EU2 - Rattrapage 2012-13.

Exercice 1 1. A est symétrique : toutes ses val. propres sont réelles.

2. $\begin{vmatrix} 5 & -1 & 2 \\ -1 & 5 & 2 \\ 2 & 2 & 2 \end{vmatrix} \begin{matrix} Eq1 \\ \to 5Eq2 \\ \to -\frac{5}{2}Eq3 \end{matrix} \begin{vmatrix} 5 & -1 & 2 \\ -5 & 25 & 10 \\ -5 & -5 & -5 \end{vmatrix}$

$\begin{matrix} Eq1 \\ \to Eq2+Eq1 \\ Eq3+Eq1 \end{matrix} \begin{vmatrix} 5 & -1 & 2 \\ 0 & 24 & 12 \\ 0 & -6 & -3 \end{vmatrix} \begin{matrix} Eq1 \\ \to Eq2/12 \\ + Eq3/3 \end{matrix} \begin{vmatrix} 5 & -1 & 2 \\ 0 & 2 & 1 \\ 0 & -2 & -1 \end{vmatrix}$

D'où $\begin{matrix} Eq1 \\ Eq2 \\ Eq3+Eq2 \end{matrix} \begin{vmatrix} \boxed{5} & -1 & 2 \\ 0 & \boxed{2} & 1 \\ 0 & 0 & 0 \end{vmatrix}$ $\begin{cases} 5x - y = -2z \\ 2y = -z \end{cases}$

$\begin{cases} 10x - 2y = -4z \\ 2y = -z \end{cases}$ $10x = -5z$, $x = -\frac{1}{2}z$, $y = -\frac{z}{2}$

$E_0 = \{(x,y,z) \in \mathbb{R}^3, (-\frac{1}{2}z, -\frac{1}{2}z, z), z \in \mathbb{R}\}$

$E_0 = \mathrm{Vect}\{(-1,-1,2)\}$.

$\begin{vmatrix} -1 & -1 & 2 \\ -1 & -1 & 2 \\ 2 & 2 & -4 \end{vmatrix}$ $Eq3 = -2\,Eq1$.

$x + y - 2z = 0$.

$\{x = -y + 2z,\ y, z \in \mathbb{R}\}$

$E_6 = \{(x,y,z) \in \mathbb{R}, (-y+2z, y, z), y, z \in \mathbb{R}\}$

$E_6 = \mathrm{Vect}\{(-1,1,0), (2,0,1)\}$.

3. $u_1 = \frac{1}{\sqrt{6}}(-1,-1,2)$ Base O.N. de E_0.

On utilise Gram-Schmidt pour construire une B.O.N. de E_6 :

$v_2 = (-1,1,0)$

$v_3 = (2,0,1) + \alpha(-1,1,0) = (2-\alpha, \alpha, 1)$

$\langle v_3, v_2 \rangle = -2+\alpha+\alpha = -2+2\alpha = 0$; $\alpha = 1$

$v_2 = (-1,1,0)$, $v_3 = (1,1,1)$

D'où B.O.N de E_6 : $u_2 = \frac{1}{\sqrt{2}}(-1,1,0)$; $u_3 = \frac{1}{\sqrt{3}}(1,1,1)$

4. u_1 est orthogonal à u_2 et u_3 comme vect. propres associés à 2 val. propres distinctes ; d'où $\{u_1, u_2, u_3\}$ est une B.O.N. de $\mathbb{R}^3$ formée de vect. propres.

5. $f(v,v') = [x'\ y'\ z']\ A \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$f(v,v') = 5xx' + 5yy' + 2zz' - (xy'+x'y) + 2(xz'+x'z) + 2(yz'+y'z)$

6. $q(v) = f(v,v) = 5x^2 + 5y^2 + 2z^2 - 2xy + 4xz + 4yz$

7. $B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{pmatrix}$

8. $q(v) = 6Y^2 + 6Z^2$

9. $q(u_1) = 0$ et donc f n'est pas un produit scalaire

Exercice 2

1. $\partial_x g(x,y) = 6x^2 + 2y$

$\partial_y g(x,y) = 2x - 2y$

$Hg(x,y) = \begin{bmatrix} 12x & 2 \\ 2 & -2 \end{bmatrix}$

2. Points critiques.

$\begin{cases} 6x^2 + 2y = 0 \\ 2x - 2y = 0 \end{cases} \longrightarrow 6x^2 + 2x = 0.$

$x(3x+1) = 0, \quad x = 0$ et $x = -\frac{1}{3}$.

D'où les pts critiques $(0,0)$ et $(-1/3, -1/3)$.

$Hg(x,y) = \begin{bmatrix} 0 & 2 \\ 2 & -2 \end{bmatrix}$, $p(\lambda) = \det(Hg(0,0) - \lambda I_2) = -\lambda(-2-\lambda) - 4$.

$p(\lambda) = \lambda^2 + 2\lambda - 4 = 0$; $\lambda_1 \lambda_2 = -4$ d'où

$\lambda_1 < 0 < \lambda_2$, Point col en $(0,0)$.

