Correction feuille de TD 4

Exc. 1

1) $\begin{cases} x(t) = t - \text{ch}\,t \cdot \text{sh}\,t \\ y(t) = 2\,\text{ch}\,t \end{cases}$, $r(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$, $t > 0$

<u>Rappels</u> : $\text{sh}\,t = \dfrac{e^t - e^{-t}}{2}$, $\text{ch}\,t = \dfrac{e^t + e^{-t}}{2}$, $(\text{sh}\,t)' = \text{ch}\,t$, $(\text{ch}\,t)' = \text{sh}\,t$.

$$\ell = \int_0^t \| r'(s) \| \, ds \quad , \text{ où } \| r'(s) \| = \sqrt{(x'(s))^2 + (y'(s))^2}$$

$$x'(s) = 1 - \text{ch}'(s) \cdot \text{sh}(s) - \text{ch}(s) \cdot \text{sh}'(s)$$
$$= 1 - (\text{sh}(s))^2 - (\text{ch}(s))^2 = 1 - \frac{e^{2s} + e^{-2s}}{2} = 1 - \text{ch}(2s).$$

$$y'(s) = 2\,\text{sh}(s)$$

$$\hookrightarrow \sqrt{(x'(s))^2 + (y'(s))^2} = \sqrt{1 + \text{ch}^2(2s) \underbrace{- 2\,\text{ch}(2s)}_{e^{2s} + e^{-2s}} + \underbrace{4\,\text{sh}^2(s)}_{= e^{2s} + e^{-2s} - 2}} = \sqrt{\text{ch}^2(2s) - 1}$$

On a $\text{ch}^2(2s) - 1 = \text{sh}^2(2s)$ (<u>exc</u>). $\Rightarrow \sqrt{\text{ch}^2(2s) - 1} = \underset{>0}{\text{sh}(2s)}$ car $s > 0$.

$$\hookrightarrow \ell = \int_0^t \text{sh}(2s)\, ds = \frac{1}{2} \left[\text{ch}(2s) \right]_0^t = \frac{\text{ch}(2t) - 1}{2} = \text{sh}^2 t.$$

2) $\begin{cases} x(\theta) = r(\theta) \cos\theta = \text{th}\left(\frac{\theta}{2}\right) \cdot \cos\theta \\ y(\theta) = r(\theta) \sin\theta = \text{th}\left(\frac{\theta}{2}\right) \cdot \sin\theta. \end{cases}$

<u>Rappel</u> : $\text{th}'(t) = \dfrac{1}{\text{ch}^2(t)}$ (<u>exc</u>).

$$\Rightarrow \begin{cases} x'(\theta) = \dfrac{1}{2\,\text{ch}^2\left(\frac{\theta}{2}\right)} \cdot \cos\theta - \text{th}\left(\frac{\theta}{2}\right) \cdot \sin(\theta) \\ y'(\theta) = \dfrac{1}{2\,\text{ch}^2\left(\frac{\theta}{2}\right)} \sin\theta + \text{th}\left(\frac{\theta}{2}\right) \cdot \cos\theta \end{cases}$$

$$\hookrightarrow (x'(\theta))^2 + (y'(\theta))^2 = \frac{\cos^2\theta}{4\,\text{ch}^4\left(\frac{\theta}{2}\right)} + \left(\text{th}\left(\frac{\theta}{2}\right)\right)^2 \sin^2\theta - \cancel{\frac{1}{\text{ch}^2\left(\frac{\theta}{2}\right)} \text{th}\left(\frac{\theta}{2}\right) \sin\theta \cos\theta}$$
$$+ \frac{\sin^2\theta}{4\,\text{ch}^4\left(\frac{\theta}{2}\right)} + \left(\text{th}\left(\frac{\theta}{2}\right)\right)^2 \cos^2\theta + \cancel{\frac{1}{\text{ch}^2\left(\frac{\theta}{2}\right)} \text{th}\left(\frac{\theta}{2}\right) \sin\theta \cos\theta}$$

$$= \frac{1}{4\,\text{ch}^4\left(\frac{\theta}{2}\right)} + \text{th}^2\left(\frac{\theta}{2}\right) =$$

