j} M^{\alpha j} < +\infty$.

Replacing f by f^n in the above argument lowers the constant M to $M_n = (\sup ||D_x f^n||)^{1/n}$. Letting $n \to +\infty$ yields the desired upper bound. □

Example 1.2 shows that the bound in this theorem is optimal. One can also establish bounds in the other direction using the infimum of the differential on the Julia set. These imply in particular that the affine Green's functions G_c of quadratic maps $f_c(z) = z^2 + c$ with $c \in \mathbb{R}$ are Hölder continuous of exponent α_c with $\alpha_c \to 0$ as $c \to +\infty$.

Examples 1.2. Consider the quadratic family of holomorphic endomorphisms of the Riemann sphere $f_c : \mathbb{P}^1 \to \mathbb{P}^1$, given by quadratic polynomials in some affine chart,

$$f_c(z) = z^2 + c.$$

We let $G_c(z) = g_{f_c}[1 : z]$ denote the affine Green's function.

1) If $c=0$ one easily computes $G_0(z)=\log^+|z|$, $J_{f_0}=S^1$ is the unit circle, and $\chi_{top}(f_0)=\log 2$. If $c=-2$, then f_{-2} is semi-conjugate to f_0, and one can compute its iterates explicitly. This yields $J_{f_{-2}}=[-2,2]$, $\chi_{top}(f_{-2})=2\log 2$ and

$$g_{-2}(z)=\log\max\left(\left|\frac{z+\sqrt{z^2-4}}{2}\right|;\left|\frac{z-\sqrt{z^2-4}}{2}\right|\right).$$

Observe that this is a Hölder-continuous function of exponent $1/2$.

2) If $|c|\leq 2$, one can easily show that the Julia set J_{f_c} is always contained in the closed disk centered at the origin and of radius 2. We infer $\chi_{top}(f_c)\leq 2\log 2$. Note that this disk contains the Mandelbrot set $\mathcal{M}$, i.e. the set of parameter values c such that J_{f_c} is connected.

More generally, if f is any monic centered polynomial of degree λ with connected Julia set, it was proved by X. Buff [Bu] that

$$\chi_{top}(f)\leq\sup_{J_f}|f'|\leq 2\log\lambda.$$

We also have bounds from below. If μ is any invariant ergodic measure such that $\log||(Df)^{\pm 1}||\in L^1(\mu)$, then its Lyapunov exponent $\chi_\mu(f)$ satisfies $\chi_\mu(f)\leq\chi_{top}(f)$. In particular if $\mu=\omega+dd^cg_f=dd^cG_f$ is the Brolin-Lyubich measure, then

$$\log\lambda\leq\log\lambda+\sum_{f'(c)=0}G_f(c)=\chi_\mu(f)\leq\chi_{top}(f).$$

Remark 1.3. For $X=\mathbb{P}^1$, the Hölder continuity of the dynamical Green's functions was first established by N. Sibony (see [CG], Theorem 8.3.2). It was then generalized to endomorphisms of $\mathbb{P}^k$ by J.-Y. Briend [Bri] and M. Kosek [K].

As we explain below, the elementary proof given above applies to other manifolds. Slightly modifying the proof shows also that if $(f_t)_{t\in M}$ is a holomorphic family of endomorphisms of the same degree λ, then the Green's function $(x,t)\mapsto g_{f_t}(x)$ is Hölder continuous with respect to the parameter t.

Consider a holomorphic endomorphism $f:X\to X$ of some compact Kähler manifold X. Then f^* respects complex conjugation and bidegree of forms, and therefore restricts to a linear action on $H^{1,1}_{\mathbb{R}}(X):=H^2(X,\mathbb{R})\cap H^{1,1}(X)$.

Assume that $f^*\eta=\lambda\eta$ for some $\lambda>1$ and $\eta\in H^{1,1}_{\mathbb{R}}(X)$. Then if ω is a smooth closed $(1,1)$-form representing η, we have $\gamma\in\mathcal{C}^\infty(X,\mathbb{R})$ such that

$$\frac{1}{\lambda}f^*\omega=\omega+dd^c\gamma.$$

Pulling back by f^n yields

$$\frac{1}{\lambda^n}(f^n)^*\omega\longrightarrow T_\eta=\omega+dd^cg_\omega,\ \text{where } g_\omega=\sum_{j\geq 0}\frac{1}{\lambda^j}\gamma\circ f^j.$$

Observe that the dynamical Green's current T_η only depends on η: if ω' also represents η, then $\omega'=\omega+dd^cu$ for some smooth function u. Hence $u\circ f^n/\lambda^n\to 0$ uniformly on X, and

$$\frac{1}{\lambda^n}(f^n)^*\omega'=\frac{1}{\lambda^n}(f^n)^*\omega+dd^c\left(\frac{1}{\lambda^n}u\circ f^n\right)\to T_\eta.$$

The same proof as above shows that **the Green's function g_ω is Hölder continuous**.

In concluding this section, we recall (see e.g. [Meo], section I) that if S is any positive closed $(1,1)$-current on X, then we may define the pullback f^*S. Namely, we write $S = \eta + dd^c u$, where η is a smooth closed $(1,1)$-form and u is a quasi-plurisubharmonic (henceforth 'qpsh') function, and we set $f^*S = f^*\eta + dd^c u \circ f$. The result is another positive closed $(1,1)$-current on X.

A cohomology class η is *pseudoeffective* if it can be represented by a *positive* closed current S of bidegree $(1,1)$. In this case, f^*S is a well-defined positive closed current of bidegree $(1,1)$ on X which represents $f^*\eta$. Thus f^* preserves the cone $H^{1,1}_{psef}(X) \subset H^{1,1}_{\mathbb{R}}(X)$ of pseudoeffective classes. Because the pseudoeffective cone is closed, convex and strict (i.e., contains no lines), it follows from Perron-Frobenius theory that there exists an invariant class $\eta \in H^{1,1}_{psef}(X)$ associated to the spectral radius $\lambda = \varrho_{f^*} \geq 1$ of $f^*|_{H^{1,1}_{\mathbb{R}}(X)}$. When $\varrho_{f^*} > 1$ (which is equivalent [Gr] to saying that f has positive entropy), it is reasonable to hope that the associated current T_η will itself be positive. In the next section, we pursue the construction of T_η, and this hope in particular, for a much larger class of maps.

2. Green's functions for meromorphic maps

Let X be a compact Kähler manifold of dimension k. When $f : X \to X$ is merely meromorphic (i.e., rational, when X is projective), the construction of dynamical Green's currents is a more delicate task, due to the presence of points of indeterminacy. Nevertheless, Green's currents have been constructed in some particular meromorphic cases (see [S] for the case $X = \mathbb{P}^k$, [DF] for the case of birational surface maps, [G 1] for the case of Hirzebruch surfaces, and [G 2] for a slightly more general context). In this section we use ideas of [BD] to provide a very general construction.

We let I_f denote the indeterminacy locus, i.e. the set of points at which f is not holomorphic. This is an analytic subset of X of codimension ≥ 2. We let $\Gamma_f \subset X \times X$ denote the graph of f and $\tilde{\Gamma}_f$ denote a desingularization of it. We have a commutative diagram

$$\begin{tikzcd} & \tilde{\Gamma}_f \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ X \arrow[rr, "f"] & & X \end{tikzcd}$$

where π_1, π_2 are holomorphic maps. We always assume that f is dominant, i.e. that its Jacobian determinant does not vanish identically (in any coordinate chart).

Given a smooth closed real form ω of bidegree $(1,1)$ on X, we set $f^*\omega := (\pi_1)_*(\pi_2^*\omega)$, where we push $\pi_2^*\omega$ forward by π_1 as a current. Observe that $f^*\omega$ is actually a form with L^1_{loc}-coefficients which coincides with the usual smooth pull-back $(f_{|X\setminus I_f})^*\omega$ in $X \setminus I_f$. Thus the definition does not depend on the choice of desingularization. Also f^* preserves boundaries and thus induces an action $f^* : H^{1,1}_{\mathbb{R}}(X) \to H^{1,1}_{\mathbb{R}}(X)$ given by

$$\{\omega\} \mapsto \{f^*\omega\}.$$

Our formula for $f^*\omega$ may also be applied to pull back (differences of) positive closed $(1,1)$-currents S: given $S \geq 0$, one uses the construction described at the end of the previous section to define π_2^*S. Then, as in the holomorphic case, $f^*S := \pi_{1*}\pi_2^*S$ is also a positive closed $(1,1)$-current. It follows again that f^* preserves the pseudoeffective cone and that there exists $\eta \in H^{1,1}_{psef}(X)$ such that $f^*\eta = \lambda\eta$ where $\lambda = \varrho_{f^*} \geq 1$ is the spectral radius of $f^*|_{H^{1,1}_{\mathbb{R}}(X)}$.

