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Ergodic properties of rational mappings with large topological degree

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Abstract

Let X be a projective manifold and $f : X \to X$ a rational mapping with large topological degree, $d_t > \lambda_{k-1}(f) :=$ the $(k-1)^{\text{th}}$ dynamical degree of f. We give an elementary construction of a probability measure μ_f such that $d_t^{-n}(f^n)^*\Theta \to \mu_f$ for every smooth probability measure Θ on X. We show that every quasiplurisubharmonic function is μ_f-integrable. In particular μ_f does not charge either points of indeterminacy or pluripolar sets, hence μ_f is f-invariant with constant jacobian $f^*\mu_f = d_t\mu_f$. We then establish the main ergodic properties of μ_f: it is mixing with positive Lyapunov exponents, preimages of "most" points as well as repelling periodic points are equidistributed with respect to μ_f. Moreover, when $\dim_{\mathbb{C}} X \leq 3$ or when X is complex homogeneous, μ_f is the unique measure of maximal entropy.

Introduction

Let X be a projective algebraic manifold and ω a Hodge form on X normalized so that $\int_X \omega^k = 1$, $k = \dim_{\mathbb{C}} X$. Let $f : X \to X$ be a rational mapping. We shall always assume in the sequel that f is dominating; i.e., its jacobian determinant does not vanish identically in any coordinate chart. We let I_f denote the indeterminacy locus of f (the points where f is not holomorphic): this is an algebraic subvariety of codimension ≥ 2. We let d_t denote the topological degree of f: this is the number of preimages of a generic point.

Define $f^*\omega^k$ to be the trivial extension through I_f of $(f_{|X\setminus I_f})^*\omega \wedge \cdots \wedge (f_{|X\setminus I_f})^*\omega$. This is a Radon measure of total mass d_t. When $d_t > \lambda_{k-1}(f)$ (see Section 1 below), we give an elementary construction of a probability measure μ_f such that $d_t^{-n}(f^n)^*\omega^k \to \mu_f$. We show that every quasiplurisubharmonic function is μ_f-integrable (Theorem 2.1). In particular μ_f does not charge pluripolar sets. This answers a question raised by Russakovskii and Shiffman [RS 97] which was addressed by several authors (see [HP 99], [FG 01], [G 02], [Do 01], [DS 02]). This also shows that μ_f is an invariant measure with positive entropy $\geq \log d_t > 0$. Thus f has positive topological entropy.

Building on the work of Briend and Duval [BD 01], we then establish the main ergodic properties of μ_f: it is mixing with positive Lyapunov exponents, preimages of "most" points as well as repelling periodic points are equidistributed with respect to μ_f (Theorem 3.1). Moreover, when $\dim_{\mathbb{C}} X \leq 3$ or when the group of automorphisms $\mathrm{Aut}(X)$ acts transitively on X, μ_f is the unique measure of maximal entropy (Theorem 4.1).

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1. Numerical invariants

In this section we define and establish inequalities between several numerical invariants. This involves some technicalities because our mappings are not holomorphic and also, the psef/nef-cones are not well understood in dimension ≥ 4. What follows is quite simple when f is holomorphic (the only nontrivial part, the link between entropy and dynamical degrees, goes back to Gromov [Gr 77]). When $X = \mathbb{P}^k$, a clean treatment of the dynamical degrees is given by Russakovskii and Shiffman in [RS 97]: the situation is simpler since $\mathbb{P}^k$ is a complex homogeneous manifold whose cohomology vector spaces $H^{l,l}$ are all one-dimensional.

1.1. *Dynamical degrees.* Given a smooth form α of bidegree (l,l), $1 \leq l \leq k$, we define the pull-back of α by f in the following way: let $\Gamma_f \subset X \times X$ denote the graph of f and consider a desingularization $\tilde{\Gamma}_f$ of Γ_f. We have a commutative diagram

$$\begin{tikzcd} & \tilde{\Gamma}_f \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ X \arrow[rr, "f"] & & X \end{tikzcd}$$

where π_1, π_2 are holomorphic maps. We set $f^*\alpha := (\pi_1)_*(\pi_2^*\alpha)$ where we push forward the smooth form $\pi_2^*\alpha$ by π_1 as a current. Note that $f^*\alpha$ is actually a form with L^1_{loc}-coefficients which coincides with the usual smooth pull-back $(f_{|X\setminus I_f})^*\alpha$ on $X \setminus I_f$; thus the definition does not depend on the choice of desingularization. In other words, $f^*\alpha$ is the trivial extension, as current, of $(f_{|X\setminus I_f})^*\alpha$ through I_f.

This definition induces a linear action on the cohomology space $H^{l,l}(X,\mathbb{R})$ which preserves $H^{l,l}_a(X,\mathbb{R})$, the subspace generated by complex subvarieties of codimension l. We let $H^{l,l}_{psef}(X,\mathbb{R})$ denote the closed cone generated by effective cycles.

Definition 1.1. Set $\delta_l(f) := \int_X f^*\omega^l \wedge \omega^{k-l}$. We define the l^{th}-*dynamical degree* of f to be

$$\lambda_l(f) := \liminf_{n\to+\infty} \; [\delta_l(f^n)]^{1/n}.$$

This definition clearly does not depend on the choice of the Kähler form ω. Observe that for $l = k$, $\lambda_k(f)$ is the topological degree of f, i.e. the number of preimages of a generic point, which we shall preferably denote by $d_t(f)$ (or simply d_t when no confusion can arise).

PROPOSITION 1.2. i) *The sequence* $l \mapsto \lambda_l(f)/\lambda_{l+1}(f)$ *is nondecreasing,* $0 \le l \le k-1$; *i.e.*, $\log \lambda_l$ *is a concave function of* l. *In particular if* $d_t = \lambda_k(f) > \lambda_{k-1}(f)$, *then* $d_t > \lambda_{k-1}(f) > \cdots > \lambda_1(f) > 1$.

ii) *There exists* $C > 0$ *such that for all dominating rational self-maps* $f, g : X \to X$,

$$\delta_1(g \circ f) \le C\delta_1(f)\delta_1(g).$$

In particular $\delta_1(f^{n+m}) \le C\delta_1(f^n)\delta_1(f^m)$ *so that* $\lambda_1(f) = \lim[\delta_1(f^n)]^{1/n}$. *Moreover* $\lambda_1(f)$ *is invariant under birational conjugacy.*

iii) *Let* $r_1(f)$ *denote the spectral radius of the linear action induced by* f^* *on* $H^{1,1}_a(X,\mathbb{R})$ *and set* $\lambda_1'(f) = \limsup r_1(f^n)^{1/n}$. *There exists* $C > 0$ *and for every* $\varepsilon > 0$ *there exists* $C_\varepsilon > 0$, *such that for all* n,

$$0 \le r_1(f^n) \le C\delta_1(f^n) \le C_\varepsilon[\lambda_1'(f) + \varepsilon]^n.$$

In particular $\lambda_1(f) = \lambda_1'(f)$.

Proof. i) It is equivalent to prove that $\lambda_{l+1}(f)\lambda_{l-1}(f) \le \lambda_l(f)^2$ for all $1 \le l \le k-1$. This is a consequence of $\delta_{l+1}(f^n)\delta_{l-1}(f^n) \le \delta_l(f^n)^2$, which follows from Teissier-Hovanskii mixed inequalities: it suffices to apply Theorem 1.6.C_1 of [Gr 90] in the graph $\tilde{\Gamma}_{f^n}$ to the smooth semi-positive forms $\pi_1^*\omega^i$ and $\pi_2^*\omega^{k-i}$.

ii) Let $f, g : X \to X$ be dominating rational self-maps. It is possible to define f^*T for any positive closed current T of bidegree $(1,1)$ (see [S 99]). In particular, $f^*(g^*\omega)$ is a globally well defined positive closed current of bidegree $(1,1)$ on X which coincides with $(g\circ f)^*\omega$ in $X \setminus I_f \cup f^{-1}(I_g)$. Now $(g\circ f)^*\omega$ is a form with L^1_{loc} coefficients, thus it does not charge the proper algebraic subset $I_f \cup f^{-1}(I_g)$. Therefore we have an inequality between these two currents,

$$(g \circ f)^*\omega \le f^*(g^*\omega) \tag{$\dagger$}$$

and the same inequality holds in $H^{1,1}_{psef}(X,\mathbb{R})$. Note that ($\dagger$) does not hold in general if we replace $[\omega]$ by the class of an effective divisor (see Remark 1.4 below).