$\text{th}^2(x) = 1 - \dfrac{1}{\text{ch}^2(x)}$ car $\text{ch}^2(x) - \text{sh}^2(x) = 1$

$$= \frac{1}{4\,\text{ch}^4\left(\frac{\theta}{2}\right)} + 1 - \frac{1}{\text{ch}^2\left(\frac{\theta}{2}\right)} = \left(1 - \frac{1}{2\,\text{ch}^2\left(\frac{\theta}{2}\right)}\right)^2$$

$$\hookrightarrow \ell = \int_0^\theta \sqrt{(x'(t))^2 + (y'(t))^2}\, dt = \int_0^\theta \left(1 - \frac{1}{2\,\text{ch}^2\left(\frac{t}{2}\right)}\right) dt = \left[t - \text{th}\left(\frac{t}{2}\right)\right]_0^\theta = \boxed{\theta - \text{th}\left(\frac{\theta}{2}\right)}.$$

$\boxed{\text{Exc. 3}}$

1) $T(s) = \begin{bmatrix} \cos\theta(s) \\ \sin\theta(s) \end{bmatrix}$ car $\|T(s)\| = 1$; $T'(s) = \begin{bmatrix} -\sin\theta(s)\cdot\theta'(s) \\ \cos\theta(s)\cdot\theta'(s) \end{bmatrix}$

$$\Rightarrow K(s) = \|T'(s)\| = \sqrt{\sin^2\theta(s)\cdot(\theta'(s))^2 + \cos^2\theta(s)\cdot(\theta'(s))^2} = \theta'(s) > 0.$$

2) si $K(s) = \frac{1}{R} \Rightarrow \theta'(s) = \frac{1}{R} \Rightarrow \theta(s) = \frac{s}{R} + \theta_0$

$$\Rightarrow T(s) = \begin{bmatrix} \cos\left(\frac{s}{R} + \theta_0\right) \\ \sin\left(\frac{s}{R} + \theta_0\right) \end{bmatrix}.$$

D'un autre côté : $T(s) = r'(s) = \begin{bmatrix} x'(s) \\ y'(s) \end{bmatrix}$

car s abscisse curviligne

$$\hookrightarrow \begin{cases} x'(s) = \cos\left(\frac{s}{R} + \theta_0\right) \\ y'(s) = \sin\left(\frac{s}{R} + \theta_0\right) \end{cases} \Rightarrow \begin{cases} x(s) = \sin\left(\frac{s}{R} + \theta_0\right)\cdot R + x_0 \\ y(s) = -\cos\left(\frac{s}{R} + \theta_0\right)\cdot R + y_0 \end{cases}$$

$\Rightarrow (x(s) - x_0)^2 + (y(s) - y_0)^2 = R^2 \Rightarrow (x(s), y(s)) \in$ un cercle (de rayon R et centre (x_0, y_0))

3) si $K(s) = 0 \Rightarrow \theta'(s) = 0 \Rightarrow \theta(s) = \theta_0$

$$\Rightarrow T(s) = \begin{bmatrix} \cos\theta_0 \\ \sin\theta_0 \end{bmatrix} \Rightarrow \begin{cases} x'(s) = \cos(\theta_0) \\ y'(s) = \sin(\theta_0) \end{cases}$$

$$\Rightarrow \begin{cases} x(s) = \cos(\theta_0)\cdot s + x_0 \\ y(s) = \sin(\theta_0)\cdot s + y_0 \end{cases}$$

$$\Rightarrow \begin{bmatrix} x(s) \\ y(s) \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + s\cdot\begin{bmatrix} \cos(\theta_0) \\ \sin(\theta_0) \end{bmatrix}$$

$\Rightarrow (x(s); y(s)) \in$ la droite passant par (x_0, y_0) et de vecteur directeur $\begin{bmatrix} \cos(\theta_0) \\ \sin(\theta_0) \end{bmatrix}$

