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DESINGULARIZATION OF QUASIPLURISUBHARMONIC FUNCTIONS

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Let T be a positive closed current of bidegree $(1,1)$ on a compact complex surface. We show that for all $\varepsilon > 0$, one can find a finite composition of blow-ups π such that π^*T decomposes as the sum of a divisorial part and a positive closed current whose Lelong numbers are all less than ε.

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1. Introduction

Let X be a compact complex surface (i.e. $\dim_{\mathbb{C}} X = 2$). It is a classical result that if $\mathcal{C}$ is a complex (singular) curve in X, then one can find $\pi : \tilde{X} \to X$ a finite composition of blow-ups such that the strict transform $\tilde{\mathcal{C}}$ of $\mathcal{C}$ is smooth.

Let $[\mathcal{C}]$ denote the current of integration along the curve $\mathcal{C}$. It has Lelong number 0 at points in $X \setminus \mathcal{C}$, 1 at regular points of $\mathcal{C}$, and Lelong number ≥ 2 at every singular point. Thus the above result can be seen as an attenuation of singularities, in the sense that $[\tilde{\mathcal{C}}]$ has all its Lelong numbers ≤ 1. More generally, given T a positive closed current of bidegree $(1,1)$ on X, one can wonder whether it is possible to attenuate its singularities. Denote by $\nu(T,p)$ the Lelong number of the current T at point p (we refer the reader to [4] for basic facts on positive currents and Lelong numbers). The aim of this note is to give an elementary proof of the following result:

Theorem 1.1. *Let T be a positive closed current of bidegree $(1,1)$ on a compact complex surface X. For any $\varepsilon > 0$, there exists $\pi : \tilde{X} \to X$ a finite composition of blow-ups such that*

$$\pi^*T = \sum_{j=1}^{s} c_j [\tilde{\mathcal{C}}_j] + \tilde{T},$$

where the $\tilde{\mathcal{C}}_j$'s are smooth curves with normal crossings, $c_j \geq 0$ and $\tilde{T}$ is a positive closed current of bidegree $(1,1)$ on $\tilde{X}$ such that $\sup_{x \in \tilde{X}} \nu(\tilde{T}, x) < \varepsilon$.

A semi-local version of this result was first obtained by Mimouni [11]. Theorem 1.1 has been also proved independently by Favre and Jonsson (see [7, Theorem 7.2]) by using an interesting (but difficult) analysis on the "valuative tree" [6]. Their proof also works in a local context. The proof we present here is quite elementary and follows the approach of Mimouni.

When $T = [\mathcal{C}]$ is the current of integration along a complex curve, then $c_j = 1$ automatically and $\tilde{T} = 0$ as soon as $\varepsilon \leq 1$, so Theorem 1.1 is indeed a generalization of the desingularization of curves.

When the surface X is *Kähler*, there is an alternative way of stating this result. Let us recall that a quasiplurisubharmonic function (qpsh for short) is an upper-semi-continuous L^1-function φ on X whose curvature is bounded from below by a smooth form, say

$$dd^c\varphi \geq -\omega,$$

where ω is a smooth closed $(1,1)$-form on X and $d = \partial + \overline{\partial}$, $d^c = \frac{i}{\pi}(\bar{\partial} - \partial)$ are both real differential operators. Such functions are locally given as the difference of a plurisubharmonic function and a smooth function. Thus qpsh functions, once normalized, are in 1-to-1 correspondence with positive closed currents of bidegree $(1,1)$ on X: one can associate to φ the positive current $T_\varphi = \omega + dd^c\varphi \geq 0$. Conversely if T is a positive closed current of bidegree $(1,1)$ on X, then T is cohomologous to a smooth closed $(1,1)$-form ω, hence $T = \omega + dd^c\varphi$ for some qpsh function φ (uniquely determined up to an additive constant): this is the celebrated "dd^c-lemma" of Kähler geometry (see e.g. [8, p. 149]). Our result can therefore be interpreted as an attenuation of singularities for qpsh functions. This is of practical importance in complex geometry, for example, where such functions arise as positive (singular) metrics of holomorphic line bundles. We refer the reader to [5, 9] for more information on this point of view.

