Corrections F4

Exercice 1 1. $\underline{r}(t) = a(t - \sin t, 1 - \cos t)$, $\underline{r}'(t) = a(1 - \cos t, \sin t)$

$\underline{r}'(t) = 2a\sin\frac{t}{2}\left(\sin\frac{t}{2}, \cos\frac{t}{2}\right)$; $\|\underline{r}'(t)\| = 2a\sin\frac{t}{2}$

(car $0 \leq \frac{t}{2} \leq \pi$) ; d'où $l = \int_0^{2\pi} \|\underline{r}'(t)\| dt = 8a$.

2. $\underline{T}(t)$ existe si $\|\underline{r}'(t)\| \neq 0$ d'où $t \neq 0, 2\pi$.

$\underline{T}(t) = \underline{r}'(t) / \|\underline{r}'(t)\| = \left(\sin\frac{t}{2}, \cos\frac{t}{2}\right)$

$\underline{N}(t)$ existe si $\|\underline{T}'(t)\| \neq 0$; $\underline{T}'(t) = \frac{1}{2}\left(\cos\frac{t}{2}, -\sin\frac{t}{2}\right)$

d'où $\underline{N}(t)$ existe si $t \neq 0, 2\pi$; $\underline{N}(t) = \left(\cos\frac{t}{2}, -\sin\frac{t}{2}\right)$

$\underline{r}''(t) = a(\sin t, \cos t)$; $|\underline{r}''(t) \times \underline{r}'(t)| = 2a^2 \sin\frac{t}{2} \det\begin{bmatrix} \sin t & \sin\frac{t}{2} \\ \cos t & \cos\frac{t}{2}\end{bmatrix}$

$|\underline{r}''(t) \times \underline{r}'(t)| = 2a^2\sin^2\frac{t}{2}$; $K(t) = 1/4a\sin\frac{t}{2}$; $C_t = r_t + \frac{1}{K(t)}\underline{N}(t)$

$C_t = a(t - \sin t, 1 - \cos t) + 4a\sin\frac{t}{2}\left(\cos\frac{t}{2}, -\sin\frac{t}{2}\right)$

$C_t = a\left(t + \sin t, -2\sin^2\frac{t}{2}\right)$

3. $s(t) - s(0) = \int_0^t \|\underline{r}'(\sigma)\| d\sigma = \int_0^t 2a\sin\frac{\sigma}{2} d\sigma = 4a\left(1 - \cos\frac{t}{2}\right) = 8a\sin^2\frac{t}{4}$

D'où en prenant $s(0) = 0$, $t = 4\arcsin\sqrt{\frac{s}{8a}}$ car $0 \leq \frac{t}{4} \leq \frac{\pi}{2}$.

4. $\underline{T}(s) = \left(\sin\left(2\arcsin\sqrt{\frac{s}{8a}}\right), \cos\left(2\arcsin\sqrt{\frac{s}{8a}}\right)\right)$

$\underline{N}(s) = \left(\cos\left(2\arcsin\sqrt{\frac{s}{8a}}\right), -\sin\left(2\arcsin\sqrt{\frac{s}{8a}}\right)\right)$

$K(s) = 1/4a\sin\left(2\arcsin\sqrt{\frac{s}{8a}}\right) = 1/8a\sqrt{1 - \frac{s}{8a}}\sqrt{\frac{s}{8a}}$

$\underline{T}'(s) = \left(\cos\left(2\arcsin\sqrt{\frac{s}{8a}}\right), -\sin\left(2\arcsin\sqrt{\frac{s}{8a}}\right)\right) \Big/ \left(8a\sqrt{1 - \frac{s}{8a}}\sqrt{\frac{s}{8a}}\right)$

On a bien $\underline{T}'(s) = K(s)\,\underline{N}(s)$.

Exercice 2 1. $\underline{T}(s) = (\cos\theta(s), \sin\theta(s))$, $\underline{T}'(s) = \theta'(s)\left(-\sin\theta(s), \cos\theta(s)\right)$

$\|\underline{T}'(s)\| = \theta(s)$

2. $\theta'(s) = \frac{1}{R}$, $\theta(s) = \theta_0 + \frac{s - s_0}{R}$; $\underline{T}(s) = (x'(s), y'(s)) = \left(\cos\left(\theta_0 + \frac{s-s_0}{R}\right), \sin\left(\theta_0 + \frac{s-s_0}{R}\right)\right)$

$x(s) = x_0 + R\sin\left(\theta_0 + \frac{s - s_0}{R}\right)$, $y(s) = y_0 - R\cos\left(\theta_0 + \frac{s - s_0}{R}\right)$

$(x - x_0)^2 + (y - y_0)^2 = R^2$: éq. d'un arc de cercle.

3. $\theta'(s) = 0$; $\theta(s) = \theta_0$; $x'(s) = \cos\theta_0$, $y'(s) = \sin\theta_0$

$x(s) = x_0 + s\cos\theta_0$, $y(s) = y_0 + s\sin\theta_0$; Eq. pr. d'une droite

Exercice 3 1. Equation du plan tangent

$$\det\begin{bmatrix} x - r\cos\theta & \cos\theta & -r\sin\theta \\ y - r\sin\theta & \sin\theta & r\cos\theta \\ z - f(r,\theta) & \partial_r f & \partial_\theta f \end{bmatrix} = \begin{aligned} & x(\sin\theta\,\partial_\theta f - r\partial_r f\cos\theta) \\ & - y(\cos\theta\,\partial_\theta f + r\partial_r f\sin\theta) \\ & + z r = r f - r^2\partial_r f \end{aligned}$$

2. $x = y = 0$, $z = \text{cste}$, $\theta = \text{cste}$; $r\partial_r f - f = -z$

Sol Gle : $f(r,\theta) = r\,v(\theta) + z$; v fct qcq.