An argument of M. Gromov [Gr] implies that f has zero topological entropy when $\varrho_{f^*} = 1$. In the sequel we assume to the contrary that $\varrho_{f^*} > 1$. Any smooth form ω representing η may be written as a difference of positive forms. Hence we can iterate f^* and try to construct a Green's current $T_\eta = \lim_{n\to\infty} f^{n*}\omega$ associated to η.

We immediately face some problems. First, the action $f^*|_{H^{1,1}_{\mathbb{R}}(X)}$ is not necessarily compatible with the dynamics: it might happen, as in the following example, that $(f^n)^*$ is different from $(f^*)^n$ for some $n \in \mathbb{N}$.

Example 2.1. The polynomial endomorphism of $\mathbb{C}^2$,

$$(z_1, z_2) \mapsto (z_1 z_2, z_1),$$

extends to a meromorphic endomorphism $f : \mathbb{P}^2 \to \mathbb{P}^2$ of the complex projective plane. The extended map is given in homogeneous coordinates by

$$f[z_0 : z_1 : z_2] = [z_0^2 : z_1 z_2 : z_0 z_1],$$

where $(z_0 = 0)$ denotes the line at infinity. Observe that the indeterminacy locus I_f consists of the single point $[0 : 0 : 1]$. Since $H^{1,1}_{\mathbb{R}}(\mathbb{P}^2) \simeq \mathbb{R}$ is one-dimensional, the linear action f^* is multiplication by 2, so that $(f^*)^2$ is multiplication by 4. On the other hand, a simple computation shows that $(f^2)^*$ is multiplication by 3.

If one extends the polynomial map above to a meromorphic map g on $X = \mathbb{P}^1 \times \mathbb{P}^1$, one can check that $(g^*)^n = (g^n)^*$ for all $n \in \mathbb{N}$. This motivates the following.

Definition 2.2. The mapping $f : X \to X$ is said to be 1-stable if $(f^n)^* = (f^*)^n$ for all $n \in \mathbb{N}$.

Remark 2.3. The notion of a 1-stable map has been studied by several authors in the past decade, where it has been variously called *generic* [FS 3], *minimally separating* [Di 1], *algebraically stable* [S], or 1-*regular* [BK].

It was shown in [DF] that when f is a birational surface map, one can always make a birational change of coordinates so that f becomes 1-stable. It is an interesting, and probably quite difficult, open question to know whether this remains true for dominant 2-dimensional meromorphic maps of 'small' (i.e. less than $\rho(f^*|_{H^{1,1}_{\mathbb{R}}(X)})$) topological degree. Favre and Jonsson, in a recent paper [FJ] concerning polynomial maps of $\mathbb{C}^2$, have proposed a very different approach to issues concerning 1-stability.

We assume from now on that $f : X \to X$ is a 1-stable meromorphic map. Let $\eta \in H^{1,1}_{psef}(X)$ be such that $f^*\eta = \lambda\eta$, with $\lambda = \varrho_{f^*} > 1$. Let ω be a smooth closed real $(1,1)$-form representing η. By the dd^c-lemma again, there exists $\gamma_\omega \in L^1(X)$ such that

$$\frac{1}{\lambda} f^*\omega = \omega + dd^c\gamma_\omega. \tag{2.1}$$

Since f is 1-stable, we can pull this equation back by f^n to get

$$\frac{1}{\lambda^n} f^{n*}\omega = \omega + dd^c g_n^\omega, \quad \text{where} \quad g_n^\omega := \sum_{j=0}^{n-1} \frac{1}{\lambda^j}\gamma_\omega \circ f^j.$$

The second problem we face is that γ_ω is not smooth when f is meromorphic, so it is not obvious that the sequence g_n^ω converges in $L^1(X)$. This convergence is the content of our next result, which is a slight refinement of Theorem A in the introduction. Recall that a class $\eta \in H^{1,1}_{\mathbb{R}}(X)$ is *nef* if it is the limit of Kähler classes.

Theorem 2.4. *If η is nef, then the sequence (g_n^ω) converges in $L^1(X)$. Let g_ω be the limit, and define*

$$T_\eta := \omega + dd^c(g_\omega) = \lim_{n\to+\infty} \frac{1}{\lambda^n}(f^n)^*\omega.$$

Then T_η is a closed current satisfying $f^*T_\eta = \lambda T_\eta$. If λ is a simple eigenvalue of f^*, then T_η is positive.

Proof. **Step 1.** The first step of the proof consists in showing that γ_ω is bounded from above on X. Rewriting (1) in the desingularized graph yields

$$dd^c\left(\gamma_\omega \circ \pi_1\right) = \frac{1}{\lambda}\pi_1^*\pi_{1*}\pi_2^*\omega - \pi_1^*\omega.$$

Since π_1 is a local isomorphism away from its exceptional divisor $\mathcal{E}(\pi_1) \subset \tilde{\Gamma}_f$, we have that $R := \pi_1^*\pi_{1*}\pi_2^*\omega - \pi_2^*\omega$ is a closed current of bidegree $(1,1)$ supported on $\mathcal{E}(\pi_1)$.

Lemma 2.5. *R is positive.*

Proof. If ω (and hence $\pi_2^*\omega$) is a non-negative form, then so is $\pi_1^*\pi_1^*\pi_2^*\omega$. Since $\pi_2^*\omega$ agrees with the latter form outside $\mathcal{E}(\pi_1)$ and does not charge $\mathcal{E}(\pi_1)$ itself, it follows that R is positive.

Dropping the non-negativity assumption, we observe that R depends only on the cohomology class of ω: if $\sigma = dd^c u$ is a cohomologically trivial $(1,1)$-form, then

$$\pi_1^*\pi_{1*}\sigma - \sigma = dd^c(\pi_*\pi^*u - u) = dd^c 0 = 0.$$

Since ω is nef, we may approximate ω *in cohomology* by non-negative $(1,1)$-forms ω_j. Pullback and pushforward are continous operators on currents, so $R_j := \pi_1^*\pi_{1*}\pi_2^*\omega_j - \pi_2^*\omega_j$ converges weakly to R. That is, R is a limit of positive currents and therefore positive. ☐

Continuing with the proof of Theorem 2.4, we have

$$dd^c(\gamma_\omega \circ \pi_1) = R + \pi_2^*\omega - \pi_1^*\omega,$$

so that $\gamma_\omega \circ \pi_1$ is a qpsh function on $\tilde{\Gamma}_f$. Therefore $\gamma_\omega \circ \pi_1$ is bounded from above on $\tilde{\Gamma}_f$, as is γ_ω on X.

Step 2. It is now clear that we can add a constant to γ_ω to get $\gamma_\omega \le 0$. Therefore

$$g_n^\omega = \sum_{j=0}^{n-1} \frac{1}{\lambda^j}\gamma_\omega \circ f^j$$

is a decreasing sequence of L^1 functions. We claim that it converges, i.e., that $g_\omega = \sum_{j\ge 0} \lambda^{-j}\gamma_\omega \circ f^j$ belongs to $L^1(X)$.

The claim can be established using tricky integration by parts as in [G 2, pp. 2377-2378]. Instead, we follow here a trick of N. Sibony (who treats the case $X = \mathbb{P}^k$ in [S]). Since ω is nef, we can choose Kähler forms ω_j whose classes converge to that of ω. The mass of ω_j is controlled by its cohomology class, so the sequence (ω_j) must accumulate on some positive closed current $\tilde{\omega}$ cohomologous to ω.

Considering Cesaro means of $\lambda^{-j}(f^j)^*\tilde{\omega}$ and extracting a limit, we can produce a positive closed $(1,1)$-current σ which is cohomologous to ω and satisfies $f^*\sigma = \lambda\sigma$. Now $\sigma = \omega + dd^c v$ for some qpsh function v on X. Invariance of σ allows us to arrange

$$\gamma_\omega = v - \frac{1}{\lambda} v \circ f$$

by adding a constant to v. Pulling this equation back by f^n then gives

$$g_n^\omega = v - \frac{1}{\lambda^n} v \circ f^n \geq v - \frac{\sup_X v}{\lambda^n}.$$

Thus, $v \leq g_\omega \leq 0$ and in particular $g_\omega \in L^1(X)$. The current $T_\eta := \omega + dd^c(g_\omega)$ clearly satisfies $f^* T_\eta = \lambda T_\eta$.