Let N be a norm on $H^{l,l}(X,\mathbb{R})$. There exists $C_1 > 0$ such that for all class $\alpha \in H^{l,l}_{psef}(X,\mathbb{R})$, $N(\alpha) \le C_1 \int \alpha \wedge \omega^{k-l}$. We infer from ($\dagger$) and the continuity of $(\alpha,\beta)\mapsto\int\alpha\wedge\beta$ that

$$\delta_1(g\circ f)\leq\int f^*(g^*\omega)\wedge\omega^{k-1}=\int g^*\omega\wedge f_*\omega^{k-1}\leq C\delta_1(g)\delta_1(f).$$

Note that we have used the fact that $f_*[\omega^{k-1}]\in H^{k-1,k-1}_{psef}(X,\mathbb{R})$ (see below for the definition of f_* and related properties). We infer from the latter inequality that the sequence $(\delta_1(f^n))$ is quasisubmultiplicative, hence the lim inf can be replaced by a lim (or an inf) in the definition of $\lambda_1(f)$. Moreover if g is birational, we get

$$\delta_1(g\circ f^n\circ g^{-1})\leq C\delta_1(g)\delta_1(g^{-1})\delta_1(f^n);$$

hence $\lambda_1(g\circ f\circ g^{-1})=\lambda_1(f)$; i.e., $\lambda_1(f)$ is a birational invariant.

iii) Observe that $H^{1,1}_{psef}(X,\mathbb{R})$ is a closed convex cone with nonempty interior which is strict (i.e. $H^{1,1}_{psef}(X,\mathbb{R})\cap -H^{1,1}_{psef}(X,\mathbb{R})=\{0\}$) and preserved by f^*. Therefore there exists, for all $n\in\mathbb{N}$, a class $[\theta_n]\in H^{1,1}_{psef}(X,\mathbb{R})$ such that $(f^n)^*[\theta_n]=r_1(f^n)[\theta_n]$. This can be thought of as a Perron-Frobenius-type result (see Lemma 1.12 in [DF 01]).

Fix a basis $[\omega_1]=[\omega],[\omega_2],\dots,[\omega_s]$ of $H^{1,1}_a(X,\mathbb{R})$, where the $\omega_j's$ are smooth forms such that $\omega_j\leq\omega$. We normalize $\theta_n=\sum_j\alpha_{j,n}\omega_j$ so that $||[\theta_n]||:=\max_j|\alpha_{j,n}|=1$; thus $\theta_n\leq s\omega$. Observe that $[\theta]\mapsto\int\theta\wedge\omega^{k-1}$ is a continuous form on $H^{1,1}_a(X,\mathbb{R})$ which is positive on $H^{1,1}_{psef}(X,\mathbb{R})$. Therefore there exists $C>0$ such that $||[\theta]||\leq C\int\theta\wedge\omega^{k-1}$, for all $[\theta]\in H^{1,1}_{psef}(X,\mathbb{R})$. This yields the first inequality:

$$\begin{aligned}r_1(f^n)&=r_1(f^n)||[\theta_n]||\leq Cr_1(f^n)\int\theta_n\wedge\omega^{k-1}\\&=C\int(f^n)^*\theta_n\wedge\omega^{k-1}\leq Cs\int(f^n)^*\omega\wedge\omega^{k-1}.\end{aligned}$$

Conversely, fix $\varepsilon>0$ and $p>1$ such that $r_1(f^p)\leq(\lambda_1'(f)+\varepsilon/2)^p$. Fix a norm N on $H^{1,1}_a(X,\mathbb{R})$. Since $[\theta]\mapsto\int_X\theta\wedge\omega^{k-1}$ defines a continuous linear form on $H^{1,1}_a(X,\mathbb{R})$, there exists $C_N>0$ such that for all $[\theta]$, $|\int_X\theta\wedge\omega^{k-1}|\leq C_NN([\theta])$. Set $\tilde{N}(f):=\sup_{N([\theta])=1}N(f^*[\theta])$. It follows from (†) that $N((f^n)^*[\omega])\leq N(f^*(\dots f^*[\omega])\dots)$, hence

$$0\leq\int(f^n)^*\omega\wedge\omega^{k-1}\leq C_N[\tilde{N}(f^p)]^qN([(f^r)^*\omega]),$$

where $n=pq+r$. Now for every $\varepsilon>0$ one can find a norm N_ε on $H^{1,1}_a(X,\mathbb{R})$ such that $r_1(f^p)\leq\tilde{N}_\varepsilon(f^p)\leq r_1(f^p)+\varepsilon/2$. This yields iii). □

Remark 1.3. It is remarkable that the mixed inequalities $\lambda_{l+1}\lambda_{l-1}\leq\lambda_l^2$ contain all previously known inequalities, e.g. $\lambda_{l+l'}(f)\leq\lambda_l(f)\lambda_{l'}(f)$ (which are proved by Russakovskii and Shiffman [RS 97] when $X=\mathbb{P}^k$).

Remark 1.4. One should be aware that simple inequalites like (†) are false if we replace $[\omega]$ by the class of an effective divisor (in particular, Lemma 3 in [Fr 91] is wrong). Here is a simple 2-dimensional counterexample: consider $\sigma : Y \to Y$ a biholomorphism of some projective surface Y with a nontrivial 2-cycle $\{p, \sigma(p)\}$. Let $\pi : X \to Y$ be the blow-up of Y at point p, $E = \pi^{-1}(p)$ and $q = \pi^{-1}(\sigma(p))$. Set $f = \pi^{-1} \circ \sigma \circ \pi : X \to X$. This is a rational self-map of X such that $I_f = \{q\}$, $f(q) = E$, $f(E) = q$. Therefore $f^*[E] = 0$, so $f^*(f^*[E]) = 0$ while $(f \circ f)^*[E] = [E]$ (contradicting Lemma 3 in [Fr 91]).

We define similarly the push-forward by f as $f_*\alpha := (\pi_2)_*(\pi_1^*\alpha)$. This induces a linear action on the cohomology spaces $H^{l,l}(X, \mathbb{R})$ which is dual to that of f^* on $H^{k-l,k-l}(X, \mathbb{R})$. The push-forward of any positive closed current of bidegree $(1, 1)$ is well defined and yields a positive closed current of bidegree $(1, 1)$ on X. Therefore $H^{1,1}_{psef}(X, \mathbb{R})$ is preserved by f_* (by duality, the dual cone $H^{k-1,k-1}_{nef}(X, \mathbb{R})$ is preserved by f^*). We have a $(\dagger)'$ inequality

$$(\dagger') \qquad (g \circ f)_*\omega \leq g_*(f_*\omega).$$

This yields results on $\lambda_{k-1}(f)$ analogous to those obtained for $\lambda_1(f)$. We summarize this in the following:

Proposition 1.5. *The dynamical degree $\lambda_{k-1}(f)$ is invariant under birational conjugacy and satisfies*

$$\lambda_{k-1}(f) = \lim[\delta_{k-1}(f^n)]^{1/n} = \lim[r_{k-1}(f^n)]^{1/n},$$

where $r_{k-1}(f)$ denotes the spectral radius of the linear action induced by f^* on $H^{k-1,k-1}_a(X, \mathbb{R})$.

Remark 1.6. When $2 \leq l \leq k-2$ (hence $k = \dim_{\mathbb{C}} X \geq 4$), it seems unlikely that the cone $H^{l,l}_{psef}(X, \mathbb{R})$ (or its dual $H^{k-l,k-l}_{nef}(X, \mathbb{R})$) is preserved by f^* (or f_*), unless f is holomorphic. It follows however from previous proofs that if $H^{l,l}_{psef}(X, \mathbb{R})$ is f^*-invariant and $f^*[\omega^l] \leq f^*(\ldots f^*[\omega^l]) \ldots)$, then we get similar information on $\lambda_l(f)$. These conditions are satisfied if e.g. X is a complex homogeneous manifold.