Correction Exercice 4 – feuille 4

1. Repère de Frenet : $\underline{N}'(s) = -\frac{1}{s\sqrt{2}}\left(\underline{T}(s) + \underline{B}(s)\right)$

$$\underline{N}''(s) = \frac{1}{s^2\sqrt{2}}\left(\underline{T}'(s) + \underline{B}'(s)\right) - \frac{1}{s\sqrt{2}}\left(\underline{T}'(s) + \underline{B}'(s)\right)$$

$$= -\frac{1}{s}\underline{N}'(s) - \frac{1}{s\sqrt{2}}\left(\frac{1}{s\sqrt{2}} + \frac{1}{s\sqrt{2}}\right)\underline{N}(s) = -\frac{1}{s}\underline{N}'(s) - \frac{1}{s^2}\underline{N}(s)$$

$s^2\underline{N}''(s) + s\underline{N}'(s) + \underline{N}(s) = 0$. Comme $\underline{T}'(s) = \frac{1}{s\sqrt{2}}\underline{N}(s)$

$$s^2\left(s\sqrt{2}\,\underline{T}'\right)''(s) + s\left(s\sqrt{2}\,\underline{T}'\right)'(s) + s\sqrt{2}\,\underline{T}'(s) = 0.$$

$$s^2\left(s\sqrt{2}\,T''' + 2\sqrt{2}\,T''\right) + s\left(s\sqrt{2}\,T'' + \sqrt{2}\,T'\right) + s\sqrt{2}\,T' = 0.$$

$$s^3\sqrt{2}\,T''' + 3\sqrt{2}\,s^2T'' + 2\sqrt{2}\,sT' = 0\,;\quad s\sqrt{2}\left(s^2T''' + 3sT'' + 2T'\right) = 0.$$

$s\sqrt{2}\left(s^2T'' + sT' + T\right)' = 0$; on a alors par continuité en 0

$$\boxed{\left(s^2T'' + sT' + T\right)' = 0.}$$

Il s'ensuit que $s^2\underline{T}'' + s\underline{T}' + \underline{T} = \underline{K}$ où $\underline{K}$ est un vecteur constant de $\mathbb{R}^3$.

2. On pose $z(u) = y(s)$ avec $s = e^u$ d'où $z'(u) = y'(s)\,\partial_u s = y'(s)\,s$.

$z''(u) = (y'(s)\,s)'\,s = y''s^2 + y's$; $z''(u) + z(u) = k$

Sol. de l'éq. ss. 2nd membre $v(u) = c\cos u + d\sin u$.

Sol. Part. $w(u) = k$; d'où $z(u) = c\cos u + d\sin u + k$.

3. On a ainsi $\underline{T}(s) = \underline{C}\cos u + \underline{D}\sin u + \underline{K}$.

$$\eta'_u = \eta'_s\,s = \underline{T}(s)\,s = e^u\left(\underline{C}\cos u + \underline{D}\sin u + \underline{K}\right)$$

$$\eta_u = \int e^u\left(\underline{C}\cos u + \underline{D}\sin u + \underline{K}\right)du + E$$

Pour trouver une primitive $\int (\underline{C} \cos u + \underline{D} \sin u) e^u \, du$, on opère composante par composante ; $(c \cos u + d \sin u) e^u$ étant cette composante, on cherche cette primitive sous la forme $(a \cos u + b \sin u) e^u$. On a ainsi

$$\left(e^u \left(a \cos u + b \sin u \right) \right)' = e^u \left(a \cos u + b \sin u - a \sin u + b \cos u \right)$$

$$= e^u \left((a+b) \cos u + (b-a) \sin u \right).$$

Il suffit ainsi de résoudre le système linéaire

$$\begin{cases} a+b = c \\ -a+b = d \end{cases} \quad d'où \quad a = \frac{c-d}{2} \; ; \; b = \frac{c+d}{2}$$

D'où le paramétrage

$$\boxed{\Omega_u = e^u \left(\underline{A} \cos u + \underline{B} \sin u + \underline{K} \right) + \underline{E}}$$

$\underline{A}$, $\underline{B}$, $\underline{K}$, $\underline{E}$ étant 4 vecteurs constants de $\mathbb{R}^2$