Step 3. It remains to prove that when λ is a simple eigenvalue of f^*, then T_η is positive. Taking $p = \dim_{\mathbb{C}} X$, we observe that it suffices to show that

$$\langle T_\eta, \chi\sigma \rangle \geq 0,$$

where χ is a smooth non-negative cutoff function supported on a coordinate chart $U \subset X$ and σ is a positive $(p-1,p-1)$-form that is constant with respect to coordinates on U.

Lemma 2.6. *If λ is a simple eigenvalue of f^*, then some subsequence of $(f_*^n(\chi\sigma)/\lambda^n)$ converges weakly to a positive closed $(p-1,p-1)$-current S.*

Proof. Note first that if ω_0 is a Kähler form on X, then $0 \leq \chi\sigma \leq C\omega_0^{p-1}$ for $C > 0$ large enough. Hence assuming λ is a simple eigenvalue of f^*, we obtain uniform control on the mass of $\lambda^{-j} f_*^n(\chi\sigma)$ as follows:

$$\int \omega_0 \wedge f_*^n(\chi\sigma) \leq C \int \omega_0 \wedge f_*^n \omega_0^{p-1} = C \int f^{n*}\omega_0 \wedge \omega_0^{p-1} \leq C'\lambda^n.$$

In particular, the sequence $(f_*^n\, \chi\sigma/\lambda^n)$ has weak limit points. These will be positive by continuity, so we need only show that they are also closed. For this we employ a well-known argument of Bedford and Smillie (see [BS]) to show that the mass of $\partial f_*^n(\chi\sigma)$ is no larger than $C\lambda^{n/2}$. Specifically, we let φ be any real test 1-form and estimate

$$\begin{aligned}
|\langle \partial f_*^n(\chi\sigma), \varphi\rangle| &= \left| \int f^{n*}\varphi \wedge d\chi \wedge \sigma \right| \\
&\leq \left(\int f^{n*}\varphi \wedge J f^{n*}\varphi \wedge \sigma \right)^{1/2} \left(\int d\chi \wedge d^c\chi \wedge \sigma \right)^{1/2} \\
&\leq C \left(\int f^{n*}\omega_0 \wedge \sigma \right)^{1/2} \left(\int d\chi \wedge d^c\chi \wedge \sigma \right)^{1/2} \leq C'\lambda^{n/2}.
\end{aligned}$$

Note that J here is the complex structure operator on real cotangent vectors. Moreover, all integrals may be interpreted as taking place away from the set $I(f)$ where $f^{n*}\varphi$ might be singular. Having established the desired control on $\partial f_*^n(\chi\sigma)$, we are done. $\square$

Using the subsequence from the lemma, we have

$$\langle T_\eta, \chi\sigma\rangle = \lim_{j\to\infty} \lambda^{-n_j} \langle f^{n_j *}\omega, \chi\sigma\rangle = \lim_{j\to\infty} \lambda^{-n_j} \left\langle \omega, f_*^{n_j}(\chi\sigma)\right\rangle = \langle \omega, S\rangle = \eta\cdot\{S\} \geq 0,$$

where the last inequality comes from the assumption that η is nef. $\square$

It remains to understand when our hypotheses are satisfied. When $\dim_{\mathbb{C}} X = 2$, the cone $H^{1,1}_{nef}(X)$ is preserved by f^*, so the invariant class η is automatically nef (Proposition 1.11 in [DF]). Moreover, λ^2 is never less than the topological degree of f; when it is strictly larger, it is a simple eigenvalue of f^* (Remark 5.2 in [DF]).

Corollary 2.7. *If $\dim_{\mathbb{C}} X = 2$, the sequence (g_n^ω) always converges in $L^1(X)$. If λ^2 exceeds the topological degree of f, then the associated closed invariant current $T_\eta := \omega + dd^c g_\omega$ is positive.*

For the rest of the paper, we focus exclusively on the case $\dim_{\mathbb{C}} X = 2$ of maps on complex surfaces. Recall from [S] that a point $x \in X$ is said to be *normal* if there exist neighborhoods U of x and V of I_f, such that $f^n U \cap V = \emptyset$ for all $n \in \mathbb{N}$. The set of normal points is denoted by $\mathcal{N}_f$: this is the set of points which remain 'locally uniformly' away from the indeterminacy locus under iteration. The proof of Theorem 1.1 applies straightforwardly to show that g_ω is Hölder continuous in $\mathcal{N}_f$ (this is Theorem 1.7.1 in [S]). Complex Hénon mappings are polynomial automorphisms of $\mathbb{C}^2$ which extend to $\mathbb{P}^2$ as 1-stable maps of positive entropy. For such mappings the set of normal points is $\mathcal{N}_f = \mathbb{P}^2 \setminus I_f$; hence the dynamical Green's function is Hölder continuous off the indeterminacy locus. This result was first proved in [FS 1].

3. Sub-Hölder continuity

The set $\mathcal{N}_f$ might well be empty for a given meromorphic map f, and g_ω can be very discontinuous in general (see, e.g., example 1.11 in [GS] and example 6.1 below). In this section we consider some families of rational surface mappings that permit weaker, though still 'Hölder-like' control on the modulus of continuity of the dynamical Green's function

$$g_\omega := \sum_{j=0}^{+\infty} \frac{1}{\lambda^j} \gamma_\omega \circ f^j.$$

Of course this can be done only off the extended indeterminacy locus,

$$\mathcal{I}_f := \overline{\bigcup_{n\geq 0} I_{f^n}},$$

since g_ω usually has positive Lelong number at every point of I_{f^n}.

Hölder continuity of g_ω at p requires that the orbit of p uniformly avoid the indeterminacy locus (normal points). Weaker kinds of continuity of g_ω can be established by simply requiring that $f^n(p)$ not approach I_f too rapidly: see [FG], [G 1], [GS] for the case of weakly-regular polynomial endomorphisms of $\mathbb{C}^k$; and [Di 2] for birational maps of $\mathbb{P}^2$ that are *separating*, i.e. such that $\mathcal{I}_f \cap \mathcal{I}_{f^{-1}} = \emptyset$.

Here we present a unified approach to estimating the modulus of continuity of g_ω in $X \setminus \mathcal{I}_f$. It applies to a class of rational maps large enough to encompass both weakly-regular endomorphisms and separating birational surface maps. Our main dynamical assumption is as follows. *There exists* $C > 1$ *and* $\beta \in [1, \lambda[$ *such that*

$$\frac{1}{C}\,[d(x,\mathcal{I}_f)]^\beta \le d(fx,\mathcal{I}_f), \quad \textit{for all } x \in X \setminus \mathcal{I}_f. \tag{3.1}$$

Note that this estimate is stated in terms of the extended indeterminacy set $\mathcal{I}_f$ rather than just the indeterminacy set I_f. This is because I_f is rarely invariant under f^{-1}, whereas one always has $f^{-1}(\mathcal{I}_f - I_{f^{-1}}) \subset \mathcal{I}_f$.

We shall rely on three further estimates, all of which hold independently of the above assumption. The first gives us pointwise control on γ_ω:

$$\gamma_\omega(x) \ge C \log d(x, I_f) + C'. \tag{3.2}$$

This follows from Proposition 1.2 in [BD], which, despite the birational context of that paper, remains valid for arbitrary meromorphic surface maps. The other two estimates are local bounds on the Lipschitz constants of f and γ_ω. Namely, one can check by computing in local charts that there exist $m_1, m_2 > 0$ such that for all $x, y \in X \setminus I_f$,

$$|\gamma_\omega(x) - \gamma_\omega(y)| \le \frac{Cd(x,y)}{[d_{I_f}(x,y)]^{m_1}} \quad\text{and}\quad d(f(x), f(y)) \le \frac{Cd(x,y)}{[d_{I_f}(x,y)]^{m_2}}, \tag{3.3}$$

where $d_{I_f}(x,y) := \min\{d(x,I_f), d(y,I_f)\}$ is the distance from the pair $\{x,y\}$ to the indeterminacy locus. We similarly denote the distance to $\mathcal{I}_f$ by $d_{\mathcal{I}_f}(x,y)$. For convenience, we take the constant $C > 0$ to be the same in (2), (3), and (4). It follows directly from (4) that

$$|\gamma_\omega \circ f^j(x) - \gamma_\omega \circ f^j(y))| \le \frac{C^{j+1} d(x,y)}{[d_{I_f}(f^j x, f^j y)]^{m_1} \Pi_{l=0}^{j-1} [d_{I_f}(f^l x, f^l y)]^{m_2}}. \tag{3.4}$$

It follows from (2) and (4) and the fact that $d_{\mathcal{I}_f} \le d_{I_f}$ that

$$\frac{1}{d_{\mathcal{I}_f}(f^j x, f^j y)} \le \frac{C^{1+\beta+\cdots+\beta^{j-1}}}{[d_{\mathcal{I}_f}(x,y)]^{\beta^j}}. \tag{3.5}$$

We will use these bounds to obtain control on the modulus of continuity of g_ω. We treat the cases $\beta = 1$ and $\beta > 1$ separately since they are quite different.

