Correction TD 2 → 2IC Maths

2014-2015

$\boxed{\text{Exc. 5}}$ 1) $\phi(x,y) = (u,v)$, $\begin{cases} u = x^2 - y^2 - 2xy \\ v = y \end{cases}$

Pour $(u,v) \in V$, on va m.q. $\exists ! (x,y) \in \Omega$ t.q. $\phi(x,y) = (u,v)$.

$$\begin{cases} u = x^2 - y^2 - 2xy \\ v = y \end{cases} \Rightarrow \begin{cases} x^2 - 2xy - y^2 - u = 0 \\ v = y \end{cases} \Rightarrow \begin{cases} x^2 - 2vx - v^2 - u = 0 \\ \underbrace{y = v} \end{cases}$$

$\Delta = 4v^2 + 4(v^2 + u) = 4(u + 2v^2) > 0$ car $(u,v) \in V \Rightarrow \exists$ solutions

$\Rightarrow x = \dfrac{2v \pm 2\sqrt{u+2v^2}}{2} = v \pm \sqrt{u+2v^2}$

La seule sol. $x < y$ est $\underbrace{x = v - \sqrt{u^2+2v^2}}$

$\Rightarrow \phi$ est bijective et $\phi^{-1}(u,v) = \left(v - \sqrt{u^2+2v^2}, v\right)$.

$\phi \in C^\infty$ et $\phi^{-1} \in C^\infty$ car chacune des composantes sont de classe C^∞ (thm. généraux).

2) $f(x,y) = g(u,v)$, $\left.\begin{array}{l} g = f \circ \phi^{-1} \\ f \in C^1 \\ \phi^{-1} \in C^\infty \end{array}\right\} \Rightarrow g \in C^1$

Règle de la chaîne :

$$\underline{\partial_x f} = \partial_u g \cdot \partial_x u + \partial_v g \cdot \partial_x v = \partial_u g \cdot (2x - 2y) + \partial_v g \cdot 0 = \underline{2(x-y) \cdot \partial_u g}$$
$$= 2\partial_u g \cdot \left(v - \sqrt{u^2 + 2v^2} - v\right)$$

$$\underline{\partial_y f} = \partial_u g \cdot \partial_y u + \partial_v g \cdot \partial_y v = \partial_u g \cdot (-2y - 2x) + \partial_v g \cdot 1$$
$$= \underline{-2(x+y) \cdot \partial_u g + \partial_v g}.$$

Quand on remplace $\partial_x f$ et $\partial_y f$ dans l'équation on obtient

$$2(x+y)\cancel{(x-y)}\,\partial_u g - 2(x+y)\cancel{(x-y)} \cdot \partial_u g + \cancel{(x-y)}\,\partial_v g = 0. \quad \begin{pmatrix} x < y \\ x - y \neq 0 \end{pmatrix}$$

$\Leftrightarrow \partial_v g = 0$

$\Rightarrow \underline{g(u,v) = h(u), \text{ avec } h \in C^1}$.

3) $f(x,y) = h(u) = h(x^2 - y^2 - 2xy)$, avec $h \in C^1$.

$\boxed{\text{Exc. 6}}$ 1) $(u,v) = \phi(x,t) = (x-ct,\, x+ct)$

$\phi : \mathbb{R}^2 \to \mathbb{R}^2$ bij. de classe C^∞.

$g(u,v) = f(x,t) = f\circ\phi^{-1}(u,v)$

$\phi^{-1}(u,v) = (x,t) = \left(\frac{u+v}{2}, \frac{v-u}{2c}\right)$

$\left.\begin{array}{l}\phi^{-1}\in C^\infty\\ f\in C^2\end{array}\right\} \Rightarrow g\in C^2.$

$f(x,t) = g(u,v)$

$\partial_x f = \partial_u g\cdot\partial_x u + \partial_v g\cdot\partial_x v = \partial_u g + \partial_v g$

$\partial_t f = \partial_u g\cdot\partial_t u + \partial_v g\cdot\partial_t v = -c\,\partial_u g + c\,\partial_v g = c(\partial_v g - \partial_u g).$

$\partial^2_{xx} f = \partial_x(\partial_x f) = \partial_u(\partial_u g + \partial_v g) + \partial_v(\partial_u g + \partial_v g) = \underline{\partial^2_{uu} g + 2\partial^2_{uv} g + \partial^2_{vv} g}$

$\partial^2_{tt} f = \partial_t(\partial_t f) = c\left[c\,\partial_v(\partial_v g - \partial_u g) - c\,\partial_u(\partial_v g - \partial_u g)\right]$

$= \underline{c^2\left(\partial^2_{vv} g - 2\partial^2_{uv} g + \partial^2_{uu} g\right)}.$

Quand on remplace dans l'éq. (1) on obtient

$\partial^2_{uv} g = 0, \quad \forall (u,v)\in\mathbb{R}^2.$

2) $\partial_u(\partial_v g) = 0 \Rightarrow \partial_v g = a(v) \Rightarrow g(u,v) = A(v) + B(u)$

avec $A, B \in C^2$.

$\Rightarrow \underline{f(x,t) = g(u,v) = A(x-ct) + B(x+ct)}$

3) $f(x,0) = \cos x \Rightarrow A(x) + B(x) = \cos x$

$\partial_t f(x,0) = 0$

$\partial_t f(x,t) = A'(x-ct)\cdot(-c) + B'(x+ct)\cdot c$.

$\partial_t f(x,0) = 0 \Rightarrow -A'(x) + B'(x) = 0 \Rightarrow B'(x) = A'(x).$

$A(x) + B(x) = \cos x \Rightarrow A'(x) + B'(x) = -\sin x$

Comme en plus $A'(x) = B'(x) \Rightarrow A'(x) = B'(x) = -\frac{\sin x}{2}.$

$\Rightarrow A(x) = \frac{\cos x}{2} + k, \quad B(x) = \frac{\cos x}{2} - k, \quad k\in\mathbb{R}$

$\Rightarrow \boxed{f(x,t) = \frac{1}{2}\cos(x-ct) + \frac{1}{2}\cos(x+ct)}$