1.2. *Topological entropy.* For $p \in X$, we define $f(p) = \pi_2\pi_1^{-1}(p)$ and $f^{-1}(p) = \pi_1\pi_2^{-1}(p)$: these are proper algebraic subsets of X. Note that $I_f = \{p \in X \,/\, \dim f(p) > 0\}$. We set $I_f^- := \{p \in X \,/\, \dim f^{-1}(p) > 0\}$ and let $\mathcal{C}_f$ denote the critical set of f, i.e. the closure of the set of points in $X \setminus I_f$ where $Jf(p) = 0$. Clearly $I_f^- \subset f(\mathcal{C}_f)$ and $I_{f^n}^- \subset f^n(I_f^-)$; thus

$$\cup_{n\geq 1} I_{f^n}^- \subset \mathrm{PC}(f) := \cup_{n\geq 1} f^n(\mathcal{C}_f) := \text{postcritical set of } f.$$

Observe that for $a \in X \setminus \cup_{n\geq 0} I_{f^n}^-$, we can define for all $n \geq 0$ the probability measures $d_t^{-n}(f^n)^*\varepsilon_a$. Here ε_a denotes the Dirac mass at point a. Therefore if

ν is a probability measure on X which does not charge $\mathrm{PC}(f)$, we can define

$$\nu_n := \frac{1}{d_t^n}(f^n)^*\nu = \int \frac{1}{d_t^n}(f^n)^*\varepsilon_a d\nu(a).$$

The latter are again probability measures which do not charge $\mathrm{PC}(f)$ since $f(\mathrm{PC}(f)) \subset \mathrm{PC}(f)$. We will prove, when $d_t > \lambda_{k-1}(f)$, that the ν_n's converge to an invariant measure μ_f (Theorem 3.1).

We now give a definition of entropy which is suitable for our purpose (this definition differs slightly from that of Friedland [Fr 91]). Observe that for all $n \geq 0$, $I_{f^n} \subset f^{-n}(I_f)$. We set

$$\Omega_f := X \setminus \cup_{n\in\mathbb{Z}} f^n(I_f).$$

This is a totally invariant subset of X such that f^n is holomorphic at every point for all $n \geq 0$. Following Bowen's definition [Bo 73] we define the topological entropy of f relative to $Y \subset \Omega_f$ to be

$$h_{\mathrm{top}}(f_{|Y}) := \sup_{\varepsilon>0} \overline{\lim} \frac{1}{n} \log \max\{\sharp F \,/\, F\ (n,\varepsilon)\text{-separated set in } Y\},$$

where F is said to be (n,ε)-separated if $d_n(x,y) \geq \varepsilon$ whenever $(x,y) \in F^2$, $x \neq y$. Here $d_n(x,y) = \max_{0\leq j\leq n-1} d(f^j(x), f^j(y))$ for some metric d on X. We define $h_{\mathrm{top}}(f) := h_{\mathrm{top}}(f_{|\Omega_f})$. These definitions clearly do not depend on the choice of the metric.

Given ν an ergodic probability measure such that $\nu(\Omega_f) = 1$, we define the metric entropy of ν following Brin-Katok [BK 83]: for almost every $x \in \Omega_f$,

$$h_\nu(f) := \sup_{\varepsilon>0} \underline{\lim} -\frac{1}{n}\nu(B_n(x,\varepsilon)),$$

where $B_n(x,\varepsilon) = \{y \in \Omega_f \,/\, d_n(x,y) < \varepsilon\}$. One easily checks that the topological entropy dominates any metric entropy:

$$h_{\mathrm{top}}(f) \geq \sup\{h_\nu(f),\ \nu \text{ ergodic with } \nu(\Omega_f) = 1\}.$$

However it is not clear whether the reverse inequality holds, as it does for nonsingular mappings. More generally if Y is a Borel subset of Ω_f such that $\nu(Y) > 0$, then $h_\nu(f) \leq h_{\mathrm{top}}(f_{|Y})$. This is what Briend and Duval call the *relative variational principle* [BD 01].

Let $\Gamma_n = \{(x, f(x), \dots, f^{n-1}(x)),\ x \in \Omega_f\}$ be the iterated graph of f and set

$$\mathrm{lov}(f) := \overline{\lim} \frac{1}{n} \log(\mathrm{Vol}(\Gamma_n)) = \overline{\lim} \frac{1}{n} \log\left(\int_{\Gamma_n} \omega_n^k\right),$$

where $\omega_n = \sum_{i=1}^n \pi_i^*\omega$, π_i being the projection $X^n \to X$ on the i^{th} factor. A well-known argument of Gromov [Gr 77] yields the estimate $h_{\mathrm{top}}(f) \leq \mathrm{lov}(f)$. When f is a holomorphic endomorphism (i.e. when $I_f = \emptyset$), a simple cohomological computation yields $\mathrm{lov}(f) = \max_{1\le j\le k} \log \lambda_j(f)$. Such computation is more delicate for mappings which are merely meromorphic. The following lemma will be quite useful in our analysis.

LEMMA 1.7. *Assume* $\dim_{\mathbb{C}} X \le 3$ *or* X *is a complex homogeneous manifold. Fix* $\varepsilon > 0$. *Then there exists* $C_\varepsilon > 0$ *such that*

$$0 \le \int_{\Omega_f} (f^{n_1})^*\omega \wedge \cdots \wedge (f^{n_{k-1}})^*\omega \wedge \omega \le C_\varepsilon [\max_{1\le j\le k-1} \lambda_j(f) + \varepsilon]^{\max n_i},$$

for all $(n_1, \ldots, n_{k-1}) \in \mathbb{N}^{k-1}$.

Proof. We can assume $n_1 \le \cdots \le n_{k-1}$ without loss of generality.

When $k = \dim_{\mathbb{C}} X \le 2$ everything is clear. Assume $k = 3$. Then $\int_{\Omega_f} (f^{n_1})^*\omega \wedge (f^{n_2})^*\omega \wedge \omega \le \int_X \omega \wedge (f^{n_2-n_1})^*\omega \wedge (f^{n_1})_*\omega$. Here we use the fact that $(f^{n_2-n_1})^*\omega \wedge (f^{n_1})_*\omega$ is a globally well defined positive closed current of bidegree $(2,2)$ on X. This follows from the intersection theory of positive currents (see [S 99]), since $(f^{n_2-n_1})^*\omega$ and $(f^{n_1})_*\omega$ have continuous potentials outside a set of codimension ≥ 2. Using Propositions 1.2 and 1.5, we thus get, for $\varepsilon > 0$ fixed,

$$\begin{aligned} 0 &\le \int_{\Omega_f} (f^{n_1})^*\omega \wedge (f^{n_2})^*\omega \wedge \omega \le CN((f^{n_2-n_1})^*[\omega])N((f^{n_1})_*[\omega]) \\ &\le C_\varepsilon[\lambda_1(f) + \varepsilon]^{n_2-n_1}[\lambda_2(f) + \varepsilon]^{n_1} \le C_\varepsilon \max_{j=1,2}[\lambda_j(f) + \varepsilon]^{n_2}. \end{aligned}$$

When $\dim_{\mathbb{C}} X \ge 4$, it becomes more difficult to define and control the positivity of $(f^{i_1})^*\omega \wedge (f^{i_2})^*\omega \wedge (f^{i_3})_*\omega$ on $X \setminus \Omega_f$. However, when X is a complex homogeneous manifold (i.e. when the group of automorphisms $\mathrm{Aut}(X)$ acts transitively on X), one can regularize every positive closed current T within the same cohomology class and get this way an approximation of T by smooth *positive* closed forms $T_\varepsilon \simeq T$ (see [Hu 94]). Proceeding as above and replacing each singular term $(f^n)^*\omega$, $(f^m)_*\omega$ by a smooth approximant, we see that Fatou's lemma yields the desired inequality (this argument is used in [RS 97] to obtain related inequalities). □

COROLLARY 1.8. *Assume* $\dim_{\mathbb{C}} X \le 3$ *or* X *is complex homogeneous. Then*

$$h_{\mathrm{top}}(f) \le \mathit{lov}(f) \le \max_{1\le j\le k} \log \lambda_j(f).$$

