Correction CC1 de MAS 1 et 2, 2013-2014

Exc. 1

1) a) $p_{00} = 1$, et $p_{0j} = 0$, $\forall j \neq 0$
$p_{NN} = 1$, et $p_{Nj} = 0$, $\forall j \neq N$.

b) $p_{i,i-1} = \mathbb{P}(X_{n+1} = i-1 \mid X_n = i) = \mathbb{P}\left(\begin{matrix}\text{le 1er individu}\\ \text{choisi est}\\ \text{de type A}\end{matrix} \text{ et } \begin{matrix}\text{le 2ème individu}\\ \text{est de type}\\ a\end{matrix}\right)$

$$= i \cdot k \cdot \frac{N-i}{N-1},$$

avec k la cte. de proportionnalité qui vérifie

$$\underbrace{i \cdot k}_{\text{la proba que le 1er individu choisi soit A}} + \underbrace{(N-i) r \cdot k}_{\text{la proba que le 1er individu choisi soit a}} = 1 \Rightarrow k = \frac{1}{i + (N-i) r}$$

$$\Rightarrow p_{i,i-1} = \frac{i(N-i)}{(N-1)(i+(N-i)r)}.$$

$$p_{i,i+1} = \mathbb{P}(X_{n+1} = i+1 \mid X_n = i) = \mathbb{P}\left(\begin{matrix}\text{1er individu}\\ a\end{matrix} \text{ et } \begin{matrix}\text{2ème}\\ \text{individu A}\end{matrix}\right)$$

$$= (N-i) r \cdot k \cdot \frac{i}{N-1} = \frac{i(N-i) r}{(N-1)(i+(N-i)r)} = r\, p_{i,i-1}.$$

$$p_{i,i} = \mathbb{P}(X_{n+1} = i \mid X_n = i) = \mathbb{P}((a,a)) + \mathbb{P}((A,A))$$

$$= (N-i) \cdot r \cdot k \cdot \frac{N-i-1}{N-1} + i \cdot k \cdot \frac{i-1}{N-1}.$$

$$= \frac{(N-i)(N-i-1) \cdot r + i(i-1)}{(N-1)(i+(N-i)r)}.$$

$$p_{i,i-1} + p_{ii} + p_{i,i+1} = \frac{i(N-i) + i(N-i) r + (N-i)(N-i-1) r + i(i-1)}{(N-1)(i+(N-i)r)}$$

$$= \frac{iN - \cancel{i^2} + (N-i) r \cdot (N-1) + \cancel{i^2} - i}{(N-1)(i+(N-i)r)} = \underline{1}.$$

c) $\mathbb{E}(X_{n+1} \mid X_n = i) = (i-1) p_{i,i-1} + i\, p_{i,i} + (i+1) p_{i,i+1}$

$= \cancel{(i-1)} p_{i,i-1} + i(1 - \cancel{p_{i,i-1}} - \cancel{p_{i,i+1}}) + \cancel{i\, p_{i,i+1}} + p_{i,i+1}$

$= \underset{\substack{\| \\ r\, p_{i,i-1}}}{p_{i,i+1}} - p_{i,i-1} + i = \underbrace{(r-1)}_{\geq 0} \underbrace{p_{i,i-1}}_{\geq 0} + i \geq i \quad , \forall i$

$\Rightarrow \mathbb{E}(X_{n+1} \mid X_n) \geq X_n \quad , \forall n.$

En plus, $\mathbb{E}(X_{n+1} \mid \mathcal{F}_n) = \mathbb{E}(X_{n+1} \mid X_n)$ car $(X_n)_n$ C.M.

$\Rightarrow \mathbb{E}(X_{n+1} \mid \mathcal{F}_n) \geq X_n, \forall n \Rightarrow (X_n)_n$ <u>sous-martingale</u>.

d) $(X_n)_n$ sous-martingale bornée $\Rightarrow$ converge p.s. vers une v.a. X_∞ quand $n \to \infty$.

Comme 0 et N sont des états absorbants, $X_\infty \in \{0, N\}$.

e) $f_0 = \mathbb{P}(X_\infty = 0 \mid X_0 = 0) = 1$

$f_N = \mathbb{P}(X_\infty = 0 \mid X_0 = N) = 0.$

$f_i = \mathbb{P}(X_\infty = 0 \mid X_0 = i) = \mathbb{P}(X_\infty = 0 \mid X_1 = i-1, X_0 = i) \times \mathbb{P}(X_1 = i-1 \mid X_0 = i)$
$+ \mathbb{P}(X_\infty = 0 \mid X_1 = i, X_0 = i) \times \mathbb{P}(X_1 = i \mid X_0 = i)$
$+ \mathbb{P}(X_\infty = 0 \mid X_1 = i+1, X_0 = i) \times \mathbb{P}(X_1 = i+1 \mid X_0 = i)$

propriété de Markov
$= f_{i-1} \times p_{i,i-1} + f_i \times \underbrace{p_{i,i}}_{\substack{\| \\ 1 - p_{i,i-1} - p_{i,i+1}}} + f_{i+1} \times p_{i,i+1}.$