3.1. The case $\beta = 1$.

Our aim here is to prove the following.

Theorem 3.1. *Let* $f : X \to X$ *be a* 1*-stable map which satisfies (2) with* $\beta = 1$. *Then there exists* $\alpha > 0$ *such that for all* $x, y \in X \setminus \mathcal{I}_f$,

$$|g_\omega(x) - g_\omega(y)| \le C_{x,y} \exp(-\alpha\sqrt{|\log d(x,y)|}),$$

where $(x,y) \mapsto C_{x,y} > 0$ *is locally uniformly bounded in* $X \setminus \mathcal{I}_f$.

We need the following elementary lemma whose proof is left to the reader.

Lemma 3.2. *Fix* $\alpha \in]0,1[$ *and set, for* $0 \le t \le 1$,

$$h_\alpha(t) := \exp(-\alpha\sqrt{|\log t|}).$$

Then for all $t \in [0,1]$ *and for all* $A \ge 1$,

$$0 \le t \le e\, h_\alpha(t) \ \textit{and}\ 0 \le h_\alpha(At) \le \exp(\alpha\sqrt{\log A})h_\alpha(t).$$

Proof of Theorem 3.1. Let $x, y \in X - \mathcal{I}_f$ be given. Since $\beta = 1$, we infer from (3) and (6) that $|\gamma \circ f^j(x)| \leq Cj$, where C depends only on $d(x, \mathcal{I}_f)$. In particular, if $t > 1$, then

$$\frac{|\gamma \circ f^j(x) - \gamma \circ f^j(y)|}{t^j} \leq M$$

for all $j \geq 0$ and some constant M depending on t and $d_{\mathcal{I}_f}(x, y)$. Moreover, from (5) and (6), we obtain the alternative upper bound

$$\frac{|\gamma_\omega \circ f^j(x) - \gamma_\omega \circ f^j(y)|}{t^j} \leq \frac{C^{m_2 j^2 + m_1 j + j + 1}}{t^j [d_{\mathcal{I}_f}(x, y)]^{m_1 + m_2 j}} d(x, y).$$

It follows therefore from Lemma 3.2 that

$$\frac{|\gamma_\omega \circ f^j(x) - \gamma_\omega \circ f^j(y)|}{M t^j} \leq e \exp\left(\alpha \sqrt{(m_2 j^2 \log C + Aj + B)}\right) h_\alpha(d(x, y)),$$

where A and B depend on t and $d_{\mathcal{I}_f}(x, y)$. Thus

$$|g_\omega(x) - g_\omega(y)| \leq \sum_{j \geq 0} \frac{M t^j}{\lambda^j} \frac{|\gamma_\omega \circ f^j(x) - \gamma_\omega \circ f^j(y)|}{M t^j} \leq C_{x,y,t} h_\alpha \circ d(x, y),$$

where the series defining $C_{x,y,t}$ converges as soon as $\alpha < \log(\lambda/t)[m_2 \log C]^{-1/2}$ since it is comparable to

$$\sum_{j \geq 0} \frac{t^j}{\lambda^j} \exp\left[j\alpha\sqrt{m_2 \log C}\right] < +\infty.$$

Observe also that the dependence of $C_{x,y}$ on (x, y) only involves $\log d_{\mathcal{I}_f}(x, y)$; hence it is bounded on compact subsets of $X \setminus \mathcal{I}_f$. $\square$

We now want to provide examples of rational mappings satisfying the assumptions of Theorem 3.1. Observe first that any *separating* birational 1-stable self-map of a compact Kähler surface X satisfies (2): this was observed by the first author who proved Theorem 3.1 in this context (see Theorem 5.3 in [Di 2]). The reader will find several examples of such birational mappings in [Di 1]. Note that the proof given above greatly simplifies the proof given in [Di 2]. It also applies to non-birational mappings, as the following example shows.

Example 3.3. Consider the meromorphic map $f : \mathbb{P}^2 \to \mathbb{P}^2$, given in local coordinates on $\mathbb{C}^2$ by

$$f : (z_1, z_2) \in \mathbb{C}^2 \mapsto (P(z_1), Q(z_1) + R(z_2)) \in \mathbb{C}^2,$$

where P, Q, R are polynomials of degree p, q, λ with $\lambda = pq > \max(p, q)$. The indeterminacy locus consists of the single point

$$I_f = [0 : 0 : 1] = \{z_0 = z_2 = 0\},$$

where $\{z_0 = 0\}$ denotes the line at infinity, written in homogeneous coordinates. Observe that f contracts the line at infinity to the superattracting fixed point $[0 : 1 : 0] \notin I_f$. Therefore f is 1-stable on $\mathbb{P}^2$ and $\mathcal{I}_f = I_f$. The map f is an example of a weakly-regular polynomial endomorphism of $\mathbb{C}^2$ (see [GS]). It corresponds to the critical case of mappings considered in [G 1], since the topological degree of f,

$$d_t(f) = pq = \lambda,$$

coincides with its first dynamical degree $\lambda > 1$. Computations similar to those in [G 1] show that f satisfies (2) with $\beta = 1$.

3.2. **The case $\beta > 1$.**

Theorem 3.4. *Assume f satisfies (2) with $1 < \beta < \lambda$. Fix $\alpha \in [0, \frac{\log \lambda}{\log \beta} - 1[$. Then for all $x, y \in X \setminus \mathcal{I}_f$,*

$$|g_\omega(x) - g_\omega(y)| \leq \frac{C_\alpha(x,y)}{1 + |\log d(x,y)|^\alpha},$$

where $(x,y) \mapsto C_\alpha(x,y)$ is locally uniformly bounded in $X \setminus \mathcal{I}_f$.

We need the following elementary lemma whose proof is left to the reader.

Lemma 3.5. *Set $\varphi_\alpha(t) := [1 + |\log t|^\alpha]^{-1}$, for $0 \leq t \leq 1$.*

Then there exists $C' > 1$ independent of α such that for all $t \in [0,1]$ and for all $A \geq e$,

$$0 \leq t \leq C'\varphi_\alpha(t) \quad \textit{and} \quad 0 \leq \varphi_\alpha(At) \leq C'(\log A)^\alpha \varphi_\alpha(t).$$

Proof of Theorem 3.4. Let $x, y \in X - \mathcal{I}_f$ be given. Since $\beta > 1$ it follows from (3) and (6) that

$$|\gamma_\omega \circ f^j(x) - \gamma_\omega \circ f^j(y)| \leq \max\{|\gamma_\omega \circ f^j(x)|, |\gamma_\omega \circ f^j(y)|\} \leq M\beta^j,$$

for all $j \in \mathbb{N}$ and some constant M depending on $d_{\mathcal{I}_f}(x,y)$. From (5) and (6) we have the alternative bound

$$|\gamma_\omega \circ f^j(x) - \gamma_\omega \circ f^j(y)| \leq \left[\frac{C_2}{d_{\mathcal{I}_f}(x,y)^{m_3}}\right]^{\beta^j} d(x,y),$$

where $m_3 = m_1 + m_2/(\beta - 1)$ and $\log C_2 = [1 + m_1/(\beta-1) + m_2/(\beta-1)^2] \log C$. We infer using Lemma 3.5,

$$\frac{|\gamma_\omega \circ f^j(x) - \gamma_\omega \circ f^j(y)|}{M\beta^j} \leq C\beta^{\alpha j}\varphi_\alpha \circ d(x,y)$$

for all $j \in \mathbb{N}$. Once again C depends on x and y only through $d_{\mathcal{I}_f}(x,y)$. The factor of $M\beta^j$ ensures that the right side remains bounded above by 1; i.e., it guarantees that the hypothesis of Lemma 3.5 is satisfied.