2. Proof of the Theorem

In the sequel we let $\mathcal{T}(X)$ denote the cone of positive closed currents of bidegree $(1,1)$ on X. Basic facts on positive currents can be found, e.g. in [4]. Of crucial importance is the following decomposition result of Siu [12]: if $T \in \mathcal{T}(X)$, then T can be written

$$T = \sum_{j \geq 0} c_j[\mathcal{C}_j] + T_0,$$

where the $\mathcal{C}_j$'s are (singular) complex (closed) curves in X, the c_j's are non-negative constants, and $T_0 \in \mathcal{T}(X)$ does not charge any proper analytic subset of X. Equivalently, the set

$$E^+(T_0) := \{x \in X / \nu(T_0, x) > 0\}$$

is at most countable. We shall say in the sequel that a current T_0 is **diffuse** when it does not charge curves.

We start by recalling a classical result linking the Lelong number of a current T at a point p and Lelong numbers of the total transform π^*T under the blow-up π at point p.

Lemma 2.1. *Let $\pi : \tilde{X} \to X$ be the blow up of X at point p and let $E = \pi^{-1}(p)$ denote the exceptional divisor. If $T \in \mathcal{T}(X)$, then*

$$\pi^*T = \nu(T,p)[E] + \tilde{T},$$

where $\tilde{T} \in \mathcal{T}(\tilde{X})$ does not charge E.

Moreover, for all $q \in \tilde{X}$, $\nu(\tilde{T}, q) \leq \nu(T, \pi(q))$. In particular, if T is diffuse, so is $\tilde{T}$.

Proof. It follows from Siu's decomposition result that $\pi^*T = c[E] + \tilde{T}$ where $c \geq 0$ and $\tilde{T} \in \mathcal{T}(\tilde{X})$ does not charge E. Note that π is a local biholomorphism near each point $q \in \tilde{X} \setminus E$, hence $\nu(\tilde{T}, q) = \nu(T, \pi(q))$.

Let us recall that $E \simeq \mathbb{P}^1$ is the set of complex lines in X through p. For $q \in E$, $\nu(\pi^*T, q)$ can be interpreted as the Kiselman number of T with weight $(2,1)$ with respect to the axes $q, q^\perp$, viewed as a system of local coordinates at p. We leave the computation to the reader and denote this generalized Lelong number by $\nu_{Kis}(T; (q, q^\perp); (2,1))$. It follows from the basic properties of Kiselman numbers that

1. $\nu_{Kis}(T; (q, q^\perp); (2,1)) = \nu(T,p)$ for all but countably many $q \in E$. In particular $c = \nu(T,p)$;
2. $\nu(\pi^*T, q) = \nu_{Kis}(T; (q, q^\perp); (2,1)) \leq 2\nu(T,p)$, hence $\nu(\tilde{T}, q) \leq \nu(T,p)$ for all $q \in E$.

Let us emphasize that these numbers depend on the choice of local coordinates (actually on the choice of axes), however this ambiguity is fixed here by the choice of the direction $q \in E \simeq \mathbb{P}^1$. We refer the reader to [10, 4] for basic facts on Kiselman numbers. □

The next lemma is an elementary observation that is crucial for the proof of Theorem 1.1.

Lemma 2.2. *Assume $T \in \mathcal{T}(X)$ is diffuse. Then*

$$\{T\}^2 \geq 0,$$

where $\{T\}^2$ denotes the self-intersection of $\{T\} \in H^2(X, \mathbb{R})$, the cohomology class of T.