Proof. By definition $\mathrm{Vol}(\Gamma_n) = \sum_{0\le i_1,\ldots,i_k\le n-1} \int_{\Omega_f} (f^{i_1})^*\omega \wedge \cdots \wedge (f^{i_k})^*\omega$. Assume $i_1 \le \cdots \le i_k$ and fix $\varepsilon > 0$. Then

$$\begin{aligned} \int_{\Omega_f} (f^{i_1})^*\omega \wedge \cdots \wedge (f^{i_k})^*\omega &= d_t(f)^{i_1} \int_{\Omega_f} (f^{i_2-i_1})^*\omega \wedge \cdots \wedge (f^{i_k-i_1})^*\omega \wedge \omega \\ &\le C_\varepsilon d_t(f)^{i_1} [\max_{1\le j\le k-1} \lambda_j(f) + \varepsilon]^{i_k-i_1} \le C_\varepsilon [\max_{1\le j\le k} \lambda_j(f) + \varepsilon]^n. \end{aligned}$$

Therefore $\mathrm{Vol}(\Gamma_n) \leq C_\varepsilon n^k [\max \lambda_j(f) + \varepsilon]^n$, hence $\mathrm{lov}(f) \leq \log[\max \lambda_j(f) + \varepsilon]$. When $\varepsilon \to 0$ we have the desired inequality. □

We will also need a relative version of this estimate.

COROLLARY 1.9. *Assume* $\dim_{\mathbb{C}} X \leq 3$ *or* X *is complex homogeneous. Let* Y *be a proper subset of* Ω_f. *If* Y *is algebraic then*

$$h_{\mathrm{top}}(f_{|Y}) \leq lov(f_{|Y}) \leq \max_{1 \leq j \leq k-1} \log \lambda_j(f).$$

In the general case, we simply get

$$h_{\mathrm{top}}(f_{|Y}) \leq \overline{\lim} \frac{1}{n} \log(Vol(\Gamma_n|Y)_\varepsilon),$$

where $\varepsilon > 0$ *is fixed*, $\Gamma_n|Y$ *denotes the restriction of* Γ_n *to* Y *and* $(\Gamma_n|Y)_\varepsilon$ *is the* ε*-neighborhood of* $\Gamma_n|Y$ *in* Γ_n.

2. A canonical invariant measure μ_f

THEOREM 2.1. *Let* $f : X \to X$ *be a rational mapping such that* $d_t(f) > \lambda_{k-1}(f)$. *Then there exists a probability measure* μ_f *such that if* Θ *is any smooth probability measure on* X,

$$\frac{1}{d_t(f)^n}(f^n)^*\Theta \longrightarrow \mu_f,$$

where the convergence holds in the weak sense of measures. Moreover:

i) *Every quasiplurisubharmonic function is in* $L^1(\mu_f)$. *In particular* μ_f *does not charge pluripolar sets and* $\log^+ ||Df^{\pm 1}|| \in L^1(\mu_f)$.

ii) $f^*\mu_f = d_t(f)\mu_f$; *hence* μ_f *is invariant* $f_*\mu_f = \mu_f$.

iii) $h_{\mathrm{top}}(f) \geq h_{\mu_f}(f) \geq \log d_t(f) > 0$. *In particular* μ_f *is a measure of maximal entropy when* $\dim_{\mathbb{C}} X \leq 3$ *or when* X *is complex homogeneous.*

Proof. Fix a a noncritical value of f and $r > 0$ such that f admits $d_t = d_t(f)$ well defined inverse branches on $B(a, r)$. Fix Θ a smooth probability measure with compact support in $B(a, r)$. Then $d_t^{-1} f^*\Theta$ is a smooth probability measure on X. Since X is Kähler, the dd^c-lemma (see [GH 78, p. 149]) yields

$$\frac{1}{d_t} f^*\Theta = \Theta + dd^c(S),$$

where S is a smooth form of bidegree $(k-1, k-1)$. Replacing S by $S + C\omega^{k-1}$ if necessary, we can assume $0 \leq S \leq C\omega^{k-1}$ for some constant $C > 0$. We now take the pull-back of the previous equation by f, as explained in Section 1.

Recall that $(f^n)^* dd^c S = dd^c (f^n)^* S$ for all n (because $(\pi_1)_*, \pi_2^*$ commute with d, d^c). We infer, by induction, that

$$\frac{1}{d_t^n}(f^n)^*\Theta = \Theta + dd^c S_n, \quad S_n = \sum_{j=0}^{n-1} \frac{1}{d_t^j}(f^j)^* S.$$

Indeed observe that $(f^{n+1})^*\Theta = (f^n)^*(f^*\Theta)$, since these are the pull-backs of smooth forms; they are smooth and coincide in $X \setminus \left(I_{f^n} \cup I_{f^{n+1}}\right)$, hence they coincide everywhere since they have L^1_{loc}-coefficients. Therefore

$$\frac{1}{d_t^{n+1}}(f^{n+1})^*\Theta = \frac{1}{d_t^n}(f^n)^*\left(\frac{1}{d_t} f^*\Theta\right) = \frac{1}{d_t^n}(f^n)^*(\Theta + dd^c S) = \Theta + dd^c S_{n+1}.$$

The sequence of positive currents (S_n) is increasing since $(f^j)^* S \geq 0$. Setting $||S_n|| := \int_X S_n \wedge \omega$, we get

$$0 \leq ||S_n|| \leq C \sum_{j=0}^{n-1} \frac{1}{d_t^j} \int_{\Omega_f} (f^j)^* \omega^{k-1} \wedge \omega \leq C_\varepsilon \sum_{j \geq 0} \left(\frac{\lambda_{k-1}(f) + \varepsilon}{d_t}\right)^j < +\infty,$$

using Proposition 1.5 with $\varepsilon > 0$ small enough. Therefore (S_n) converges towards some positive current S_∞; hence

$$\frac{1}{d_t^n}(f^n)^*\Theta \longrightarrow \mu_f := \Theta + dd^c S_\infty.$$

Observe that if Θ' is another smooth probability measure, then $\Theta' = \Theta + dd^c R$, for some smooth form R of bidegree $(k-1, k-1)$. Since $||(f^n)^* R|| = o(d_t^n)$, we have again $d_t^{-n}(f^n)^*\Theta' \to \mu_f$.

Let φ be a quasiplurisubharmonic (qpsh) function on X, i.e. an upper semi-continuous function which is locally given as the sum of a plurisubharmonic function and a smooth function. Translating and rescaling φ if necessary, we can assume $\varphi \leq 0$ and $dd^c\varphi \geq -\omega$. It follows from a regularization result of Demailly (see [De 99]) that there exist $C > 0$ and $\varphi_\varepsilon \leq 0$ a smooth sequence of functions pointwise decreasing towards φ such that $dd^c\varphi_\varepsilon \geq -C\omega$. Using Stokes' theorem we get

$$0 \leq \int (-\varphi_\varepsilon) d\mu_f = \int (-\varphi_\varepsilon)\Theta + \int S_\infty \wedge (-dd^c\varphi_\varepsilon) \leq \int (-\varphi_\varepsilon)\Theta + C \int S_\infty \wedge \omega,$$

since $S_\infty \geq 0$. The monotone convergence theorem thus implies

$$0 \leq \int_X (-\varphi) d\mu_f \leq \int_X (-\varphi)\Theta + C \int_X S_\infty \wedge \omega < +\infty.$$

Since any pluripolar set is included in the $-\infty$ locus of a qpsh function, μ_f does not charge pluripolar sets. In particular $\mu_f(I_f) = 0$; hence $f_*\mu_f = \mu_f$; i.e. μ_f is an invariant probability measure. Similarly $\mu_f(I_f^-) = 0$ so that $f^*\mu_f = d_t \mu_f$; i.e. μ_f has constant jacobian d_t.

