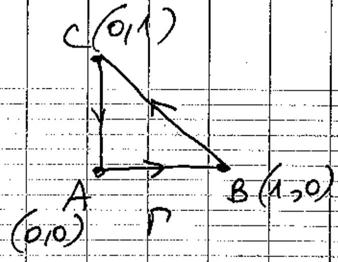
Feuille de TD 4

Exercice 6

$$\int_{\Gamma\circlearrowleft} (x^2+y^2)\,dx + (x^2-y^2)\,dy$$



$$= \int_{\overrightarrow{AB}} \{\dots\} + \int_{\overrightarrow{BC}} \{\dots\} + \int_{\overrightarrow{CA}} \{\dots\}$$

- $\displaystyle\int_{\overrightarrow{AB}} (x^2+y^2)\,dx + (x^2-y^2)\,dy = \int_0^1 \{(x^2+0)\cdot 1 + (x^2-0)\cdot 0\}\,dx = \int_0^1 x^2\,dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}$

paramétrisation de $\overrightarrow{AB}$: $[0,1] \ni x \longmapsto \begin{bmatrix} x \\ 0 \end{bmatrix}$

- Param. de $\overrightarrow{BC}$: $[0,1] \ni t \longmapsto \begin{bmatrix} (1-t)\,x_B + t\,x_C \\ (1-t)\,y_B + t\,y_C \end{bmatrix} = \begin{bmatrix} 1-t \\ t \end{bmatrix}$

$$\Rightarrow \int_{\overrightarrow{BC}} (x^2+y^2)\,dx + (x^2-y^2)\,dy = \int_0^1 \{[(1-t)^2+t^2]\cdot(-1) + [(1-t)^2-t^2]\cdot 1\}\,dt$$

$$= -\int_0^1 2t^2\,dt = \left[-\frac{2t^3}{3}\right]_0^1 = -\frac{2}{3}$$

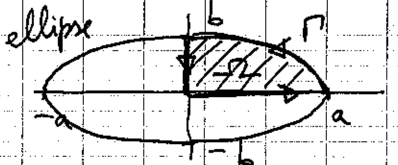
- Param. de $\overrightarrow{CA}$: $[0,1] \ni y \longmapsto \begin{bmatrix} 0 \\ 1-y \end{bmatrix}$

$$\int_{\overrightarrow{CA}} (x^2+y^2)\,dx + (x^2-y^2)\,dy = \int_0^1 \{[0+(1-y)^2]\cdot 0 + [0-(1-y)^2](-1)\}\,dy$$

$$= \int_0^1 (1-y)^2\,dy = \left[-\frac{(1-y)^3}{3}\right]_0^1 = \frac{1}{3}$$

$$\hookrightarrow \int_{\Gamma\circlearrowleft} (x^2+y^2)\,dx + (x^2-y^2)\,dy = \frac{1}{3} - \frac{2}{3} + \frac{1}{3} = \boxed{0}$$

Exercice 7 $\Omega = \left\{ (x,y) \in (\mathbb{R}^+)^2 : \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1 \right\}$ $a, b > 0$



1) $\int_\Omega (2x^3 - y)\,dx\,dy$

$\Psi : \begin{cases} x = a\,r\cos\theta \\ y = b\,r\sin\theta \end{cases}$, $r \in [0,1]$, $\theta \in \left[0, \frac{\pi}{2}\right]$

$$\int_\Omega (2x^3 - y)\,dx\,dy = \int_{r=0}^{1}\int_{\theta=0}^{\pi/2} \left(2a^3r^3\cos^3\theta - br\sin\theta\right) \cdot abr\,d\theta\,dr$$

$$\left[\det(\Psi'(r,\theta)) = \begin{vmatrix} a\cos\theta & -ar\sin\theta \\ b\sin\theta & br\cos\theta \end{vmatrix} = abr \right]$$

$$= 2a^4 b \left(\int_0^1 r^4\,dr\right) \cdot \left(\int_0^{\pi/2} \cos^3\theta\,d\theta\right) - ab^2 \left(\int_0^1 r^2\,dr\right)\left(\int_0^{\pi/2} \sin\theta\,d\theta\right)$$

$$= 2a^4 b \cdot \frac{1}{5} \cdot \frac{2}{3} - ab^2 \cdot \frac{1}{3} \cdot \underbrace{\left[-\cos\theta\right]_0^{\pi/2}}_{=1}$$

$$= \boxed{\frac{ab}{3}\left(\frac{4a^3}{5} - b\right)}$$

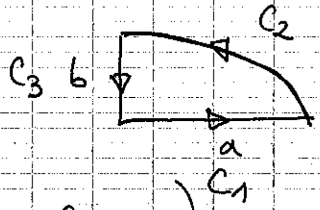
On a ... $\int_0^{\pi/2} \cos^3\theta\,d\theta = \frac{2}{3}$!

$$\int_0^{\pi/2} \cos^3\theta\,d\theta = \int_0^{\pi/2} \cos^2 x \cdot (\sin x)'\,dx = \int_0^{\pi/2} (1 - \sin^2 x) \cdot (\sin x)'\,dx$$

$$\underset{\sin x = t}{=} \int_0^1 (1 - t^2)\,dt = \left[t - \frac{t^3}{3}\right]_0^1 = \frac{2}{3}.$$