Proof. It follows from the work of Demailly [3] that there exists $C > 0$ such that for all $\varepsilon > 0$, one can find closed real currents T_ε cohomologous to T such that

1. $T_\varepsilon \geq -C\varepsilon\omega$, where ω is a fixed hermitian metric;
2. T_ε is smooth in $X \setminus E_\varepsilon(T)$, where $E_\varepsilon(T)$ is the finite set of points x such that $\nu(T,x) \geq \varepsilon$;
3. $T_\varepsilon \to T$ in the weak sense of currents and $\nu(T_\varepsilon, x) \to \nu(T,x)$.

We infer that the currents T_ε have local potentials whose gradients are locally in L^2. One can therefore define the pointwise wedge-product of T_ε with itself (see [1, 2]) which is bounded from below by $-C^2\varepsilon^2\omega^2$. On cohomology classes this yields

$$\{T\}^2 = \lim\{T_\varepsilon\}^2 \geq 0. \qquad \square$$

Proof of Theorem 1.1. Fix $T \in \mathcal{T}(X)$ and $\varepsilon > 0$. It follows froms Siu's decomposition result that

$$T = \sum_{j\geq 0} c_j[\mathcal{C}_j] + T_0,$$

where the $\mathcal{C}_j$'s are (possibly singular) curves, $c_j \geq 0$ and T_0 is a diffuse current.

Step 1. Set

$$E_\varepsilon(T_0) := \{x \in X \,/\, \nu(T_0, x) \geq \varepsilon\}.$$

It is another result of Siu [12] that $E_\varepsilon(T_0)$ is a closed proper analytic subset of X. Since T_0 is diffuse, this is a finite set of points. If this set is empty, this is the end of Step 1. Otherwise we label these points, $E_\varepsilon(T_0) = \{p_1^0, \dots, p_{s_0}^0\}$. Let $\pi_1 : X_1 \to X$ denote the blow-up of X at points $p_1^0, \dots, p_{s_0}^0$. By Lemma 2.1,

$$\pi_1^* T_0 = \sum_{j=1}^{s_0} \nu\big(T_0, p_j^0\big)\big[E_j^1\big] + T_1,$$

where $E_j^1 = \pi_1^{-1}\big(p_j^0\big)$ and $T_1 \in \mathcal{T}(X_1)$ is a diffuse current. Let us recall that $\{\pi_1^* T_0\} \cdot \big\{E_j^1\big\} = 0$ and $\big\{E_j^1\big\}^2 = -1$, so it follows from Lemma 2.2 that

$$\sum_{j=1}^{s_0} \nu\big(T_0, p_j^0\big)^2 = \{T_0\}^2 - \{T_1\}^2 \leq \{T_0\}^2.$$

Consider now $E_\varepsilon(T_1)$. If this set is empty we stop here, otherwise $E_\varepsilon(T_1) = \{p_1^1, \dots, p_{s_1}^1\}$ is finite. Let $\pi_2 : X_2 \to X_1$ denote the blow-up of X_1 at these points. Then

$$\pi_2^* T_1 = \sum_{j=1}^{s_1} \nu\big(T_1, p_j^1\big)\big[E_j^2\big] + T_2,$$

where $E_j^2 = \pi_2^{-1}\big(p_j^1\big)$ and $T_2 \in \mathcal{T}(X_2)$ is again a diffuse current. We infer from Lemma 2.2

$$\sum_{j=1}^{s_1} \nu\big(T_1, p_j^1\big)^2 + \sum_{j=1}^{s_0} \nu\big(T_0, p_j^0\big)^2 = \{T_0\}^2 - \{T_2\}^2 \leq \{T_0\}^2.$$

Observe that T_2 can be written

$$T_2 = (\pi_1 \circ \pi_2)^* T_0 - \sum_{j=1}^{s_1'} \alpha_j^2 \big[E_j^2\big],$$

where the E_j^2's denote, for $1 + s_1 \leq j \leq s_1' = s_0 + s_1$, the strict transform of the E_j^1's under π_2.