It follows from the Rohlin-Parry formula (see [P 69]) that $h_{\mu_f}(f) \geq \log d_t$. Since $\mu_f(\Omega_f) = 1$, we get in particular $h_{\text{top}}(f) \geq \log d_t > 0$. This is reminiscent of the well-known result of Misiurewicz and Przytycki that the topological entropy of a $\mathcal{C}^1$-smooth endomorphism of a compact manifold is minorated by $\log d_t$ (see [KH 95]). When $\dim_{\mathbb{C}} X \leq 3$ or when X is complex homogeneous, we get

$$h_{\mu_f}(f) \leq h_{\text{top}}(f) \leq \max_{1 \leq j \leq k} \log \lambda_j(f) = \log d_t,$$

by Proposition 1.2 and Corollary 1.8; hence μ_f is a measure of maximal entropy. □

3. First ergodic properties of μ_f

In this section we adapt the work of Briend and Duval [BD 01] to establish some ergodic properties of μ_f.

THEOREM 3.1. *Let f, μ_f be as in Theorem 2.1. Then the following hold:*

i) *If ν is a probability measure which does not charge the postcritical set* $\mathrm{PC}(f) := \cup_{j \geq 1} f^j(\mathcal{C}_f)$, *then* $d_t^{-n}(f^n)^*\nu \to \mu_f$.

ii) *The measure μ_f is mixing.*

iii) *If we let $\chi_k \geq \cdots \geq \chi_1$ denote the Lyapunov exponents of μ_f, then*

$$\chi_1 \geq \frac{1}{2}\log(d_t/\lambda_{k-1}(f)) > 0.$$

iv) *Let* $\mathrm{RPer}_n(f)$ *denote the set of repelling periodic points of order n. They are equidistributed with respect to μ_f if* $\limsup(\sharp \mathrm{RPer}_n(f)/d_t^n) \leq 1$. *The latter holds when* $\dim_{\mathbb{C}} X \leq 3$ *or when X is complex homogeneous.*

Remark 3.2. When $X = \mathbb{P}^k$ and f is holomorphic (i.e. when $I_f = \emptyset$), the measure μ_f was constructed by Hubbard and Papadopol [HP 94] and Fornæss and Sibony [FS 94]. The latter also proved ii) and a weaker version i′) of i): they showed the existence of an exceptional pluripolar set $\mathcal{E}_f \subset \mathbb{P}^k$ such that $d_t^{-n}(f^n)^*\varepsilon_a \to \mu_f$ if $a \notin \mathcal{E}_f$. The remaining assertions iii), iv) were established by Briend and Duval [BD 99], [BD 01], who also proved that the exceptional set $\mathcal{E}_f$ is actually a totally invariant algebraic subset of $\mathrm{PC}(f)$.

When $X = \mathbb{P}^k$ but f is merely meromorphic, the measure μ_f was constructed by Russakovskii and Shiffman [RS 97] by proving i′). Following [BD 01], we actually show that $\mathcal{E}_f$ is a subset of $\mathrm{PC}(f)$. Note however, that one can not expect $\mathcal{E}_f$ to be algebraic in the meromorphic case.

This result heavily relies on the following lemma. We thank Julien Duval for explaining to us the construction of inverse branches on balls.

LEMMA 3.3. *Set* $V_l = \cup_{j=1}^{l} f^j(\mathcal{C}_f)$, *where* $\mathcal{C}_f$ *denotes the critical set of* f. *Fix* $\varepsilon > 0$ *and an embedding* $X \subset \mathbb{CP}^N$. *Fix* $1 < \delta < d_t/\lambda_{k-1}$, δ *arbitrarily close to* d_t/λ_{k-1}. *Then there exists* $l \gg 1$ *such that the following hold*:

i) *For every holomorphic disk* $\overline{\Delta} \subset L \cap X \setminus V_l$, *where* L *is a generic projective linear subspace of codimension* $\dim_{\mathbb{C}} X - 1$ *in* $\mathbb{CP}^N$, *there are* $(1-\varepsilon)d_t^n$ *inverse branches of* f^n $(n \geq l)$ *whose images* Δ_i^{-n} *satisfy*

$$\operatorname{diam}(\Delta_i^{-n}) \leq C\delta^{-n/2},$$

where C *is independent of* n.

ii) *For every ball* $\overline{B} \subset X \setminus V_l$, *there are* $(1-\varepsilon)d_t^n$ *inverse branches of* f^n *on* B, $n \geq l$, *whose images* B_i^{-n} *satisfy*

$$\operatorname{diam}(B_i^{-n}) \leq C\delta^{-n/2}.$$

Proof. Fix $\varepsilon > 0$ small and $\delta = d_t/(\lambda_{k-1}(f) + \varepsilon)$.

i) Let $V_1 = f(\mathcal{C}_f)$ denote the set of critical values of f. Let D be an algebraic curve on X which is not included in $\mathrm{PC}(f)$. Then $V_1 \cap f^{-n}D$ is finite (possibly empty) for all $n \geq 0$. Let α be a closed smooth form of bidegree $(k-1, k-1)$ which is cohomologous to $[D]$. Then $\alpha \leq C_D\omega^{k-1}$ for some constant $C_D > 0$. Note that $C_D = C_1$ can be chosen independent of D if we restrict ourselve to curves D which are the trace on X of projective linear subspaces of $\mathbb{P}^N$; in this case we can choose $\alpha = (\omega_{\mathrm{FS}}^{k-1})_{|X}$, where ω_{FS} denotes the Fubini-Study Kähler form on $\mathbb{P}^N$. We assume in the sequel that V_1 is a hypersurface of X (in general V_1 may have codimension ≥ 2 in X; in this case we simply replace V_1 by some hypersurface $\tilde{V}_1$ containing V_1). Let β be a closed smooth $(1,1)$-form cohomologous to $[V_1]$, $\beta \leq C_2\omega$. Then

$$\begin{aligned}\sharp V_1 \cap f^{-n}D &= \int [V_1] \cap (f^n)^*[D] \leq C_1 \int [V_1] \wedge (f^n)^*\omega^{k-1} \\ &\leq C_1C_2 \int \omega \wedge (f^n)^*\omega^{k-1} \leq C_\varepsilon[\lambda_{k-1}(f) + \varepsilon]^n = C_\varepsilon\delta^{-n}d_t^n,\end{aligned}$$

where the last inequality follows from Proposition 1.5.

Since $\overline{\Delta} \cap V_l = \emptyset$, there are d_t^l well defined inverse branches f_i^{-l} of f^l on Δ. Set $\Delta_i^{-l} = f_i^{-l}\Delta$. We can further define d_t inverse branches of f on Δ_i^{-l} if $\Delta_i^{-l} \cap V_1 = \emptyset$. It follows from the computation above that at most $C_\varepsilon\delta^{-l}d_t^l$ of the Δ_i^{-l}'s may intersect V_1. Therefore we can define $d_t^{l+1}(1 - C_\varepsilon\delta^{-l})$ inverse branches of f^{l+1} on Δ. A straightforward induction shows that we can define $d_t^n(1 - C_\varepsilon\delta^{-l}\sum_{j\geq 0}\delta^{-j}) \geq d_t^n(1 - \varepsilon/2)$ inverse branches of f^n on Δ, if we fix l large enough so that $C_\varepsilon\delta^{-l}(1 - \delta^{-1}) < \varepsilon/2$. Let I_ε^n denote the set of indices such that f_i^{-n} is well defined on Δ. Now,

$$\sum_{i\in I_\varepsilon^n} \operatorname{Area}(\Delta_i^{-n}) = \sum_{i\in I_\varepsilon^n} \int [f_i^{-n}(\Delta)] \wedge \omega \leq \int (f^n)^*[D] \wedge \omega \leq C_\varepsilon[\lambda_{k-1}(f) + \varepsilon]^n.$$

Therefore

$$\sharp\left\{i \in I_\varepsilon^n \,/\, \mathrm{Area}(\Delta_i^{-n}) > \frac{2C_\varepsilon}{\varepsilon}\delta^{-n}\right\} \leq \frac{\varepsilon}{2} d_t^n;$$

hence for $(1-\varepsilon)d_t^n$ inverse branches f_i^{-n}, we get an upper bound $\mathrm{Area}(\Delta_i^{-n}) \leq C'_\varepsilon \delta^{-n}$. It is now a standard fact that the area controls the diameter of slightly smaller disks $\tilde{\Delta}_i^{-n} = f_i^{-n}(\tilde{\Delta})$,

$$\mathrm{diam}(\tilde{\Delta}_i^{-n}) \leq C''_\varepsilon \delta^{-n/2}.$$

We refer the reader to the appendix in [BD 01] where this is proved using the notion of extremal length. Note that when the Δ_i^{-n}'s are included in a relatively compact ball of some affine chart (i.e. if we already know that $\mathrm{diam}(\tilde{\Delta}_i^{-n})$ is small enough), this follows from Cauchy's formula.