Now we conclude that

$$\begin{aligned}|g_\omega(x) - g_\omega(y)| &\leq \sum_{j=0}^{\infty} M \left(\frac{\beta}{\lambda}\right)^j \frac{|\gamma_\omega \circ f^j(x) - \gamma_\omega \circ f^j(y)|}{M\beta^j} \\ &\leq C\varphi_\alpha(d(x,y)) \sum_{j=0}^{\infty} \left(\frac{\beta}{\lambda}\right)^j \beta^{\alpha j}.\end{aligned}$$

The sum on the right side converges as soon as $\beta^{\alpha+1} < \lambda$, i.e. as soon as $\alpha < \log\lambda/\log\beta - 1$. □

Weakly-regular polynomial endomorphisms of $\mathbb{C}^k$ as considered in [FG], [G 1], [GS] provide several examples of mappings which satisfy the assumptions of Theorem 3.4. Let us recall that such control on the modulus of continuity of g_ω yields integrability of $\log^+ ||Df||$ with respect to special invariant measures, as well as estimates on the pointwise dimension of these measures (see [S], [Di 2]).

4. Sobolev regularity

It can happen that the extended indeterminacy locus $\mathcal{I}_f$ is very large (see Section 6 for an example where $\mathcal{I}_f = X$). Our aim in this section is to consider weaker regularity properties for the functions g_ω which can hold even across points of indeterminacy. To simplify **we restrict ourselves to the case where f is a bimeromorphic surface map**; i.e., $\dim_{\mathbb{C}} X = 2$ and there is a meromorphic map $f^{-1} : X \to X$ such that $f \circ f^{-1} = \mathrm{Id}$.

In this case it was proved by the first author and C. Favre [DF] that one can always make a birational change of coordinates so that $f : X \to X$ becomes 1-stable. Moreover the spectral radius λ of $f^* : H^{1,1}_{\mathbb{R}}(X) \to H^{1,1}_{\mathbb{R}}(X)$ is a simple eigenvalue as soon as $\lambda > 1$. Since f^* preserves the pseudoeffective cone and f_* is intersection adjoint to f^*, there are classes $\{\omega^\pm\} \in H^{1,1}_{psef}(X)$, unique up to positive multiples, such that

$$f^*\{\omega^+\} = \lambda\{\omega^+\} \text{ and } (f^{-1})^*\{\omega^-\} = \lambda\{\omega^-\}.$$

By Theorem 2.4 applied to f and f^{-1}, there are positive closed currents

$$T^\pm = \omega^\pm + dd^c g^\pm, \text{ where } g^\pm = \sum_{j\geq 0} \frac{1}{\lambda^j}\gamma^\pm \circ f^{\pm j}$$

and the functions $\gamma^\pm \in L^1(X)$ satisfy $\lambda^{-1}(f^\pm)^*\omega^\pm = \omega^\pm + dd^c\gamma^\pm$. We shall assume moreover that

$$\{\omega^+\} \cdot f(p) > 0 \text{ and } \{\omega^-\} \cdot f^{-1}(q) > 0, \tag{4.1}$$

for all $p \in I_f$, $q \in I_{f^{-1}}$. This can always be arranged (see Proposition 4.1 in [BD]).

The main result of this section identifies geometric conditions equivalent to the statement that the gradients $\nabla g^\pm$ belong to $L^2(X)$. Since qpsh functions are always in $L^1(X)$, this is equivalent to saying that g^+ and g^- belong to the Sobolev space $W^{1,2}(X)$.

Theorem 4.1. *Let $f : X \to X$ be a birational map with $\lambda = \varrho_{f^*} > 1$. Then the following conditions are equivalent:*

1. $\nabla g^+ \in L^2(\omega^- \wedge \Omega)$;
2. *for all* $p \in I_f$, $\sum_{n\geq 0} \frac{\gamma^- \circ f^{-n}(p)}{\lambda^{2n}} > -\infty$;
3. $\sum_{n\geq 0} \lambda^{-2n} \log \mathrm{dist}(I_{f^{-1}}, f^{-n} I_f) > -\infty$.

Observe that when ω^- is Kähler, then (1) means precisely that g^+ has gradient in $L^2(X)$ with respect to the Lebesgue measure. The class $\{\omega^-\}$ is automatically Kähler when, for instance, $X = \mathbb{P}^2$. This condition should be compared to the slightly stronger condition studied in [BD]: g^- is finite at each point of I_f. This is equivalent to

$$\sum_{n\geq 0} \frac{\gamma^- \circ f^{-n}(p)}{\lambda^n} > -\infty$$

for all $p \in I_f$. As in [BD], these equivalent conditions are symmetric in f and f^{-1}; i.e., one can interchange the roles played by f and its inverse f^{-1} and obtain three further equivalent conditions.

Proof. The equivalence between (2) and (3) follows from the fact that under the assumption (4.1), γ^- is a smooth function in $X \setminus I_{f^{-1}}$ with logarithmic singularities at points of indeterminacy of f^{-1}. More precisely, there exist constants

$A, B, A', B' > 0$ such that

$$A \log \operatorname{dist}(x, I_{f^{-1}}) - B \leq \gamma^-(x) \leq A' \log \operatorname{dist}(x, I_{f^{-1}}) - B'.$$

We refer the reader to [BD] for a proof of this fact.

It is a simple exercise to check that condition (2) is equivalent to the finiteness of the sum $\sum_{n\geq 0} \lambda^{-n} g_n^-(p)$ at all points $p \in I_f$. Therefore the equivalence between (1) and (2) is a consequence of the next lemma. $\square$

Lemma 4.2.

$$\int_X dg_n^+ \wedge d^c g_n^+ \wedge \omega^- = \sum_{p\in I_f} \sum_{j=0}^{n-1} c_j(p) \frac{|g_j^-(p)|}{\lambda^j} + O(1),$$

where the constants $c_j(p)$ are positive and uniformly bounded away from 0 and ∞.

Proof. Set $T_n^+ := \lambda^{-n}(f^n)^*\omega^+$. For $0 \leq j \leq n-1$ we have

$$\int (\gamma^+ \circ f^j)\, T_n^+ \wedge \omega^- = \int \gamma^+ T_{n-j}^+ \wedge \frac{f_*^j \omega^-}{\lambda^j} = \int \gamma^+ T_{n-j}^+ \wedge \omega^- + A_{n,j}, \tag{4.2}$$

where

$$A_{n,j} := \int \gamma^+ T_{n-j}^+ \wedge dd^c g_j^- = \int \left[\frac{1}{\lambda} f^*\omega^+ - \omega^+\right] \wedge g_j^- T_{n-j}^+.$$

Observe that the currents $f^*\omega^+, T_{n-j}^+$ both have positive Lelong numbers at points in I_f; thus

$$\frac{1}{\lambda} f^*\omega^+ \wedge T_{n-j}^+ = \sum_{p\in I_f} c_{n-j}(p)\delta_p + \frac{1}{\lambda^2} f^*\left(\omega^+ \wedge T_{n-j-1}^+\right),$$

where δ_p denotes the Dirac mass at point p and the c_j's are positive constants. Observe that $f^*\omega^+ \wedge T_j \to f^*\omega^+ \wedge T^+$, since g_j^+ decreases towards g^+. Since T^+ has positive Lelong number at all points of indeterminacy, it follows that the measure $f^*\omega^+ \wedge T^+$ has a Dirac mass $c_\infty(p) > 0$ at each point $p \in I_f$. Therefore $c_j(p) \to c_\infty(p) > 0$. In particular, the sequences $(c_j(p))$ are uniformly bounded away from zero and infinity. We infer

$$A_{n,j} = \sum_{p\in I_f} c_{n-j}(p) g_j^-(p) + \frac{1}{\lambda^2} \int (g_j^- \circ f^{-1})\, \omega^+ \wedge T_{n-(j+1)}^+ - \int g_j^- \omega^+ \wedge T_{n-j}^+.$$

Observe that $\lambda^{-1} g_j^- \circ f^{-1} = g_{j+1}^- - \gamma^-$. Thus multiplying (8) by λ^{-j} and summing from $j = 0$ to $n-1$ yields

$$\int g_n^+ T_n^+ \wedge \omega^- = \sum_{p\in I_f} \sum_{j=0}^{n-1} c_{n-j}(p) \frac{g_j^-(p)}{\lambda^j} + \int \frac{g_n^-}{\lambda^n} \omega^+ \wedge \omega^+ - \int \gamma^- T^+ \wedge \omega^+ + M_n,$$

where

$$M_n := \sum_{j=0}^{n-1} \frac{1}{\lambda^j} \int \gamma^+ T_{n-j}^+ \wedge \omega^+ - \sum_{j=0}^{n-1} \frac{1}{\lambda^{j+2}} \int \gamma^- \omega^+ \wedge T_{n_j-1}^+$$

is a bounded sequence. Note to conclude that

$$\begin{aligned} \int (-g_n^+) T_n^+ \wedge \omega^- &= \int (-g_n)^+ dd^c g_n^+ \wedge \omega^- + \int (-g_n)^+ \omega^+ \wedge \omega^+ \\ &= \int dg_n^+ \wedge d^c g_n^+ \wedge \omega^- + O(1). \end{aligned}$$

$\square$

The function g^+ (resp. g^-) typically has positive Lelong number at every point of $I_f^\infty := \bigcup_{n\geq 0} I_{f^n} = \bigcup_{n\geq 0} f^{-n}(I_f)$ (resp. $I_{f^{-1}}^\infty := \bigcup_{n\geq 0} I_{f^{-n}} = \bigcup_{n\geq 0} f^n(I_{f^{-1}})$). The closer $f^{-n}I_f$ is to $I_{f^{-1}}$, the stronger the singularity of g^- at these points. The symmetric condition $\nabla g^\pm \in L^2(X)$ is thus a quantitative way of saying that the sets I_f^∞ and $I_{f^{-1}}^\infty$ stay away from each other (recall that the condition of 1-stability means precisely that $I_f^\infty \cap I_{f^{-1}}^\infty = \emptyset$). We will see in Section 6 that this condition is often but not always satisfied.

