Correction CC2 de Proba – Stat

3 IC, 2012 – 2013

Exc. 1

1) Pour toute var. aléat. X admettant une variance, on a

$$\forall \varepsilon > 0 \ : \ \mathbb{P}\left(|X-\mathbb{E}(X)| > \varepsilon\right) \leq \frac{Var(X)}{\varepsilon^2}.$$

2) $\begin{matrix} X \sim \mathcal{N}(1,8) \\ Y \sim \mathcal{N}(2,8) \end{matrix}$ indép. $\Rightarrow X+Y \sim \mathcal{N}(\underbrace{1+2}_{3}, \underbrace{8+8}_{16})$.

$\mathbb{E}(S) = \mathbb{E}(X) + \mathbb{E}(Y) = 3$

$Var(S) \underset{\uparrow \text{ car indép.}}{=} Var(X) + Var(Y) = 16$

$S \sim \mathcal{N}(3,16) \Rightarrow Z = \frac{S-3}{4} \sim \mathcal{N}(0,1)$

$$\Rightarrow \mathbb{P}(S<-1) = \mathbb{P}\left(\frac{S-3}{4} < \frac{-1-3}{4}\right) = \mathbb{P}(Z<-1) \underset{\text{par symétrie}}{=} \mathbb{P}(Z>1)$$

$$= 1 - F_Z(1) = \underline{0,159}$$

Exc. 2

1) $f_X(x) = \begin{cases} e^{-x}, & x \geq 0 \\ 0, & \text{sinon} \end{cases}$, $F_X(t) = \begin{cases} 0, & \text{si } t<0 \\ \int_0^t e^{-x}dx = 1-e^{-t}, & \text{si } t \geq 0 \end{cases}$

2) $F_Y(t) = \begin{cases} 0, & \text{si } t \leq 0 \\ \mathbb{P}(Y \leq t) = \mathbb{P}\left(\frac{X}{\mu} \leq t\right) = \mathbb{P}(X \leq \mu t) = F_X(\mu t) = 1-e^{-\mu t}, & \text{si } t \geq 0 \end{cases}$

$\hookrightarrow \underline{Y \sim Exp(\mu)}$.

3) a) $\mathbb{E}(Z) = 1 \times p + (-1) \times (1-p) = \underline{2p-1}$

$Var(Z) = \underbrace{\mathbb{E}(Z^2)}_{\substack{\shortparallel \\ 1\times p + 1 \times (1-p) = 1}} - (\mathbb{E}(Z))^2 = 1-(2p-1)^2 = \underline{4p(1-p)}$.

b) $\mathbb{P}(T>t \cap Z=1) = \mathbb{P}(XZ>t \cap Z=1) = \mathbb{P}(X>t \cap Z=1)$

$\underset{X \text{ et } Z \text{ indép.}}{=} \mathbb{P}(X>t) \times \mathbb{P}(Z=1) = (1-F_X(t)) \times p$

$= \begin{cases} e^{-t} \times p, & \text{si } t \geq 0 \\ p, & \text{si } t \leq 0 \end{cases}$

$\mathbb{P}(T>t \cap Z=-1) = \mathbb{P}(XZ>t \cap Z=-1) = \mathbb{P}(-X>t \cap Z=-1) = \mathbb{P}(X<-t \cap Z=-1)$

$\underset{X \text{ et } Z \text{ indép.}}{=} \mathbb{P}(X<-t) \times \mathbb{P}(Z=-1) = F_X(-t) \times (1-p)$

$= \begin{cases} (1-e^{t})(1-p), & \text{si } t \leq 0 \\ 0, & \text{si } t>0 \end{cases}$

$\mathbb{P}(T>t) = \mathbb{P}(T>t \cap Z=1) + \mathbb{P}(T>t \cap Z=-1) = \begin{cases} pe^{-t}, & \text{si } t \geq 0 \\ p+(1-e^{t})(1-p), & \text{si } t \leq 0 \end{cases}$

c) $F_T(t) = 1 - \mathbb{P}(T > t)$

$$= \begin{cases} 1 - p e^{-t}, & \text{si } t \geq 0 \\ 1 - p - (1-p)(1-e^{t}) = (1-p)e^{t}, & \text{si } t \leq 0 \end{cases}$$

d) $f_T(t) = F_T'(t) = \begin{cases} p e^{-t}, & \text{si } t \geq 0 \\ (1-p) e^{t}, & \text{si } t < 0 \end{cases}$

e) $\mathbb{E}(T) = \int_{-\infty}^{\infty} f_T(t)\,dt = \underbrace{\int_{-\infty}^{0} (1-p)\, t e^{t}\,dt}_{} + \underbrace{\int_{0}^{\infty} p\, t e^{-t}\,dt}_{} = -(1-p) + p = \boxed{2p-1}$

$(s = -t)$

$$(1-p)\int_{\infty}^{0} (-s) e^{-s} \cdot (-1)\,ds \qquad\qquad p \times \mathbb{E}(\mathcal{E}xpo(1))$$

$$\| \qquad\qquad\qquad\qquad \|$$

$$-(1-p)\int_{0}^{\infty} s e^{-s}\,ds \qquad\qquad p \times 1 = p$$

$$\|$$

$$-(1-p) \times \mathbb{E}(\mathcal{E}xpo(1))$$

$$\|$$

$$-(1-p) \times 1 = -(1-p)$$

$X \sim \mathcal{E}xpo(1) \Rightarrow \mathbb{E}(X) = 1 \Rightarrow$ On a bien $\mathbb{E}(T) = \mathbb{E}(Z) \times \mathbb{E}(X)$.

f) $\mathbb{P}(T > 0 \cap Z = -1) = 0 \neq \underbrace{\mathbb{P}(T > 0)}_{p} \times \underbrace{\mathbb{P}(Z = -1)}_{1-p} \Rightarrow$ T et Z ne sont pas indép.