2) $$\int_\Omega (2x^3 - y)\,dx\,dy = \int_\Omega (\partial_x g - \partial_y f)\,dx\,dy \overset{\text{Green-Riemann}}{=} \oint_{\Gamma \circlearrowleft} f(x,y)\,dx + g(x,y)\,dy =$$

$$\left[\begin{array}{ll} \partial_x g(x,y) = 2x^3 & ; \; g(x,y) = \frac{x^4}{2} \\ \partial_y f(x,y) = y & ; \; f(x,y) = \frac{y^2}{2} \end{array} \right]$$



$$= \frac{1}{2} \oint_{\Gamma \circlearrowleft} y^2\,dx + x^4\,dy = \frac{1}{2}\left(\int_{C_1} \{\ldots\} + \int_{C_2} \{\ldots\} + \int_{C_3} \{\ldots\} \right)$$

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$$\int_{C_1} y^2 dx + x^4 dy = \int_0^a (0 \cdot 1 + x^4 \cdot 0)\, dx = 0$$

param. $[0,a] \ni x \longmapsto \begin{bmatrix} x \\ 0 \end{bmatrix}$

$$\int_{C_3} y^2 dx + x^4 dy = 0$$

$$\int_{C_2} y^2 dx + x^4 dy = \int_0^{\frac{\pi}{2}} \left\{ b^2 \sin^2\theta \cdot x'(\theta) + a^4 \cos^4\theta \cdot y'(\theta) \right\} d\theta =$$

$$\left[\text{param. } \begin{cases} x(\theta) = a\cos\theta \\ y(\theta) = b\sin\theta \end{cases}, \quad \theta \in \left[0, \frac{\pi}{2}\right] \right]$$

$$= \int_0^{\pi/2} \left(b^2 \sin^2\theta \cdot (-a\sin\theta) + a^4 \cos^4\theta \cdot b\cos\theta \right) d\theta$$

$$= -ab^2 \int_0^{\pi/2} \sin^3\theta\, d\theta + a^4 b \int_0^{\pi/2} \cos^5\theta\, d\theta$$

$$\overset{!}{=} -\frac{2}{3}ab^2 + \frac{8}{15}a^4 b = \frac{2ab}{3}\left(\frac{4a^3}{5} - b\right),$$

car $$\int_0^{\pi/2} \sin^3\theta\, d\theta = -\int_0^{\pi/2} (1-\cos^2\theta) \cdot (\cos\theta)'\, d\theta$$

$$\overset{[t=\cos\theta]}{=} -\int_1^0 (1-t^2)\, dt = \int_0^1 (1-t^2)\, dt = \left[t - \frac{t^3}{3}\right]_0^1 = \frac{2}{3}.$$

$$\int_0^{\pi/2} \cos^5\theta\, d\theta = \int_0^{\pi/2} (1-\sin^2\theta)^2 (\sin\theta)'\, d\theta$$

$$\overset{t=\sin\theta}{=} \int_0^1 (1-t^2)^2\, dt = \int_0^1 (1+t^4-2t^2)\, dt$$

$$= \left[t + \frac{t^5}{5} - \frac{2t^3}{3}\right]_0^1 = 1 + \frac{1}{5} - \frac{2}{3} = \frac{8}{15}$$

On obtient donc $$\int_{\Gamma \circlearrowleft} y^2 dx + x^4 dy = \frac{2ab}{3}\left(\frac{4a^3}{5} - b\right),$$

d'où $$\int_{\Omega} (2x^3 - y)\, dx\, dy = \frac{1}{2} \int_{\Gamma \circlearrowleft} y^2 dx + x^4 dy$$

$$= \frac{ab}{3}\left(\frac{4a^3}{5} - b\right).$$

Exercice 8 $0 \leq z \leq A$, $x^2 + y^2 = 2pz$, $p > 0$.

Param. de la surface

$$M(r,\theta) = \begin{bmatrix} x(r,\theta) \\ y(r,\theta) \\ z(r,\theta) \end{bmatrix} , \quad \begin{cases} x(r,\theta) = r\cos\theta \\ y(r,\theta) = r\sin\theta \\ z(r,\theta) = \dfrac{r^2}{2p} \end{cases}$$

$$0 \leq z \leq A \Rightarrow 0 \leq r \leq \sqrt{2pA}$$

$$\theta \in [0, 2\pi] .$$

On doit calculer

$$\int_S dS = \int_0^{2\pi} \int_0^{\sqrt{2pA}} \| M'_r(r,\theta) \times M'_\theta(r,\theta) \| \, dr \, d\theta$$

$$M'_r(r,\theta) \times M'_\theta(r,\theta) = \begin{pmatrix} \cos\theta \\ \sin\theta \\ \frac{r}{p} \end{pmatrix} \times \begin{pmatrix} -r\sin\theta \\ r\cos\theta \\ 0 \end{pmatrix} =$$

$$= \begin{pmatrix} -\frac{r^2}{p}\cos\theta \\ -\frac{r^2}{p}\sin\theta \\ r \end{pmatrix}$$

$$\Rightarrow \| M'_r(r,\theta) \times M'_\theta(r,\theta) \| = \sqrt{\frac{r^4}{p^2} + r^2} = r\sqrt{\frac{r^2}{p^2} + 1} .$$

$$\hookrightarrow \text{Aire}(S) = \int_S dS = 2\pi \cdot \int_0^{\sqrt{2pA}} r\left(\sqrt{\frac{r^2}{p^2} + 1}\right) dr$$

$$\left[\frac{r^2}{p^2} = t\right] = \pi p^2 \cdot \int_0^{\frac{2A}{p}} \sqrt{t+1} \, dt = \pi p^2 \cdot \frac{2}{3}\left[(t+1)^{3/2}\right]_0^{\frac{2A}{p}}$$

$$= \boxed{\frac{2p^2\pi}{3}\left[\left(\frac{2A}{p} + 1\right)^{3/2} - 1\right]}$$