ii) Let now $B = B(p, 8r_\varepsilon)$ be a ball such that $\overline{B} \cap V_l = \emptyset$. We now construct $(1-\varepsilon)d_t^n$ inverse branches f_i^{-n} of f^n on $B(p, 4r_\varepsilon)$ such that

$$\mathrm{diam}(f_i^{-n} B(p, r_\varepsilon)) \leq C'''_\varepsilon \delta^{-n/2}.$$

There are d_t^l well defined inverse branches f_i^{-l} of f^l on $B = B(p, 8r_\varepsilon)$. Set $B_i^{-l} = f_i^{-l} B$. For $n \geq l$, we set $r_n = 1 - \rho_n$ with $\rho_n = \sum_{j=l}^n j^{-2}$. We can further define d_t inverse branches of f on $f_i^{-l}(r_{l+1}\overline{B})$ if $f_i^{-l}(r_{l+1}\overline{B}) \cap V_1 = \emptyset$. Assume $f_i^{-l}(r_{l+1}\overline{B}) \cap V_1 \neq \emptyset$; then $f^l(B_i^{-l} \cap V_1) \cap r_{l+1} B \neq \emptyset$. Let Z_l denote the analytic set $f^l(B_i^{-l} \cap V_1)$ and pick x_l a point on Z_l such that $B(x_l, 8r_\varepsilon l^{-2}) \subset B$. Thus $Z_l \cap B(x_l, 8r_\varepsilon l^{-2})$ is an analytic subset of $B(x_l, 8r_\varepsilon l^{-2})$ without boundary. It follows from Jensen's inequality that

$$\int [f^l(B_i^{-l} \cap V_1)] \wedge \omega^{k-1} \geq \int_{B(x_l, 8r_\varepsilon l^{-2})} [Z_l] \wedge \omega^{k-1} \geq C_0 (8r_\varepsilon l^{-2})^{2(k-1)},$$

for some uniform constant $C_0 > 0$. This is because Z_l has Lelong number ≥ 1 at point x_l. On the other hand,

$$\sum_i \int [f^l(B_i^{-l} \cap V_1)] \wedge \omega^{k-1} \leq \int (f^l)_*[V_1] \wedge \omega^{k-1} \leq C[\lambda_{k-1}(f) + \varepsilon]^l.$$

Therefore $\sharp\{i \,/\, f_i^{-l}(r_{l+1}\overline{B}) \cap V_1 \neq \emptyset\} \leq C' l^{4(k-1)} \delta^{-l} d_t^l$. Continuing the induction, slightly shrinking the radius of the ball at each step as indicated above, we construct $d_n := d_t^n (1 - C' \sum_{j=l}^{n-1} l^{4(k-1)} \delta^{-l} d_t^l)$ inverse branches of f^n on the ball $B_n = B(p, 8r_\varepsilon r_n)$. Now $r_n \geq 1 - \sum_{j \geq l} j^{-2}$ so that $B_n \supset B(p, 4r_\varepsilon)$ and $d_n \geq d_t^n(1 - \varepsilon/2)$ for all $n \geq l$, if l is chosen large enough.

Let $\omega' = \int [L_\theta] d\nu(\theta)$, where L_θ denotes the trace of a projective line through p and ν is the Fubini-Study volume form on the set of lines $\simeq \mathbb{P}^{N-1}$, so that ω' is a positive closed current of bidegree $(k-1, k-1)$ which is smooth in $X \setminus \{p\}$. Thus

$$0 \leq \sum_i \int_{B(p, 4r_\varepsilon)} (f_i^{-n})_* \omega' \wedge \omega \leq \int (f^n)^* \omega' \wedge \omega \leq C''[\lambda_{k-1}(f) + \varepsilon]^n.$$

We infer that $\int_{B(p,4r_\varepsilon)}(f_i^{-n})_*\omega' \wedge \omega \leq \frac{2C''}{\varepsilon}\delta^{-n}$ for at least $(1-\varepsilon)d_t^n$ inverse branches.

Let I_ε^n denote the corresponding set of indices. Set $\Delta_\theta = L_\theta \cap B(p, 4r_\varepsilon)$. For i fixed in I_ε^n, we get

$$\mathrm{Area}(f_i^{-n}\Delta_\theta) \leq \frac{4C''}{\varepsilon}\delta^{-n}$$

on a set of projective lines $A_i^n \subset \mathbb{P}^{N-1}$ of measure $\geq 1/2$. Therefore $\mathrm{diam}(f_i^{-n}\frac{1}{2}\Delta_\theta) \leq C_\varepsilon \delta^{-n/2}$ for $\theta \in A_i^n$. Now the sets A_i^n have projective capacity $\geq 1/2$, so it follows from a result of Sibony and Wong [A 81] (see also [DS 02], where this is used in a dynamical context) that

$$\mathrm{diam}\left(f_i^{-n}\frac{1}{4}\Delta_\theta\right) \leq C_\varepsilon \delta^{-n/2}$$

for every line L_θ. The desired bound on $\mathrm{diam}(f_i^{-n}B(p,r_\varepsilon))$ follows. □

Proof of Theorem 3.1. Let $a, b \in X \setminus \mathrm{PC}(f)$. We claim $d_t^n(f^n)^*(\varepsilon_a - \varepsilon_b) \to 0$. Indeed let $0 \leq \chi \leq 1$ be a test function. Fix $\varepsilon > 0$ and $l = l_\varepsilon \gg 1$ as in Lemma 3.3. Let Δ be a holomorphic disk joining a to b such that $\overline{\Delta} \cap V_l = \emptyset$. Using Lemma 3.3, we construct $(1-\varepsilon)d_t^n$ inverse branches f_i^{-n} of f^n on Δ with small diameter. Thus

$$\left|\left\langle \frac{(f^n)^*(\varepsilon_a - \varepsilon_b)}{d_t^n}, \chi \right\rangle\right| \leq 2\varepsilon \sup|\chi| + \sum_{i=1}^{(1-\varepsilon)d_t^n} \frac{|\chi \circ f_i^{-n}(a) - \chi \circ f_i^{-n}(b)|}{d_t^n} < 3\varepsilon,$$

if n is large enough so that $\mathrm{diam}(f_i^{-n}\Delta)$ is smaller than the modulus of continuity of χ with respect to ε. This proves the claim.

Now let $a \notin \mathrm{PC}(f)$. Using the identities $\mu_f = \int \varepsilon_b d\mu_f(b)$ and $f^*\mu_f = d_t\mu_f$, we get

$$\mu_f - \frac{1}{d_t^n}(f^n)^*\varepsilon_a = \frac{1}{d_t^n}(f^n)^*(\mu_f - \varepsilon_a) = \int \frac{1}{d_t^n}(f^n)^*(\varepsilon_b - \varepsilon_a)d\mu_f(b) \to 0,$$

by the dominated convergence theorem, by the fact that $\mu_f(\mathrm{PC}(f)) = 0$. Similarly, if ν is a probability measure such that $\nu(\mathrm{PC}(f)) = 0$, we get $d_t^{-n}(f^n)^*\nu = \int d_t^{-n}(f^n)^*\varepsilon_a d\nu(a) \to \mu_f$. In particular let χ be a test function. Translating and rescaling, we can assume $0 \leq \chi$ and $c_\chi := \int \chi d\mu_f = 1$ so that $\chi\mu_f$ is a probability measure. Since $\chi\mu_f(\mathrm{PC}(f)) = 0$, we obtain

$$\chi \circ f^n \mu_f = \frac{1}{d_t^n}(f^n)^*(\chi\mu_f) \to \mu_f = c_\chi \mu_f.$$

This says precisely that the measure μ_f is mixing (see [KH 95]).