Exc. 3 1) a) $X_i \sim \mathcal{B}(p) \Rightarrow \mathbb{E}(X_i) = p \Rightarrow \mathbb{E}(\hat{p}_n) = \frac{\sum \mathbb{E}(X_i)}{n} = \frac{np}{n} = p$

$\Rightarrow \hat{p}_n$ estimateur sans biais de p

Par la Loi des grands nombres,

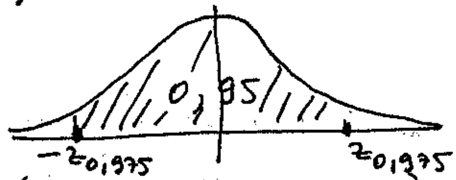
$\hat{p}_n \xrightarrow[n\to\infty]{\text{proba}} \mathbb{E}(X_i) = p \Rightarrow \hat{p}_n$ estimateur consistant de p.

b) Par le Thm. Limite Central,

$$\frac{\sqrt{n}(\hat{p}_n - m)}{\sigma} \xrightarrow[n\to\infty]{\text{loi}} \mathcal{N}(0,1), \quad \text{avec} \quad \begin{aligned} m &= \mathbb{E}(X_i) = p \\ \sigma^2 &= \mathrm{Var}(X_i) = p(1-p) \end{aligned}$$

$$\Rightarrow \hat{p}_n \underset{n \text{ grand}}{\overset{\text{loi}}{\approx}} \mathcal{N}\left(m, \frac{\sigma^2}{n}\right) = \mathcal{N}\left(p, \frac{p(1-p)}{n}\right).$$

c) On a $\frac{\sqrt{n}(\hat{p}_n - p)}{\sqrt{p(1-p)}} \underset{\text{grand}}{\overset{\text{loi}}{\approx}} \mathcal{N}(0,1)$



donc $\mathbb{P}\left(-z_{1-\frac{\alpha}{2}} \leq \frac{\sqrt{n}(\hat{p}_n - p)}{\sqrt{p(1-p)}} < z_{1-\frac{\alpha}{2}}\right) \underset{n \text{ grand}}{\approx} \mathbb{P}\left(-z_{1-\frac{\alpha}{2}} \leq Z \leq z_{1-\frac{\alpha}{2}}\right)$

avec $Z \sim \mathcal{N}(0,1)$

Ici $z_{1-\frac{\alpha}{2}} = z_{0,975}$ est t.q.

$$\mathbb{P}(Z < z_{0,975}) = 0,975 \Rightarrow z_{0,975} = 1,96$$

On a donc

$$\mathbb{P}\left(-1{,}96 \leq \frac{\sqrt{n}\,(\hat{p}_n - p)}{\sqrt{p(1-p)}} \leq 1{,}96\right) \underset{n \text{ grand}}{\simeq} 0{,}95$$

$$\Leftrightarrow \mathbb{P}\left(p \in \underbrace{\hat{p}_n \pm 1{,}96 \times \frac{\sqrt{p(1-p)}}{\sqrt{n}}}\right) \simeq 0{,}95$$

les bornes dépendent de p.

$\rightarrow$ soit on majore $p(1-p) \leq \frac{1}{4}$ $\Rightarrow IC_{0{,}95}(p) = \left[\hat{p}_n \pm \frac{1{,}96}{2\sqrt{n}}\right]$

$\rightarrow$ soit on remplace p par $\hat{p}_n$ dans $\sqrt{p(1-p)}$

$$\Rightarrow IC_{0{,}95}(p) = \left[\hat{p}_n \pm 1{,}96 \times \frac{\sqrt{\hat{p}_n(1-\hat{p}_n)}}{\sqrt{n}}\right]$$

d) $\hat{p}_n = \frac{20}{100} = 0{,}2$

$$IC_{0{,}95}(p) = \left[0{,}2 \pm 1{,}96 \times \frac{\sqrt{0{,}2 \times 0{,}8}}{\sqrt{100}}\right] = \left[0{,}2 \pm 1{,}96 \times \frac{0{,}4}{10}\right] = \left[0{,}2 \pm 0{,}0784\right].$$

2) a) $N \sim \mathcal{B}(n, p)$ avec $n = 400$ et $p = 0{,}2$
binomiale

car $N = X_1 + \dots + X_n$ avec $X_i \sim Ber(p)$ indép.

$\mathbb{E}(N) = n \times p = 400 \times 0{,}2 = 80$

$Var(N) = np(1-p) = 80 \times 0{,}8 = 64$.

b) Par le TCL : $\dfrac{X_1 + \dots + X_n - np}{\sqrt{np(1-p)}} \xrightarrow[n \to \infty]{\text{loi}} \mathcal{N}(0,1)$

donc pour n grand $N \overset{\text{loi}}{\approx} \mathcal{N}(np, np(1-p)) = \mathcal{N}(80, 64)$

c) $\mathbb{P}(N > 104) = \mathbb{P}\left(\frac{N-80}{8} > \frac{104-80}{8}\right) \approx \mathbb{P}(Z > 3)$, avec $Z \sim \mathcal{N}(0,1)$

$$= 1 - F_Z(3)$$
$$= \boxed{0{,}0013}$$