5. A canonical invariant measure

We consider here, as in the previous section, a compact Kähler surface X equipped with a Kähler form Ω, and a 1-stable bimeromorphic map $f : X \to X$ such that $\lambda = \rho_f > 1$. As before, we let $T^\pm = \omega^\pm + dd^c g^\pm$ denote the positive closed $(1,1)$-currents invariant under $f^{\pm 1}$. We normalize so that

$$\{T^+\} \cdot \{T^-\} = \{\Omega\} \cdot \{T^+\} = \{\Omega\} \cdot \{T^-\} = 1.$$

Our aim here is to define and study the measure

$$\mu_f := \text{“}T^+ \wedge T^-\text{”}.$$

5.1. **Definition of the canonical measure μ_f.** It is well known that one cannot always define the wedge product of two positive closed currents. When g^+ is integrable with respect to the trace measure of T^-, i.e. $g^+ \in L^1(T^- \wedge \Omega)$, the current g^+T^- is well defined, and we can set

$$\mu_f := \omega^+ \wedge T^- + dd^c(g^+T^-).$$

Observe that the condition is symmetric, as follows from the Stokes theorem:

$$\int_X (-g^+)T^- \wedge \Omega = \int_X (-g^-)T^+ \wedge \Omega + \int_X (g^-\omega^+ - g^+\omega^-) \wedge \Omega.$$

When the potentials $g^\pm$ have gradients in $L^2(X)$, it follows from the Cauchy-Schwarz inequality that $g^+ \in L^1(T^- \wedge \Omega)$, since

$$\begin{aligned} 0 &\leq \int -g^+T^- \wedge \Omega = \int -g^+\omega^- \wedge \Omega + \int dg^+ \wedge d^cg^- \wedge \Omega \\ &\leq \int -g^+\omega^- \wedge \Omega + \left(\int dg^+ \wedge d^cg^+ \wedge \Omega\right)^{1/2} \left(\int dg^- \wedge d^cg^- \wedge \Omega\right)^{1/2} < +\infty. \end{aligned}$$

It may happen, however, that $g^+ \in L^1(T^- \wedge \Omega)$ while $\nabla g^+ \notin L^2(X)$ (see example 6.2). We know of no example for which the function g^+ is not integrable with respect to the trace measure of T^-. Hence we have the following:

Question 5.1. Is the condition $g^+ \in L^1(T^- \wedge \Omega)$ always satisfied?

We now derive a criterion which will allow us to check the condition $g^+ \in L^1(T^- \wedge \Omega)$ for some birational mappings.

Proposition 5.2. *Let S be a positive closed $(1,1)$-current on X, whose cohomology class $\{S\}$ is Kähler. Assume*

1. $g^- \in L^1(S \wedge \Omega)$, *so that the measure $S \wedge T^-$ is well defined;*
2. $g^+ \in L^1(S \wedge T^-)$.

Then $g^+ \in L^1(T^- \wedge \Omega)$.

Proof. Let $\theta_S \geq \varepsilon\Omega$ be a Kähler form cohomologous to S. Let $\varphi_S \in L^1(X)$ be a qpsh function such that $S = \theta_S + dd^c\varphi_S$. We can assume without loss of generality that $\varphi_S \leq 0$ and $\varepsilon = 1$. Then

$$0 \leq \int_X -g^+T^- \wedge \Omega \leq \int_X -g^+T^- \wedge \theta_S = \int_X -g^+T^- \wedge S + \int_X g^+T^- \wedge dd^c\varphi_S.$$

The next to last integral is finite by assumption. The last one is finite by Stokes theorem:

$$\begin{aligned}\int g^+T^- \wedge dd^c\varphi_S &= \lim_{n\to\infty} \int g_n^+T^- \wedge dd^c\varphi_S = \lim_{n\to\infty} \int -\varphi_S T^- \wedge (-dd^c g_n^+) \\ &\leq \int_X (-\varphi_S)T^- \wedge \omega^+ < \infty.\end{aligned}$$

The first inequality holds because $-dd^c g_n^+ = \omega^+ - T^+ \leq \omega^+$. The second holds because $\omega^+ \leq C\Omega$ and $\varphi_S \in L^1(T^- \wedge \Omega) \Leftrightarrow g^- \in L^1(S \wedge \Omega)$. □

We will use this criterion in Section 6.2 when $S = [V]$ is the current of integration along an invariant irreducible curve V.

5.2. **Dynamical properties of μ_f.** We assume in the sequel that $g^+ \in L^1(T^- \wedge \Omega)$ so that $\mu_f = T^+ \wedge T^-$ is well defined.

Theorem 5.3. *The measure μ_f does not charge the indeterminacy locus I_f. It is an invariant probability measure.*

Proof. The current μ_f is a positive measure. This can be seen by locally regularizing the qpsh function g^+. It is a probability measure by our choice of normalization, and it is the weak limit of the measures

$$\mu_n := \frac{1}{\lambda^n}(f^n)^*\omega^+ \wedge T^-.$$

Let χ be a test function. Observe that T^- does not charge curves (it has zero Lelong number at each point in $X \setminus I_{f^{-1}}^\infty$; see [DF]), hence neither does μ_n. It follows therefore from the change of variables formula and the invariance $f_*T^- = \lambda T^-$ that

$$\langle \mu_n, \chi \rangle = \frac{1}{\lambda^{n+1}}\langle (f^n)^*\omega^+, \chi f_* T^- \rangle = \langle \mu_{n+1}, \chi \circ f \rangle.$$

We infer, if $\mu_f(I_f) = 0$, that $\langle \mu_f, \chi \rangle = \langle \mu_f, \chi \circ f \rangle$; hence μ_f is invariant.

It remains to prove that μ_f does not charge any point $p \in I_f$. Since f is 1-stable, we may assume that $p \notin I_{f^{-1}}^\infty$. If $p = f^{-N}(p)$ is periodic, then g^- is finite at p. Hence it follows from Proposition 5.4 below that $\mu_f(\{p\}) = 0$.

If on the other hand, p is not periodic, we can fix $N \gg 1$ and choose $r = r_N > 0$ such that $f^{-j}B(p,r) \cap f^{-k}B(p,r) = \emptyset$ for $0 \leq j,k \leq N$, $j \neq k$. Let χ be a test function such that $0 \leq \chi \leq 1$, $\chi \equiv 1$ near p, and $\operatorname{supp} \chi \subset B(p,r)$. Then

$$0 \leq \psi := \sum_{j=0}^{n-1} \chi \circ f^j \leq 1,$$

since the functions $\chi \circ f^j$ have disjoint supports. Set

$$R_n := (\psi \circ f^n)T^+ = \sum_{j=0}^{N-1} (\chi \circ f^{j+n})T^+.$$

It follows from the extremality of T^+ that the positive currents $(\chi \circ f^{j+n})T^+$ converge to $c_\chi T^+$, where $c_\chi = \int \chi d\mu_f$ (see [BD], [G 1]). Thus $R_n \to N c_\chi T^+$. Now $0 \leq R_n \leq T^+$ since $0 \leq \psi \leq 1$; hence

$$0 \leq \mu_f(\{p\}) \leq c_\chi = \int \chi d\mu_f \leq \frac{1}{N}.$$

Since N was arbitrary, we conclude that $\mu_f(\{p\}) = 0$. Thus μ_f does not charge the indeterminacy locus. □

Proposition 5.4. *Fix $p \in I_f$. Then $g^-(p)$ is finite if and only if $\log \mathrm{dist}(\cdot, p) \in L^1(\mu_f)$. In particular g^- is finite on I_f (the* [BD] *condition described after Theorem* 4.1*) if and only if $\log \mathrm{dist}(\cdot, I_f) \in L^1(\mu_f)$.*

Proof. We first characterize the [BD] condition. Afterward, we will show how to 'localize' it to individual points in I_f. Recall from [BD] that, under the assumption (4.1), $\log \mathrm{dist}(\cdot, I_f)$ is comparable to the function γ^+. Hence it suffices to analyze the condition $\gamma^+ \in L^1(\mu_f)$.