Going on by induction, we end up with

$$T_n = \pi^* T_0 - \sum_{j=1}^{s_n'} \alpha_j^n \left[E_j^n\right],$$

a diffuse current on $X_n \xrightarrow{\pi} X$, where $\pi = \pi_1 \circ \cdots \circ \pi_n$, such that

$$\sum_{i=0}^{n} \sum_{j=1}^{s_j} \nu\big(T_i, p_j^i\big)^2 \leq \{T_0\}^2.$$

Thus the series on the left-hand side is convergent, so $\sup_{x \in X} \nu(T_n, x) < \varepsilon$ for n large enough. This ends Step 1.

Step 2. Going back to our current T, what we have proved so far writes

$$\pi^* T = \sum_{j \geq O} c_j' \left[\mathcal{C}_j{}'\right] + T_n,$$

where $\sup_{x \in X_n} \nu(T_n, x) < \varepsilon$ and the $\mathcal{C}_j{}'$'s are either exceptional divisors or strict transforms of the curves $\mathcal{C}_j$. Observe now that $E_\varepsilon(\pi^* T)$ is a proper analytic subset of X_n, so only finitely many $\mathcal{C}_j{}'$'s may produce Lelong numbers $\geq \varepsilon$. Thus we can rewrite (after possibly relabelling)

$$\pi^* T = \sum_{j=O}^{s} c_j' \left[\mathcal{C}_j{}'\right] + T_n',$$

where T_n' is not necessarily diffuse anymore but satisfies $\sup_{x \in X_n} \nu(T_n', x) < \varepsilon$. It only remains to blow-up a few more times in order to desingularize the $\mathcal{C}_j{}'$'s. This may of course modify the current T_n', but it will not increase the Lelong numbers thanks to Lemma 2.1. □

Remark 2.3. In [11], Mimouni considers psh functions φ in the unit ball B of $\mathbb{C}^2$ and obtains the same attenuation result under the hypothesis that φ is locally bounded near the boundary ∂B. Note that such functions can easily be extended as qpsh functions on the complex projective space $X = \mathbb{P}^2$, hence Mimouni's result can be seen as a particular case of Theorem 1.1.

References

[1] E. Bedford and B. A. Taylor, Variational properties of the complex Monge–Ampère equation. I. Dirichlet principle, *Duke Math. J.* **45**(2) (1978) 375–403.

[2] Z. Blocki, On the definition of the Monge–Ampère operator in $\mathbb{C}^2$, *Math. Ann.* **328**(3) (2004) 415–423.

[3] J.-P. Demailly, Regularization of closed positive currents and intersection theory, *J. Algebraic Geom.* **1**(3) (1992) 361–409.

[4] J.-P. Demailly, Complex analytic and differential geometry, Free accessible book, http://www-fourier.ujf-grenoble.fr/ demailly/books.html.
[5] J.-P. Demailly and J. Kollar, Semi-continuity of complex singularity exponents and Kähler-Einstein metrics on Fano orbifolds, *Ann. Sci. Ecole Norm. Sup. (4)* **34**(4) (2001) 525–556.
[6] C. Favre and M. Jonsson, *The Valuative Tree*, Lecture Notes in Mathematics, Vol. 1853 (Springer, 2004).
[7] C. Favre and M. Jonsson, Valuative analysis of plurisubharmonic functions, preprint, www.arxiv.org/abs/math.CV/0401111.
[8] P. Griffiths and J. Harris, *Principles of Algebraic Geometry*, Pure and Applied Mathematics (Wiley-Interscience, New York, 1978).
[9] V. Guedj and A. Zeriahi, Intrinsic capacities on compact Kähler manifolds, preprint, www.arxiv.org/abs/math.CV/0401302
[10] C. Kiselman, Attenuating the singularities of plurisubharmonic functions, *Ann. Polon. Math.* **60**(2) (1994) 173–197.
[11] S. K. Mimouni, Singularités des fonctions plurisousharmoniques et courants de Liouville, Thèse de la faculté des sciences de Monastir, Tunisie (2001).
[12] Y.-T. Siu, Analyticity of sets associated to Lelong numbers and the extension of closed positive currents, *Invent. Math.* **27** (1974) 53–156.