$\Rightarrow$

$\cancel{f_i} = f_{i-1} \times p_{i,i-1} + f_i(\cancel{1} - p_{i,i-1} - p_{i,i+1}) + f_{i+1} p_{i,i+1}$

$\Rightarrow (f_{i+1} - f_i) \underbrace{p_{i,i+1}}_{r\, p_{i,i-1}} = (f_i - f_{i-1}) p_{i,i-1}$

Comme $p_{i,i-1} > 0$ pour $i \in \{1, \dots, N-1\} \Rightarrow f_{i+1} - f_i = r^{-1} \times (f_i - f_{i-1})$.

f) $(f_{i+1} - f_i)_i$ forment une suite géométrique de raison r^{-1}

$$\Rightarrow \quad f_{i+1} - f_i = r^{-i} (f_1 - f_0)$$

$$\Rightarrow \quad f_{i+1} = f_i + r^{-i} (f_1 - f_0)$$

$$\Rightarrow \quad f_{i+1} = \underbrace{f_0}_{=1} + (f_1 - f_0) \cdot \underbrace{\sum_{k=0}^{i} r^{-k}}_{\frac{1 - r^{-(i+1)}}{1 - r^{-1}}}$$

$$\Rightarrow \quad f_N = 1 + \frac{1 - r^{-N}}{1 - r^{-1}} (f_1 - \underbrace{f_0}_{=1}).$$ Mais on a $f_N = 0$,

donc on obtient que $\quad f_1 - f_0 = - \dfrac{1 - r^{-1}}{1 - r^{-N}}$.

$$\Rightarrow \quad f_i = 1 - \frac{\cancel{1 - r^{-1}}}{1 - r^{-N}} \times \frac{1 - r^{-i}}{\cancel{1 - r^{-1}}} = \underbrace{\frac{r^{-i} - r^{-N}}{1 - r^{-N}}}.$$

g) $\mathbb{E}(X_\infty \mid X_0 = i) = 0 \times f_i + N \times (1 - f_i) = N \times \dfrac{1 - r^{-i}}{1 - r^{-N}}$.

$$\mathbb{E}(X_\infty) = \sum_{i=0}^{N} \mathbb{E}(X_\infty \mid X_0 = i) \times \mathbb{P}(X_0 = i) = \frac{N}{1 - r^{-N}} \times \sum_{i=0}^{N} (1 - r^{-i}) \times C_N^i \frac{1}{2^N}$$

$$= \frac{N}{2^N (1 - r^{-N})} \times \left(\sum_{i=0}^{N} C_N^i - \sum_{i=0}^{N} C_N^i \frac{1}{r^i} \right)$$

$$= \frac{N}{2^N (1 - r^{-N})} \left(2^N - \left(1 + \frac{1}{r}\right)^N \right) = \underbrace{\frac{N}{1 - r^{-N}} \left(1 - \left(\frac{1+r}{2r}\right)^N \right)}.$$

2) $T'_n \sim$ Géométrique (p_n),

avec $p_n = \mathbb{P}($ 2 individus parmi les n ont le même parent $)$

$$= \frac{C_n^2}{C_N^2}.$$

b) $T'_{PRAC} = T'_n + T'_{n-1} + \dots + T'_2$, avec (T'_k) indép.

et $T'_k \sim \text{Géom}\left(\frac{C_k^2}{C_N^2}\right)$.

$$\mathbb{E}(T_{PRAC}) = \sum_{k=2}^{n} \mathbb{E}(T'_k) = \sum_{k=2}^{n} \frac{C_N^2}{C_k^2} = 2C_N^2 \times \sum_{k=2}^{n} \frac{1}{k(k-1)}$$

$$= 2C_N^2 \times \sum_{k=2}^{n}\left(\frac{1}{k-1} - \frac{1}{k}\right) = \underline{2C_N^2 \times \left(1 - \frac{1}{n}\right)}.$$

Exc. 2

1) $X_n = X_{n-1} + U_n$, avec U_n indép. de X_{n-1}

$\Rightarrow (X_n)_n$ est une C.M.

de probabilités de transition

$$P(x,x') = \mathbb{P}(X_n = x' \mid X_{n-1} = x) = \begin{cases} \underbrace{\tfrac{1}{2}}_{=\mathbb{P}(U_n=1)}, & \text{pour } x' = x+1 \\ \underbrace{\tfrac{1}{2}}_{=\mathbb{P}(U_n=-1)}, & \text{pour } x' = x-1 \end{cases}$$

et loi initiale $\nu = \mathcal{L}(U_1)$: $\nu(\{1\}) = \nu(\{-1\}) = \frac{1}{2}$.