3. $x = r\cos\theta$, $y = r\sin\theta$, $z = \theta$: surface engendrée par la $\frac{1}{2}$ droite parallèle au plan xy, confondue avec le $\frac{1}{2}$ axe $x \geq 0$ pour $\theta = 0$ et qui pour tout θ a une ordonnée $z = \theta$ (escalier en colimaçon). L'intersection de cette surface avec le cylindre $r = r_0$ (r, θ, z coordonnées cylindriques) est l'hélice de pas 2π qui coupe le $\frac{1}{2}$ axe des $x \geq 0$ au point $x = r_0$.

4. $x = r_0\cos\theta$, $y = r_0\sin\theta$, $z = \theta$.

$\underline{r}'(\theta) = (-r_0\sin\theta, r_0\cos\theta, 1)$; $\|\underline{r}'(\theta)\| = \sqrt{r_0^2 + 1}$

H. $\underline{r}(\theta) = (r_0\cos\theta, r_0\sin\theta, \theta)$; $\underline{r}'(\theta) = (-r_0\sin\theta, r_0\cos\theta, 1)$

$\|\underline{r}'(\theta)\| = \sqrt{r_0^2+1}$; $\underline{T}(\theta) = \underline{r}'(\theta)/\sqrt{r_0^2+1}$

$ds = \|r'(\theta)\| d\theta$; $s = s_0 + \sqrt{r_0^2+1}\,\theta$

$\underline{T}'(\theta) = \underline{T}'(s)\,\partial_\theta s = \sqrt{r_0^2+1}\;\underline{T}'(s)$

D'où $\underline{T}'(s) = \dfrac{1}{\sqrt{r_0^2+1}}\underline{T}'(\theta) = \dfrac{r_0}{r_0^2+1}(-\cos\theta, -\sin\theta, 0)$

$K(\theta) = \|\underline{T}'(s)\| = r_0/(r_0^2+1)$; $\underline{N}(\theta) = (-\cos\theta, -\sin\theta, 0)$

$\underline{B}(\theta) = \underline{T}(\theta)\times\underline{N}(\theta) = \dfrac{1}{\sqrt{r_0^2+1}}(\sin\theta, -\cos\theta, r_0)$

$\underline{B}'(\theta) = \underline{B}'(s)\,\partial_\theta s = \dfrac{1}{\sqrt{r_0^2+1}}(\cos\theta, \sin\theta, 0)$

$\underline{B}'(s) = \dfrac{1}{r_0^2+1}(\cos\theta, \sin\theta, 0) = -\omega(\theta)\,\underline{N}(\theta)$

D'où $\omega(\theta) = 1/(r_0^2+1)$.

Exercice 4 $\displaystyle\int_{\Gamma_g}\{\cdots\} = \int_{\overrightarrow{AB}}\{\cdots\} + \int_{\overrightarrow{BC}}\{\cdots\} + \int_{\overrightarrow{CA}}\{\cdots\}$

$\displaystyle\int_{\overrightarrow{AB}}\{\cdots\} = \int_0^1 \left(x^2+y^2+(x^2-y^2)y'(x)\right)\Big|_{y(x)=0} dx = \frac{1}{3}$

$\displaystyle\int_{\overrightarrow{BC}}\{\cdots\} = -\int_0^1 \left((x^2+y^2)+(x^2-y^2)y'(x)\right)\Big|_{y(x)=1-x} dx = -\frac{2}{3}$

$\displaystyle\int_{\overrightarrow{CA}}\{\cdots\} = -\int_0^1 \left((x^2+y^2)x'(y)+(x^2-y^2)\right)\Big|_{x(y)=0} dy = \frac{1}{3}$

$\displaystyle\int_\Gamma (x^2+y^2)\,dx + (x^2-y^2)\,dy = 0$

Exercice 5

1. $x = ar\cos\theta$, $y = br\sin\theta$, $0\le r\le 1$; $0\le\theta\le\pi/2$

$\displaystyle\int_\Omega (2x^2-y)\,dx\,dy = \int_{r=0}^1\int_{\theta=0}^{\pi/2}\left(2a^3r^3\cos^3\theta - br\sin\theta\right)abr\,dr\,d\theta$

$\displaystyle = \frac{ab}{3}\left(\frac{4a^3}{5} - b\right)$

2. $\displaystyle\int_\Omega (2x^2-y)\,dx\,dy = \frac{1}{2}\int_{\Gamma_g} y^2\,dx + x^4\,dy$

$\displaystyle = -\frac{1}{2}\int_0^{\pi/2}\left(y^2x'(\theta) + x^4y'(\theta)\right)\Big|_{\substack{x(\theta)=a\cos\theta\\ y(\theta)=b\sin\theta}} d\theta$;

$\displaystyle = \frac{1}{2}\int_0^{\pi/2}\left(-ab^2\sin^3\theta + a^5b\cos^5\theta\right)d\theta = \frac{ab}{3}\left(\frac{4a^3}{5} - b\right)$

Exercice 6

Paramétrisation de la surface : $x = r\cos\theta$, $y = r\sin\theta$, $z = r^2/2p$

$\underline{\Pi}'_r = (\cos\theta, \sin\theta, r/p)$, $\underline{\Pi}'_\theta = (-r\sin\theta, r\cos\theta, 0)$

$dS = r\sqrt{1+\dfrac{r^2}{p^2}}\,dr\,d\theta$

$\displaystyle S = \int_0^{\sqrt{2pA}}\left\{r\sqrt{1+\frac{r^2}{p^2}}\int_0^{2\pi}d\theta\right\}dr = \frac{2\pi}{3}p^2\left(\left(1+\frac{2A}{p}\right)^{3/2} - 1\right)$