In particular μ_f is ergodic. Moreover $\log^+ ||Df^{\pm 1}|| \in L^1(\mu_f)$ (by Theorem 2.1.i); hence μ_f has well defined (finite) Lyapunov exponents $\chi_k \geq \cdots \geq \chi_1$. It follows from Birkhoff's ergodic theorem that

$$\chi_1 = \lim_{n \to +\infty} -\frac{1}{n} \int \log ||(D_x f^n)^{-1}|| d\mu_f(x).$$

Fix $\varepsilon > 0$, $l = l_\varepsilon \gg 1$ and $x \in \operatorname{Supp} \mu_f \setminus V_l$ a generic point. Using Lemma 3.3, we construct $(1-\varepsilon)d_t^{-n}$ inverse branches f_i^{-n} of f^n on $B = B(x, r_\varepsilon)$ whose images B_i^{-n} have small diameter. Let x_i^{-n} denote the preimages of x under f^n. Since $Df_i^{-n}(x) = (Df^n(x_i^{-n}))^{-1}$, it follows from Cauchy's inequalities and Lemma 3.3 that

$$||(D_{x_i^{-n}} f^n)^{-1}|| \leq C\delta^{-n/2},$$

where δ is arbitrarily close to d_t/λ_{k-1}, C is independent of n and $1 \leq i \leq (1-\varepsilon)d_t^n$. Let

$$\widehat{\Omega_f} := \{\widehat{x} = (x_n)_{n\in\mathbb{Z}} \in \Omega_f^{\mathbb{Z}} :\ f(x_n) = x_{n+1} \text{ for all } n \in \mathbb{Z}\}$$

be the natural extension of (f, Ω_f). It is well-known that the dynamical system (Ω_f, f, μ_f) lifts to $(\widehat{\Omega_f}, \widehat{f}, \widehat{\mu_f})$, where $\widehat{f}$ denotes the shift on $\widehat{\Omega_f}$ and $\widehat{\mu_f}$ is the unique invariant probability measure on $\widehat{\Omega_f}$ such that $(\pi_n)_*\widehat{\mu_f} = \mu_f$, where π_n denotes the projection onto the n^{th} coordinate. Set $\widehat{B} := \pi_0^{-1}B$ and $\widehat{B}_\varepsilon := \{\widehat{x} \in \widehat{B} : \forall n \geq 1, x_{-n} = f_i^{-n}(x_0) \text{ for some } 1 \leq i \leq (1-\varepsilon)d_t^n\}$. Observe that

$$\widehat{B}_\varepsilon = \bigcap_{n\geq l} \searrow \widehat{f}^n \left(\cup_{i=1}^{(1-\varepsilon)d_t^n} \widehat{B_i^{-n}}\right), \text{ where } B_i^{-n} = f_i^{-n}B.$$

Therefore

$$\widehat{\mu_f}(\widehat{B_\varepsilon}) = \lim \widehat{\mu_f}\left(\cup_{i=1}^{(1-\varepsilon)d_t^n} \widehat{B_i^{-n}}\right) = \lim \sum_{i=1}^{(1-\varepsilon)d_t^n} \mu_f(B_i^{-n}) = (1-\varepsilon)\mu_f(B) > 0,$$

when $\widehat{\mu_f}$ is $\widehat{f}$-invariant, $\widehat{\mu_f}(\widehat{A}) = \mu_f(A)$, and $\mu_f(B_i^{-n}) = d_t^{-n}\mu_f(B)$ (because $(f^n)^*\mu_f = d_t^n\mu_f$ and f^n is injective on B_i^{-n}).

Set $\varphi := -\log||(D_x f)^{-1}||$ and $\widehat{\varphi} = \varphi \circ \pi_0 \in L^1(\widehat{\mu_f})$. Then $\chi_1 = \int \varphi d\mu_f = \int \widehat{\varphi} d\widehat{\mu_f}$. The measure $\widehat{\mu_f}$ is mixing since μ_f is; hence by Birkhoff's theorem,

$$\frac{1}{n}\sum_{j=0}^{n-1} \widehat{\varphi} \circ \widehat{f}^{-j}(\widehat{x}) \to \chi_1 \text{ for almost every } \widehat{x}.$$

Fix $\widehat{x}$ a generic point in $\widehat{B}_\varepsilon$. Then

$$\begin{aligned}\frac{1}{n}\sum_{j=0}^{n-1} \widehat{\varphi} \circ \widehat{f}^{-j}(\widehat{x}) &= -\frac{1}{n}\sum_{j=0}^{n-1} \log||(D_{x_{-j}} f)^{-1}|| \\ &= -\frac{1}{n}\log||D_x f_i^{-n}|| \geq \frac{\log\delta}{2} - \frac{\log C}{n},\end{aligned}$$

hence $\chi_1 \geq \frac{1}{2}\log\delta$. The desired lower bound follows when $\delta \to d_t/\lambda_{k-1}$.

Set $\nu_n := [\sharp \mathrm{RPer}_n(f)]^{-1} \sum_{x\in\mathrm{RPer}_n(f)} \varepsilon_x$ and let ν be any cluster point of ν_n. Fix $\varepsilon > 0$ and $x \in \operatorname{Supp}\mu \setminus \mathrm{PC}(f)$. Using lemma 3.3, we construct $(1-\varepsilon)d_t^n$

inverse branches f_i^{-n} of f^n on $B = B(x, r_\varepsilon)$ whose images have small diameter. We now prove the following inequality:

$$(\dagger\dagger) \qquad (1-\varepsilon)^3 \mu_f(B) \leq \nu(\overline{B}).$$

Clearly $(\dagger\dagger)$ implies $\mu_f \leq \nu$. Indeed any Borel subset A can be approximated by disjoint union of small balls satisfying $(\dagger\dagger)$; hence $(1-\varepsilon)^4 \mu_f(A) \leq \nu(\overline{A})$. One can then let $\varepsilon \to 0$. Finally since μ_f and ν are probability measures, we actually get $\mu_f = \nu$; hence $\nu_n \to \mu_f$.

It remains to prove $(\dagger\dagger)$. We can assume $\mu_f(B) > 0$. Fix $B' \subset\subset B'' \subset\subset B$ such that $\mu_f(B') \geq (1-\varepsilon)\mu_f(B)$. We consider as above $\widehat{B}_\varepsilon$ the set of histories of points in B given by the inverse branches f_i^{-n}. Since $\widehat{\mu_f}$ is mixing, we get $\widehat{\mu_f}(\widehat{f}^{-n}(\widehat{B}_\varepsilon) \cap \widehat{B'}) \to \widehat{\mu_f}(\widehat{B}_\varepsilon)\widehat{\mu_f}(\widehat{B'})$. Thus, for n large enough,

$$\begin{aligned}(1-\varepsilon)^3 \mu_f(B)^2 &\leq (1-\varepsilon)\widehat{\mu_f}(\widehat{B}_\varepsilon)\widehat{\mu_f}(\widehat{B'}) \\ &\leq \widehat{\mu_f}(\widehat{f}^{-n}(\widehat{B}_\varepsilon) \cap \widehat{B'}) \leq \sum_{i=1}^{(1-\varepsilon)d_t^n} \mu_f(B_i^{-n} \cap B').\end{aligned}$$

Observe that either $B_i^{-n} \cap B' = \emptyset$ or $B_i^{-n} \subset B'' \subset\subset B$ since $\mathrm{diam}(B_i^{-n}) \to 0$. When $B_i^{-n} \cap B' \neq \emptyset$, f_i^{-n} is thus a contraction on B. Therefore it admits a unique attracting fixed point which is henceforth a repelling periodic point of order n for f. Using again that $\mu_f(B_i^{-n}) = d_t^{-n}\mu_f(B)$, we infer

$$(1-\varepsilon)^3 \mu_f(B)^2 \leq \frac{\sharp\mathrm{RPer}_n(f)}{d_t^n}\nu_n(B)\mu_f(B).$$

Letting $n_i \to +\infty$ yields $(\dagger\dagger)$ if $\overline{\lim}\sharp\mathrm{RPer}_n(f)/d_t^n \leq 1$. Note that $d_t(f) > \max_{1 \leq j \leq k-1} \lambda_j(f)$ by Proposition 1.2. When $\dim_{\mathbb{C}} X \leq 3$ or when X is complex homogeneous, each dynamical degree $\lambda_l(f)$ equals the asymptotical growth of the spectral radii $r_l(f^n)$ of the linear action induced by f^* on $H_a^{l,l}(X, \mathbb{R})$ (see Proposition 1.2 and Remark 1.3.ii). In these cases, the upper bound on $\sharp\mathrm{RPer}_n(f)$ follows from the Lefschetz fixed point formula if f has no curve of periodic points. Note that f cannot have a curve of repelling periodic points. The bound therefore follows from a perturbation argument. □

4. Uniqueness of the measure of maximal entropy

THEOREM 4.1. *Assume* $\dim_{\mathbb{C}} X \leq 3$ *or* X *is complex homogeneous. Then the measure* μ_f *is the unique measure of maximal entropy.*

Here again we follow Briend and Duval [BD 01] who proved this result for holomorphic endomorphisms of $\mathbb{CP}^k$.