Conditionnellement aux $(X_n)_n$, les $(Y_n)_n$ sont indépendants, car les $(\varepsilon_n)_n$ le sont, et la loi de Y_n dépend seulement de X_n.

Donc $(X_n, Y_n)_n$ est une CM cachée.

Les probabilités d'observation

$$Q(x,y) = \mathbb{P}(Y_n = y \mid X_n = x) = \mathbb{P}(X_n + \varepsilon_n = y \mid X_n = x)$$

$$= \mathbb{P}(x + \varepsilon_n = y \mid X_n = x) = \mathbb{P}(\varepsilon_n = y - x) \quad \text{car } \varepsilon_n, X_n \text{ indép.}$$

$$= \begin{cases} p, & \text{si } x = y \quad (\text{car } \mathbb{P}(\varepsilon_n = 0)) \\ \frac{1-p}{2}, & \text{si } y = x-1 \quad (\text{car } \mathbb{P}(\varepsilon_n = -1)) \\ \frac{1-p}{2}, & \text{si } y = x+1 \quad (\text{car } \mathbb{P}(\varepsilon_n = +1)). \end{cases}$$

2) $P_{n+1}(x ; y_{1:n}) = \mathbb{P}(X_{n+1} = x \mid Y_{1:n} = y_{1:n})$

$= \mathbb{P}(X_{n+1} = x, X_n = x-1 \mid Y_{1:n} = y_{1:n})$

$+ \mathbb{P}(X_{n+1} = x, X_n = x+1 \mid Y_{1:n} = y_{1:n})$

$= \mathbb{P}(X_n = x-1 \mid Y_{1:n} = y_{1:n}) \times \overbrace{\mathbb{P}(X_{n+1} = x \mid X_n = x-1, \cancel{Y_{1:n} = y_{1:n}})}^{\frac{1}{2}}$

$+ \mathbb{P}(X_n = x+1 \mid Y_{1:n} = y_{1:n}) \times \underbrace{\mathbb{P}(X_{n+1} = x \mid X_n = x+1, \cancel{Y_{1:n} = y_{1:n}})}_{\frac{1}{2}}$

$= \frac{1}{2} F_n(x-1 ; y_{1:n}) + \frac{1}{2} F_n(x+1 ; y_{1:n})$.

3) $F_n(x ; y_{1:n}) = \mathbb{P}(X_n = x \mid Y_{1:n} = y_{1:n}) = \dfrac{\mathbb{P}(X_n = x, Y_n = y_n \mid Y_{1:n-1} = y_{1:n-1})}{\mathbb{P}(Y_n = y_n \mid Y_{1:n-1} = y_{1:n-1})}$

$$= \frac{\mathbb{P}(X_n = x \mid Y_{1:n-1} = y_{1:n-1}) \times \mathbb{P}(Y_n = y_n \mid X_n = x, \cancel{Y_{1:n-1} = y_{1:n-1}})}{\sum_z \mathbb{P}(X_n = z \mid Y_{1:n-1} = y_{1:n-1}) \times \mathbb{P}(Y_n = y_n \mid X_n = z, \cancel{Y_{1:n-1} = y_{1:n-1}})}$$

$$= \frac{P_n(x ; y_{1:n-1}) \times Q(x, y_n)}{\sum_{z=y_n-1}^{y_n+1} P_n(z ; y_{1:n-1}) \times Q(z, y_n)}, \quad \text{si} \quad x \in \{y_n - 1, y_n, y_n + 1\},$$

car $Q(x,y) = 0$ pour $|x-y| > 1$.

4) $$F_n(y_n ; y_{1:n}) = \frac{P_n(y_n ; y_{1:n-1}) \times Q(y_n, y_n)}{\sum_{z=y_n-1}^{y_n+1} P_n(z ; y_{1:n-1}) \times Q(z, y_n)}$$

$$= \frac{p\, P_n(y_n ; y_{1:n-1})}{p\, P_n(y_n ; y_{1:n-1}) + \frac{1-p}{2}\left(P_n(y_n - 1 ; y_{1:n-1}) + P_n(y_n + 1 ; y_{1:n-1})\right)}$$

$$= \frac{p\, P_n(y_n ; y_{1:n-1})}{p\, P_n(y_n ; y_{1:n-1}) + \frac{1-p}{2}\left(1 - P_n(y_n ; y_{1:n-1})\right)}$$

$$= \boxed{\frac{2p\, P_n(y_n ; y_{1:n-1})}{1-p + (3p-1) P_n(y_n ; y_{1:n-1})}}.$$