Suppose first that g^- is finite on I_f. It follows from Corollary 4.8 in [BD] that $g^+ \in L^1(\mu_f)$, and from the bound $g^+ \leq \gamma^+ \leq 0$ that $\gamma^+ \in L^1(\mu_f)$.

Assume now that $\gamma^+ \in L^1(\mu_f)$. It follows from the Stokes theorem that

$$\int_X (-\gamma^+) d\mu_f = O(1) + \int_X (-g^+) dd^c\gamma^+ \wedge T^- \geq O(1) + \int_X d\gamma^+ \wedge d^c\gamma^+ \wedge T^-,$$

where the $O(1)$ terms account for the fact that g^+ and γ^+ are plurisubharmonic only up to the addition of a smooth function. Therefore γ^+ has finite energy with respect to the invariant current T^-. By the Stokes theorem again,

$$\int d\gamma^+ \wedge d^c\gamma^+ \wedge T^- = O(1) + \int d\gamma^+ \wedge d^c\gamma^+ \wedge dd^c g^- = O(1) + \int (-g^-)(dd^c\gamma^+)^2.$$

Now

$$(dd^c\gamma^+)^2 = \sum_{p \in I_f} c_p \delta_p + \lambda^{-2} f^*(\omega^+ \wedge \omega^+) - 2\lambda^{-1} f^*\omega^+ \wedge \omega^+ + \omega^+ \wedge \omega^+,$$

where $c_p > 0$ and δ_p denotes the Dirac mass at point p. Therefore

$$\int_X d\gamma^+ \wedge d^c\gamma^+ \wedge T^- = O(1) - \sum_{p \in I_f} c_p g^-(p),$$

so $\gamma^+ \in L^1(\mu_f)$ implies that g^- is finite at every point of I_f.

We can now localize the previous reasoning in the following way. Fix $p \in I_f$ and $\chi \geq 0$ a test function supported near p such that $\chi \equiv 1$ in some small neighborhood of p. Thus $\varphi^+ := \chi\gamma^+$ is comparable to $\log \mathrm{dist}(\cdot, p)$. Observe that $dd^c\varphi^+$ equals $\chi dd^c\gamma^+$ up to a smooth form. By using the Stokes theorem as above we thus get

$$\int (-\varphi^+) d\mu_f = O(1) + \int (-g^-) T^+ \wedge \chi dd^c\gamma^+ = O(1) + \lambda^{-1} \int (-g^-) T^+ \wedge \chi f^*\omega^+.$$

Now $\chi T^+ \wedge f^*\omega^+ = c\delta_p + \chi f^*(T^+ \wedge \omega^+)$ for some $c > 0$. Therefore

$$\int (-\varphi^+) d\mu_f = O(1) - c\frac{g^-(p)}{\lambda} + \int (-g^- \circ f^{-1})\, T^+ \wedge \omega^+.$$

The last integral is finite since $0 \geq g^- \circ f^{-1} \geq \lambda g^-$. Thus $\varphi^+ \simeq \log \mathrm{dist}(\cdot, p)$ is integrable with respect to μ_f if and only if g^- is finite at point p. □

Theorem 5.5. *The measure μ_f is mixing.*

Proof. Let χ, ψ be test functions. We have to show that

$$\int_X \psi(\chi \circ f^n)\, d\mu_f \longrightarrow c_\chi c_\psi, \quad \text{where } c_\chi = \int_X \chi\, d\mu_f \text{ and } c_\psi = \int_X \psi\, d\mu_f.$$

It follows from the extremality of T^+ that the currents $(\chi \circ f^n)T^+$ converge weakly towards $c_\chi T^+$. For fixed j, the forms $\psi\lambda^{-j}(f^{-j})^*\omega^-$ are smooth off the finite set $I_{f^{-j}}$. Since $T^+ \wedge \lambda^{-j}(f^{-j})^*\omega^-$ does not charge this set, we infer

$$\langle (\chi \circ f^n)\, T^+, \psi\lambda^{-j}(f^{-j})^*\omega^- \rangle \longrightarrow \langle c_\chi T^+, \psi\lambda^{-j}(f^{-j})^*\omega^- \rangle,$$

as $n \to +\infty$. Since $\lambda^{-j}(f^{-j})^*\omega^- = \omega^- + dd^c g_j^-$ with g_j^- decreasing towards g^-, it follows from the Monotone Convergence Theorem that

$$\langle c_\chi T^+, \psi\lambda^{-j}(f^{-j})^*\omega^- \rangle \longrightarrow \langle c_\chi T^+, \psi T^- \rangle = c_\chi c_\psi,$$

when $j \to +\infty$. Therefore it suffices to show that the difference

$$\langle \chi \circ f^n T^+, \psi T^- \rangle - \langle \chi \circ f^n T^+, \psi\lambda^{-j}(f^{-j})^*\omega^- \rangle = \langle \chi \circ f^n T^+, \psi dd^c(g^- - g_j^-) \rangle$$

converges to 0 as $j \to +\infty$ uniformly with respect to n. By the Stokes theorem, it suffices to uniformly control the following four quantities:

$$A_{n,j} := \langle \chi \circ f^n T^+ \wedge dd^c\psi, (g^- - g_j^-) \rangle; \; B_{n,j} := \langle \psi dd^c(\chi \circ f^n) \wedge T^+, (g^- - g_j^-) \rangle;$$

and

$$C_{n,j} := \langle d\psi \wedge d^c(\chi \circ f^n) \wedge T^+, (g^- - g_j^-) \rangle; \; D_{n,j} := \langle d^c\psi \wedge d(\chi \circ f^n) \wedge T^+, (g^- - g_j^-) \rangle.$$

Observe first that

$$|A_{n,j}| \le ||\psi||_{\mathcal{C}^2} ||\chi||_{\mathcal{C}^0} \; \langle \Omega \wedge T^+, (g_j^- - g^-) \rangle.$$

To control $B_{n,j}$, we observe that $dd^c(\chi \circ f^n\, T^+)$ does not charge curves. Hence changing variables gives

$$|B_{n,j}| = \left| \langle dd^c\chi \wedge T^+, \psi \circ f^{-n}(g^- - g_{j+n}^-) \rangle \right| \le ||\psi||_{\mathcal{C}^0} ||\chi||_{\mathcal{C}^2} \; \langle \Omega \wedge T^+, (g_{j+n}^- - g^-) \rangle.$$

To control $C_{n,j}$, we use the Cauchy-Schwarz inequality and obtain

$$\begin{aligned} |C_{n,j}| &\le \langle d\psi \wedge d^c\psi \wedge T^+, (g^- - g_j^-) \rangle^{1/2} \cdot \langle d\chi \circ f^n \wedge d^c\chi \circ f^n \wedge T^+, (g^- - g_j^-) \rangle^{1/2} \\ &\le ||\psi||_{\mathcal{C}^1} ||\chi||_{\mathcal{C}^1} \; \langle \Omega \wedge T^+, (g_j^- - g^-) \rangle^{1/2} \cdot \langle \Omega \wedge T^+, (g_{j+n}^- - g^-) \rangle^{1/2}. \end{aligned}$$

The estimation for $D_{n,j}$ is similar. This shows that μ_f is mixing. □

Thanks to the preceding results, we can show that μ_f is not too concentrated.

Corollary 5.6. *The measure μ_f does not charge compact complex curves.*

Proof. Suppose first that μ_f charges some point $p \in X$. By Theorems 5.3 and 5.5, p is a fixed point not in I_f or $I_{f^{-1}}$, and μ_f is concentrated entirely at p. In particular, both functions $\log \operatorname{dist}(\cdot, I_f)$ and $\log \operatorname{dist}(\cdot, I_{f^{-1}})$ are μ_f-integrable. From Proposition 5.4 we obtain that g^+ is finite on $I_{f^{-1}}$ and g^- is finite on I_f, i.e. that the [BD] condition holds. Theorem 4.10 from [BD] then implies that μ_f does not charge points, which is a contradiction.