Proof. Let ν be an ergodic measure such that $\nu(\mathrm{PC}(f)) > 0$. Then $\nu(f^j(\mathcal{C}_f)) > 0$ for some $j \in \mathbb{N}$, so that it follows from the relative variational principle and Corollary 1.9 that

$$h_\nu(f) \le h_{\mathrm{top}}(f_{|f^j(\mathcal{C}_f)}) \le \max_{1\le j\le k-1} \log \lambda_j(f) < \log d_t(f).$$

Consider now an ergodic probability measure ν of entropy $h_\nu(f) > \max_{1\le j\le k-1}\log\lambda_j(f)$. Then ν does not charge $\mathrm{PC}(f)$; hence $d_t^{-n}(f^n)^*(\nu) \to \mu_f$. Assume $\nu \ne \mu_f$. Then ν does not have constant jacobian, i.e. $f^*\nu \ne d_t\nu$. Therefore one can construct a simply connected domain U in $X \setminus f(\mathcal{C}_f)$ with $\nu(U) = \mathrm{Vol}(U) = 1$ admitting $U_1, \dots, U_{d_t}$ preimages on which f is one-to-one and not equally well ν-distributed, say with $\nu(U_1) > \sigma > d_t^{-1}$ (see [BD 01] for more details on this construction). We are going to show that this implies $h_\nu(f) < \log d_t(f)$.

Observe that $\nu(\Omega_f) = 1$; otherwise $h_\nu(f) \le \max_{1\le j\le k-1}\log\lambda_j(f)$ by Corollary 1.9. Consider O a slightly smaller open subset of U_1 such that $O_\varepsilon \subset U_1$, where O_ε denotes the ε-neighborhood of O, and $\nu(O) > \sigma$. Set $Y = \{a \in \Omega_f : \sharp\{0 \le j \le n-1, f^j(a) \in O\} \ge n\sigma \text{ for } n \ge m\}$. It follows from Birkhoff's theorem that $\nu(Y) > 0$ for m large enough. The relative variational principle yields

$$h_\nu(f) \le h_{\mathrm{top}}(f_{|Y}) \le \limsup \frac{1}{n}\mathrm{Vol}(\Gamma_n|Y)_\varepsilon,$$

where $\Gamma_n = \{(a, \dots, f^{n-1}(a)) : a \in \Omega_f\}$ is the iterated graph of f (see Section 1). Up to a zero volume set, we get

$$(\Gamma_n|Y)_\varepsilon \subset \bigcup_{\alpha\in\Sigma_n} \Gamma_n(\alpha),$$

where $\Sigma_n = \{\alpha \in \{1, \dots, d_t\} : \sharp\{q, \alpha_q = 1\} \ge n\sigma\}$ and $\Gamma_n(\alpha) = \Gamma_n \cap (U_{\alpha_1} \times \cdots \times U_{\alpha_n})$. Indeed the U_j's form a partition of X (up to a zero volume set) and $\{\Gamma_n(\alpha)\}$ is the induced partition on Γ_n. Therefore

$$\begin{aligned}\mathrm{Vol}(\Gamma_n|Y)_\varepsilon &\le \sum_{\alpha\in\Sigma_n} \int_{\Gamma_n(\alpha)} \omega_n^k \\ &\le \sum_{i\in\{0,\dots,n-1\}^k} \sum_{\alpha\in\Sigma_n} \int_{\pi(\Gamma_n(\alpha))} (f^{i_1})^*\omega \wedge \cdots \wedge (f^{i_k})^*\omega,\end{aligned}$$

where π denotes the projection of X^n on the first factor. Fix $\varepsilon > 0$ so small that $\beta + \varepsilon < d_t$, where $\beta := \max_{1\le j\le k-1}\lambda_j(f)$. Fix $\gamma < 1$ to be chosen later and define, following a trick of Briend and Duval,

$$I = \{i \in \{0, \dots, n-1\}^k : i_1, \dots i_k \ge \gamma n\} \text{ and } II = \{0, \dots, n-1\}^2 \setminus I.$$

Fix $i \in II$ and assume $i_1 \le \cdots \le i_k$ (hence $i_1 \le \gamma n$). Since the $\pi(\Gamma_n(\alpha))$ form a partition of Ω_f, we get

$$\begin{aligned}\sum_{\alpha\in\Sigma_n}\int_{\pi(\Gamma_n(\alpha))}(f^{i_1})^*\omega\wedge\cdots\wedge(f^{i_k})^*\omega &\le \int_{\Omega_f}(f^{i_1})^*\omega\wedge\cdots\wedge(f^{i_k})^*\omega\\ &= d_t^{i_1}\int_{\Omega_f}\omega\wedge(f^{i_2-i_1})^*\omega\wedge\cdots\wedge(f^{i_k-i_1})^*\omega\\ &\le C_\varepsilon d_t^{i_1}[\beta+\varepsilon]^{i_k-i_1}\\ &\le C_\varepsilon d_t^{\gamma n}[\beta+\varepsilon]^{n(1-\gamma)},\end{aligned}$$

where the existence of C_ε is as in Lemma 1.7. Therefore

$$\sum_{i\in II}\sum_{\alpha\in\Sigma_n}\int_{\pi(\Gamma_n(\alpha))}(f^{i_1})^*\omega\wedge\cdots\wedge(f^{i_k})^*\omega\le C_\varepsilon n^k d_t^{\gamma n}[\beta+\varepsilon]^{n(1-\gamma)}.$$

Now fix $i\in I$, $\alpha\in\Sigma_n$ and set $q=[\gamma n]$. Since f^q is injective on $\pi(\Gamma_n(\alpha))$, assuming $i_1\le\cdots\le i_k$, we get

$$\begin{aligned}&\int_{\pi(\Gamma_n(\alpha))}(f^{i_1})^*\omega\wedge\cdots\wedge(f^{i_k})^*\omega\\ &\qquad=\int_{\pi(\Gamma_n(\alpha))}(f^q)^*\left((f^{i_1-q})^*\omega\wedge\cdots\wedge(f^{i_k-q})^*\omega\right)\\ &\qquad\le\int_{\Omega_f}(f^{i_1-q})^*\omega\wedge\cdots\wedge(f^{i_k-q})^*\omega\\ &\qquad\le C_\varepsilon d_t^{i_1-q}[\beta+\varepsilon]^{i_k-i_1}=\left(\frac{d_t}{\beta+\varepsilon}\right)^{i_1-q}[\beta+\varepsilon]^{i_k-q}\\ &\qquad\le C_\varepsilon d_t^{n-1-q}\le C_\varepsilon d_t^{n(1-\gamma)}.\end{aligned}$$

By Lemma 7.2 in [L 83] there exists $\rho<1$ such that $\sharp\Sigma_n\le d_t^{n\rho}$. Therefore

$$\sum_{i\in I}\sum_{\alpha\in\Sigma_n}\int_{\pi(\Gamma_n(\alpha))}(f^{i_1})^*\omega\wedge\cdots\wedge(f^{i_k})^*\omega\le C_\varepsilon n^k d_t^{\rho n}d_t^{n(1-\gamma)}.$$

Altogether this yields

$$h_\nu(f)\le\max([1+\rho-\gamma]\log d_t(f),\gamma\log d_t(f)+[1-\gamma]\log(\beta+\varepsilon)),$$

so that $h_\nu(f)<\log d_t(f)$ if we choose $\rho<\gamma<1$. □

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