5) $L(p\,;x_{1:n},y_{1:n}) = \log \mathbb{P}_p\left(X_{1:n}=x_{1:n}\,,\,Y_{1:n}=y_{1:n}\right)$

$$= \log(\nu(x_1)) + \sum_{k=1}^{n-1}\log P(x_k,x_{k+1}) + \sum_{k=1}^{n}\log Q(x_k,y_k),$$

car $\mathbb{P}_p\left(X_{1:n}=x_{1:n}\,,\,Y_{1:n}=y_{1:n}\right) = \nu(x_1)\cdot\prod_{k=1}^{n-1}P(x_k,x_{k+1})\times\prod_{k=1}^{n}Q(x_k,y_k)$.

$$Q(p|p_0) = \mathbb{E}_{p_0}\left[L(p\,;X_{1:n},y_{1:n}) \,\middle|\, Y_{1:n}=y_{1:n}\right]$$

$$= \sum_{x\in\mathbb{Z}}\log(\nu(x))\times\mathbb{P}_{p_0}(X_1=x\,|\,Y_{1:n}=y_{1:n})$$

$$+\sum_{k=1}^{n-1}\sum_{x,z\in\mathbb{Z}}\log(P(x,z))\times\mathbb{P}_{p_0}(X_k=x,X_{k+1}=z\,|\,Y_{1:n}=y_{1:n})$$

$$+\sum_{k=1}^{n}\sum_{x\in\mathbb{Z}}\log(Q(x,y_k))\times\mathbb{P}_{p_0}(X_k=x\,|\,Y_{1:n}=y_{1:n}),$$

avec $\log(0)\cdot 0 = 0$ par convention.

Maximiser $Q(p|p_0)$ revient à maximiser

$$\sum_{k=1}^{n}\sum_{x\in\mathbb{Z}}\log(Q(x,y_k))\times\mathbb{P}_{p_0}(X_k=x\,|\,Y_{1:n}=y_{1:n})$$

$$=\sum_{k=1}^{n}\sum_{x=y_k-1}^{y_k+1}\log(Q(x,y_k))\times\mathbb{P}_{p_0}(X_k=x\,|\,Y_{1:n}=y_{1:n})$$

$$=\sum_{k=1}^{n}\Big\{\log(p)\times\mathbb{P}_{p_0}(X_k=y_k\,|\,Y_{1:n}=y_{1:n}) + \log\left(\frac{1-p}{2}\right)\times\Big(\underbrace{\mathbb{P}_{p_0}(X_k=y_k-1\,|\,Y_{1:n}=y_{1:n})+\mathbb{P}_{p_0}(X_k=y_k+1\,|\,Y_{1:n}=y_{1:n})}_{=\,1-\mathbb{P}_{p_0}(X_k=y_k\,|\,Y_{1:n}=y_{1:n})}\Big)\Big\}$$

$$=\left(\sum_{k=1}^{n}\mathbb{P}_{p_0}(X_k=y_k\,|\,Y_{1:n}=y_{1:n})\right)\cdot\log(p) + \left(n-\sum_{k=1}^{n}\mathbb{P}_{p_0}(X_k=y_k\,|\,Y_{1:n}=y_{1:n})\right)\times\log\left(\frac{1-p}{2}\right)$$

$$= f(p).$$

Soit $S \overset{not}{=} \sum_{k=1}^{n} \mathbb{P}_{p_0}(X_k = y_k \mid Y_{1:n} = y_{1:n})$. 7

6) $f'(p) = \frac{1}{p} \times S - \frac{2}{1-p}(n-S) \overset{!}{=} 0$

$$\Rightarrow \quad S(1-p) - 2(n-S)p = 0$$

$$\Rightarrow \quad S - \cancel{Sp} - 2np + \cancel{2}Sp = 0 \iff p(S-2n) + S = 0$$

$$\Rightarrow \quad \hat{p} = \frac{S}{2n-S} = \boxed{\frac{\sum_{k=1}^{n} \mathbb{P}_{p_0}(X_k = y_k \mid Y_{1:n} = y_{1:n})}{2n - \sum_{k=1}^{n} \mathbb{P}_{p_0}(X_k = y_k \mid Y_{1:n} = y_{1:n})}}$$

(On peut vérifier aussi que $\hat{p}$ est un point de maximum et non pas de min.)

En effet : $f''(p) = -\frac{S}{p^2} - \frac{2}{(1-p)^2}(n-S) < 0$.