Now suppose that μ_f charges some irreducible curve $V \subset X$. Then invariance of μ_f implies that V cannot be critical for f, because $f(V - I_f) \subset I_{f^{-1}}$. Invariance also implies that μ_f almost every point is non-wandering. Hence f^k restricts to an automorphism of V for some $k \ge 0$. However, the only mixing invariant measures for automorphisms of curves are point masses concentrated at fixed points, and we have already ruled out the possibility that μ_f charges points. $\square$

In Section 6 we will see examples where $g^+ \in L^1(T^- \wedge \Omega)$ but the conditions in both Theorem 4.1 and in [BD] fail. Observe that when the condition in [BD] is not satisfied, it is unclear whether f has well-defined Lyapunov exponents (see Proposition 5.4).

6. Examples

In this section, we present two examples that complement the theorems above. The first shows that the indeterminacy orbit of a rational map can be dense in the host manifold. The second, which occupies the majority of the section, shows that the invariant measure $\mu_f = T^+ \wedge T^-$ can exist for a birational surface map even when the map fails to satisfy the equivalent conditions in Theorem 4.1.

6.1. **A rational map with dense indeterminacy orbits.** Let $Y = E \times E$ be a complex torus, where $E = \mathbb{C}/\mathbb{Z}[\zeta]$ is the elliptic curve associated to a primitive root of unity ζ of order $3, 4$ or 6. The matrix

$$A = \begin{bmatrix} d & 1 \\ 1 & d \end{bmatrix},\ d \geq 3,$$

preserves the lattice $\Lambda = \mathbb{Z}[\zeta] \times \mathbb{Z}[\zeta]$ and thus induces a holomorphic endomorphism $g : Y \to Y$ such that

$$\lambda_2(g) = (d^2 - 1)^2 \ \text{ and } \ \lambda_1(g) = (d+1)^2 < \lambda_2(g).$$

Let $\sigma : Y \to Y$ be multiplication by ζ, and let $f : X \to X$ denote the endomorphism induced by g on the rational surface X obtained by desingularizing the quotient $Y/\langle\sigma\rangle$, i.e. by blowing up at fixed points of σ.

Let a be such a fixed point. Since g has topological degree $\lambda_2(g) \geq 2$, $g^{-1}(a)$ contains preimages other than the fixed points of σ. Each point in $g^{-1}(a) \setminus Fix(\sigma)$ corresponds, in X, to a point of indeterminacy of f. Since the Lebesgue measure ν_Y of the torus Y is g-mixing, the preimages $(g^{-n}(a))_{n \in \mathbb{N}}$ are equidistributed with respect to ν_Y and therefore dense in Y. It follows that the set

$$I_f^\infty := \bigcup_{n \in \mathbb{N}} f^{-n} I_f \text{ is dense in } X.$$

Observe also that f is 1-stable: since g does not contract any curve, neither does f.

6.2. **A birational surface map with constrained indeterminacy orbits.** Our second example is a variation on one due to Favre [F]. For parameters $a, b, c \in \mathbb{C}^*$, we consider $f = f_{abc} : \mathbb{P}^2 \to \mathbb{P}^2$ defined by

$$f[x : y : z] = [bcx(-cx + acy + z) : acy(x - ay + abz) : abz(bcx + y - bz)].$$

The following facts can be verified by straightforward computation.

- f_{abc} is birational with inverse $f^{-1} = f_{a^{-1}b^{-1}c^{-1}}$.
- $I_f = \{[a : 1 : 0], [0 : b : 1], [1 : 0 : c]\}$.

- f preserves each of the lines $\{x = 0\}$, $\{y = 0\}$, $\{z = 0\}$ according to the formulas

$$[x : 1 : 0] \mapsto [-\frac{bc}{a}x : 1 : 0], \quad [0 : y : 1] \mapsto \left[0 : -\frac{ac}{b}y : 1\right], \quad [1 : 0 : z] \mapsto \left[1 : 0 : -\frac{ba}{c}z\right].$$

In particular, we have $I_f^\infty, I_{f^{-1}}^\infty \subset \{xyz = 0\}$ for all $a, b, c \in \mathbb{C}^*$. Let Ω denote the Fubini Study Kähler form on $\mathbb{P}^2$. We will spend the rest of this section proving

Theorem 6.1. *Given $s > 1$ and an irrational number $\theta \in \mathbb{R}$, let $f : \mathbb{P}^2 \to \mathbb{P}^2$ be the birational map $f = f_{abc}$ with $a = i$, $b = -se^{2\pi i\theta}$, $c = i/s$. Then f is 1-stable with invariant currents $T^\pm = \Omega + dd^c g^\pm$ such that $g^+ \in L^1(\Omega \wedge T^-)$. Moreover, for suitable choices of the irrational number θ, the condition (3) in Theorem 4.1 fails.*

In other words, the measure $\mu = T^+ \wedge T^-$ is well-defined even though the conditions in Theorem 4.1 sometimes fail.

Proof. To see that f is 1-stable, observe that the hypothesis $s > 1$ implies that $f^{-n}I_f \cap \{x = 0\} \subset \{[0 : y : 1] : |y| > 1\}$ for all $n \geq 0$, whereas $I_{f^{-1}} \cap \{x = 0\} \subset \{[0 : y : 1] : |y| < 1\}$. In particular $f^{-n}I_f \cap I_{f^{-1}} \cap \{x = 0\} = \emptyset$. Similarly, $f^{-n}I_f \cap I_{f^{-1}} \cap \{y = 0\} = \emptyset$. Finally, for each $n \geq 0$, we have $f^{-n}I_f \cap \{z = 0\} = [ie^{-2\pi in\theta} : 1 : 0]$, and since θ is irrational, these points never coincide with $I_{f^{-1}} \cap \{z = 0\} = [-i : 1 : 0]$. We conclude that $f^{-n}I_f \cap I_{f^{-1}} = \emptyset$ for all $n \geq 0$. That is, f is 1-stable. In particular $\lambda := \varrho_{f^*} = 2$.

To prove that $g^+ \in L^1(\Omega \wedge T^-)$, we apply Proposition 5.2 with S equal to the current of integration $[x = 0]$ over the line $\{x = 0\}$. Observe that $\Omega \wedge [x = 0]$ is just area measure on $\{x = 0\}$. Because g^- is qpsh it follows that either g^- is integrable with respect to $\Omega \wedge [x = 0]$, or $g^-|_{\{x=0\}} \equiv -\infty$. The latter is far from true, however. One can compute directly, for instance, that $g^- > -\infty$ at the fixed point $[0 : 0 : 1]$. Thus $g^- \in L^1(\Omega \wedge [x = 0])$.

It follows from standard arguments that local potentials for T^- must be harmonic on any open set $U \subset \{x = 0\}$ such that

- iterates of f^{-1} form a normal family on U, and
- $U \cap f^n I_{f^{-1}} = \emptyset$ for all $n \geq 0$.

The only point in $\{x = 0\}$ where iterates of f^{-1} fail to act normally is the fixed point $[0 : 0 : 1]$. Hence

$$\operatorname{supp}([x = 0] \cap T^-) \subset \overline{\bigcup_{n \geq 0} f^n I_{f^{-1}}}$$

is a compact subset of $\{[0 : y : 1] : |y| < 1\}$. Replacing f^{-1} with f, the same reasoning shows that local potentials for T^+ are harmonic on $\{[0 : y : 1] : |y| < 1\}$. Thus g^+ is uniformly bounded on $\operatorname{supp}([x = 0] \cap T^-)$, and it follows that $g^+ \in L^1([x = 0] \wedge T^-)$. Therefore by Proposition 5.2, $g^+ \in L^1(\Omega \wedge T^-)$.

To see that condition (3) in Theorem 4.1 fails for suitably chosen θ, let $h : \mathbb{R}^+ \to \mathbb{R}^+$ be a function decreasing rapidly to 0. By a Baire category argument, one can find irrational θ such that

$$2n_j\theta \mod 1 < h(n_j)$$

for infinitely many $n_j \in \mathbb{N}$. Thus, if we set $p^+ = [i : 1 : 0] \in I_f$ and $p^- = [-i : 1 : 0] \in I_f$, we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{1}{\lambda^{2n}} \log \operatorname{dist}(f^{-n} I_f, I_{f^{-1}}) &\leq \sum_{n=0}^{\infty} \frac{\log \operatorname{dist}(f^n(p^-), p^+)}{2^{2n}} \\ &\leq C \sum_{n=0}^{\infty} \frac{\log |e^{-2\pi n i} + 1|}{2^{2n}} \leq C \sum_{j=0}^{\infty} \frac{\log h(n_j)}{2^{2n_j}}.\end{aligned}$$

The last sum diverges to $-\infty$ if we take e.g. $h(x) = 2^{-2^{2n}}$, and condition (3) in Theorem 4.1 then fails. $\square$

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