$Hg(-1/3, -1/3) = \begin{bmatrix} -4 & 2 \\ 2 & -2 \end{bmatrix}$; $p(\lambda) = (4+\lambda)(2+\lambda) - 4$

$p(\lambda) = \lambda^2 + 6\lambda + 4$; 2 racines négatives

d'où g a un max. local en $(-1/3, -1/3)$

Exercice 3

1. $\partial_x f(x,y) = \partial_u g(u,v)\partial_x u + \partial_v g(u,v)\partial_x v$

$= y\,\partial_u g(u,v) - \frac{y}{x^2}\partial_v g(u,v)$

$\partial_y f(x,y) = \partial_u g(u,v)\partial_y u + \partial_v g(u,v)\partial_y v$

$= x\,\partial_u g(u,v) + \frac{1}{x}\partial_v g(u,v).$

2. $x\partial_x f(x,y) + y\partial_y f(x,y) = xy\,\partial_u g(u,v) - \frac{y}{x}\partial_v g(u,v)$

$+ xy\,\partial_u g(u,v) + \frac{y}{x}\partial_v g(u,v) = 2$

$2u\,\partial_u g(u,v) = 2$; $u\,\partial_u g(u,v) = 1$

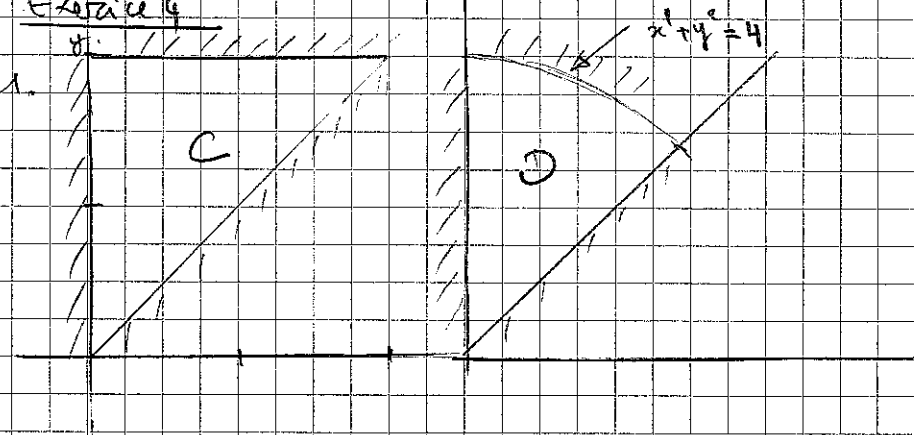
3. $\partial_u\left(g(u,v) - \ln u\right) = 0.$

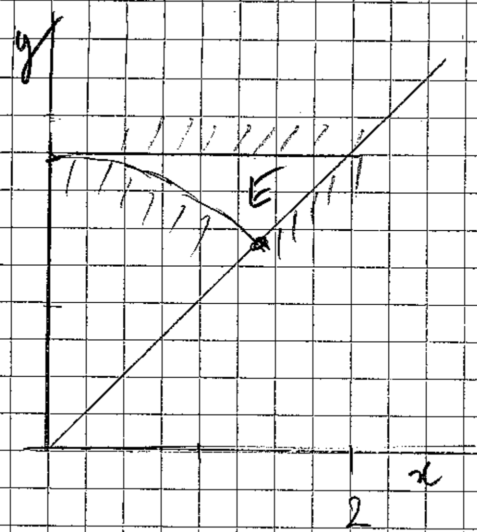
$g(u,v) = \ln u + k(v)$; k est arbitraire

D'où $f(x,y) = \ln x + \ln y + k\left(\frac{y}{x}\right).$

Exercice 4

1.





2. Aire $C = 2\times 2\,\frac{1}{2} = 2$; Aire $D = 4\,\frac{\pi}{8} = \frac{\pi}{2}$

Aire E = Aire C − Aire D $= 2 - \frac{\pi}{2}$

3. $$\int_C (x^2+y^2)\,dx\,dy = \int_0^2 \left(\int_{y=x}^2 (x^2+y^2)\,dy\right) dx$$

$$= \int_0^2 \left(x^2(2-x) + \frac{1}{3}(2^3 - x^3)\right) dx$$

$$= \int_0^2 \left(-\frac{4}{3}x^3 + 2x^2 + \frac{8}{3}\right) dx$$

$$= \left[-\frac{1}{3}x^4\right]_0^2 + \left[\frac{2}{3}x^3\right]_0^2 + \frac{16}{3}$$

$$= -\frac{16}{8} + \frac{2.8}{3} + \frac{2.8}{3} = \frac{16}{3}$$

$$\int_D (x^2+y^2)\,dx\,dy = \int_{r=0}^2 r^3 \int_{\theta=\pi/4}^{\pi/2} d\theta\,dr = \frac{\pi}{4}\,\frac{2^4}{4} = \pi$$

$$\int_E (x^2+y^2)\,dx\,dy = \int_C (x^2+y^2)\,dx\,dy - \int_D (x^2+y^2)\,dx\,dy = \frac{16}{3} - \pi$$