Correction CCR IC3

Exercice 1

1° -

2° $S = X + Y \sim N(1,3)$

$T = 3X \sim N(0,9)$

$V = X + 4 \sim N(4,1)$.

3°. $X, Y \sim \mathcal{B}er(1/2)$

. $S = X + Y \sim \mathcal{B}(2, 1/2)$

. T à valeurs dans $\{0, 3\}$

$\mathbb{P}(T = 0) = \mathbb{P}(T = 3) = 1/2$

. $V = X + 4$ à valeurs dans $\{4, 5\}$

$\mathbb{P}(V = 4) = \mathbb{P}(V = 5) = 1/2$.

4°. $X + Y$ à valeurs dans $\mathbb{N}^*$.

$$\mathbb{P}(X + Y = k) = \mathbb{P}\left(\bigcup_{l=0}^{k} \{X = l\} \cap \{Y = k - l\}\right)$$

$$= \sum_{l=0}^{k} \mathbb{P}(X = l)\, \mathbb{P}(Y = k - l)$$

$$= \sum_{l=0}^{k} e^{-\lambda_1} \frac{\lambda_1^l}{l!} e^{-\lambda_2} \frac{\lambda_2^{k-l}}{(k-l)!}$$

$$= e^{-(\lambda_1+\lambda_2)} \frac{1}{k!} \sum_{l=0}^{k} C_k^l \lambda_1^l \lambda_2^{k-l}$$

$$= e^{-(\lambda_1+\lambda_2)} \frac{(\lambda_1+\lambda_2)^k}{k!}$$

5° -

6° $\hat{\sigma}^2 = S_n^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X}_n)^2$

Exercice 2

1° $\mathbb{E}[Y] = 1 \times \frac{1}{2} + (-1) \times \frac{1}{2} = 0$.

2° $$\mathbb{P}(X \leq t \,/\, Y = 1) = \frac{\mathbb{P}(YZ \leq t \cap Y = 1)}{\mathbb{P}(Y = 1)}$$

$$= \frac{\mathbb{P}(Z \leq t)\, \mathbb{P}(Y = 1)}{\mathbb{P}(Y = 1)} = \mathbb{P}(Z \leq t).$$

De même,

$\mathbb{P}(X \leq t \,/\, Y = -1) = \mathbb{P}(Z \geq -t)$.

1. $\forall t \in \mathbb{R}$, on a

$$F_X(t) = \mathbb{P}(X \leq t)$$
$$= \mathbb{P}(X \leq t / Y=1)\mathbb{P}(Y=1) + \mathbb{P}(X \leq t / Y=-1) \times \mathbb{P}(Y=-1)$$

$$= \frac{1}{2}\mathbb{P}(Z \leq t) + \frac{1}{2}\mathbb{P}(Z \geq -t)$$
$$= \frac{1}{2} F_Z(t) + \frac{1}{2}(1 - \cancel{\mathbb{P}} F_Z(-t))$$

4°. On a alors

$$f_X(t) = F_X'(t)$$
$$= \frac{1}{2} f_Z(t) + \frac{1}{2} f_Z(-t)$$
$$= f_Z(t)$$

Ainsi $X \sim N(0,1)$.

5°. $\mathbb{E}[X] = 0 = \mathbb{E}[Y]\mathbb{E}[Z]$.

6°. $\mathbb{P}(Z > 1 \cap X > 1)$
$$= \mathbb{P}(YZ > 1 \cap X > 1)$$
$$= \mathbb{P}(Y = 1 \cap Z > 1) = \frac{1}{2}\mathbb{P}(Z>1).$$

et $\mathbb{P}(Z>1 \cap X>1)$
$$\neq \mathbb{P}(Z>1)\,\mathbb{P}(X>1).$$

X et Z ne sont donc pas indép.

<u>Exercice 3</u>

1°. $U_1 \sim \mathcal{U}[0,1]$, $\forall t \in [0,1]$
$$\mathbb{P}(U_1 \leq t) = \int_0^t dx = t.$$

$$\mathbb{E}[U_1] = \int_0^1 x\,dx = \left[\frac{x^2}{2}\right]_0^1 = \frac{1}{2}$$

$$\mathbb{E}[U_1^2] = \int_0^1 x^2\,dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}$$

$$\Rightarrow Var(U_1) = \frac{1}{3} - \left(\frac{1}{4}\right) = \frac{1}{12}.$$

2°. $\mathbb{P}(S>t) = \mathbb{P}\left(\bigcap_{i=1}^{n}\{U_i > t\}\right)$
$$= \mathbb{P}(U_1 > t)^n$$
$$= (1-t)^n.$$

3/ On a donc

$$F_S(t) = 1 - (1-t)^n, \quad \forall t \in [0,1].$$

et $f_S(t) = \begin{cases} n(1-t)^{n-1} & \text{si } t \in [0,1] \\ 0 & \text{sinon} \end{cases}$

4/ $\mathbb{E}[S] = \displaystyle\int_0^1 n\, t (1-t)^{n-1}\, dt$

$$\begin{bmatrix} u = t & u' = 1 \\ v' = n(1-t)^{n-1} & v = -(1-t)^n \end{bmatrix}$$

$$= \left[t(1-t)^n \right]_0^1 + \int_0^1 (1-t)^n\, dt$$

$$= \left[-\frac{1}{n+1} (1-t)^{n+1} \right]_0^1$$

$$= \frac{1}{n+1}.$$

Exercice 4

1°) $\bar{X}_{10} \sim N\left(710, \frac{25}{10}\right)$

En effet :

$\mathbb{E}[\bar{X}_{10}] = \mathbb{E}[X_1]$

$\mathrm{Var}(\bar{X}_{10}) = \dfrac{\mathrm{Var}(X_1)}{10}$ (indép. des X_i)

2°) -

3°) $\mathbb{P}(\bar{X}_{10} \leq t) = 5\%$

$\Leftrightarrow \mathbb{P}\left(\dfrac{\sqrt{10}(\bar{X}_{10} - 710)}{\sqrt{25}} \leq \dfrac{\sqrt{10}(t-710)}{\sqrt{25}} \right) = 5\%$

$\Leftrightarrow P\left(Z \leq \dfrac{\sqrt{10}(t-710)}{\sqrt{25}} \right) = 5\%$

où $Z \sim N(0,1)$

Or $\mathbb{P}(Z \leq c_0) = \mathbb{P}(Z \geq -c_0) = 5\%$

$\Leftrightarrow \mathbb{P}(Z \leq -c_0) = 95\%$

avec $c_0 = 1{,}65$. On choisit donc

$\dfrac{\sqrt{10}(t-710)}{\sqrt{25}} = -1{,}65 \Leftrightarrow \boxed{t = 710 - 1{,}65 \times \sqrt{\dfrac{25}{10}}}$