Correction Exercice 1, Feuille de TD 4

Exercice 1 (r,θ) sont les coordonnées polaires.

2) $r(\theta) = \operatorname{th}\left(\frac{\theta}{2}\right) \rightsquigarrow \begin{cases} x(\theta) = \operatorname{th}\left(\frac{\theta}{2}\right)\cos\theta \\ y(\theta) = \operatorname{th}\left(\frac{\theta}{2}\right)\sin\theta \end{cases}$

$$x'(\theta) = \frac{1}{2\operatorname{ch}^2\left(\frac{\theta}{2}\right)}\cos\theta - \operatorname{th}\left(\frac{\theta}{2}\right)\sin\theta = \frac{1}{2}\left(1-\operatorname{th}^2\left(\frac{\theta}{2}\right)\right)\cos\theta - \operatorname{th}\left(\frac{\theta}{2}\right)\sin\theta$$

$$y'(\theta) = \frac{1}{2\operatorname{ch}^2\left(\frac{\theta}{2}\right)}\sin\theta + \operatorname{th}\left(\frac{\theta}{2}\right)\cos\theta$$

$$= \frac{1}{2}\left(1-\operatorname{th}^2\left(\frac{\theta}{2}\right)\right)\sin\theta + \operatorname{th}\left(\frac{\theta}{2}\right)\cos\theta.$$

On a utilisé le fait que $\begin{cases} \operatorname{sh}' = \operatorname{ch} \\ \operatorname{ch}' = \operatorname{sh} \\ \operatorname{ch}^2 - \operatorname{sh}^2 = 1 \end{cases}$

$$\rightsquigarrow \left\| \begin{bmatrix} x'(\theta) \\ y'(\theta) \end{bmatrix} \right\| = \sqrt{(x'(\theta))^2 + (y'(\theta))^2} =$$

$$= \Big\{ \frac{1}{4}\left(1-\operatorname{th}^2\left(\frac{\theta}{2}\right)\right)^2\cos^2\theta + \operatorname{th}^2\left(\frac{\theta}{2}\right)\sin^2\theta - \cancel{\left(1-\operatorname{th}^2\left(\frac{\theta}{2}\right)\right)\operatorname{th}\left(\frac{\theta}{2}\right)\sin\theta\cos\theta}$$

$$+ \frac{1}{4}\left(1-\operatorname{th}^2\left(\frac{\theta}{2}\right)\right)^2\sin^2\theta + \operatorname{th}^2\left(\frac{\theta}{2}\right)\cos^2\theta + \cancel{\left(1-\operatorname{th}^2\left(\frac{\theta}{2}\right)\right)\operatorname{th}\left(\frac{\theta}{2}\right)\sin\theta\cos\theta} \Big\}^{1/2}$$

$$= \sqrt{\frac{1}{4}\left(1-\operatorname{th}^2\left(\frac{\theta}{2}\right)\right)^2 + \operatorname{th}^2\left(\frac{\theta}{2}\right)}$$

$$= \frac{1}{2}\left(1+\operatorname{th}^2\left(\frac{\theta}{2}\right)\right).$$

$$\rightsquigarrow \ell(M_0M_\theta) = \int_0^\theta \frac{1}{2}\left(1+\operatorname{th}^2\left(\frac{t}{2}\right)\right)dt = \int_0^\theta \Big[1 - \underbrace{\frac{1}{2}\left(1-\operatorname{th}^2\frac{t}{2}\right)}_{= \left(\operatorname{th}\left(\frac{t}{2}\right)\right)'}\Big]dt$$

$$= \left[t - \operatorname{th}\left(\frac{t}{2}\right)\right]_0^\theta$$

$$= \boxed{\theta - \operatorname{th}\left(\frac{\theta}{2}\right)}$$