Correction DM Proba-Stat

2 IMACS, 2011-2012

Exe. 1

1) $G_X(t) = \mathbb{E}(t^X) = \sum_{k=0}^{\infty} t^k p_k$ (*)

b) $\sum_{k=0}^{\infty} |t^k p_k| \leq \sum_{k=0}^{\infty} |t|^k < \infty$ pour tout $t \in]-1,1[$.

- pour $t = 1$ ou $t = -1$:

$$\sum_{k=0}^{\infty} |t^k p_k| = \sum_{k=0}^{\infty} p_k = 1 < \infty$$

↳ la série (*) est absolument convergente $\forall\, t \in [-1,1]$.

↳ G_X est bien définie sur $[-1,1]$.

c) $G_X'(t) = \sum_{k=1}^{\infty} k\, t^{k-1} p_k = p_1 + 2t\, p_2 + 3t^2 p_3 + \dots$

thm. de dérivation des séries entières

$\Rightarrow \boxed{G_X'(0) = p_1}$

$G_X''(t) = \sum_{k=2}^{\infty} k(k-1)\, t^{k-2} p_k$

$\Rightarrow \boxed{G_X''(0) = 2 p_2}$

$\boxed{G_X(0) = p_0}$

la dérivée d'ordre i

$$G_X^{(i)}(t) = \sum_{k=i}^{\infty} k(k-1)\dots(k-i+1)\, t^{k-i} p_k$$

On remarque que (peut montrer)

$\Rightarrow G_X^{(i)}(0) = i(i-1)\dots(i-i+1)\, p_i = i!\, p_i$

$$\Rightarrow \boxed{p_k = \frac{G_X^{(k)}(0)}{k!}}, \quad \forall\, k \geq 0.$$

d) $G'_X(1) = \sum_{k=1}^{\infty} k p_k = \mathbb{E}(X) \quad \left(= \sum_{k=0}^{\infty} k p_k \right)$

$$G''_X(1) = \sum_{k=2}^{\infty} k(k-1) p_k = \sum_{k=1}^{\infty} k(k-1) p_k = \sum_{k=1}^{\infty} k^2 p_k - \sum_{k=1}^{\infty} k p_k$$
$$= \sum_{k=0}^{\infty} k^2 p_k - \sum_{k=0}^{\infty} k p_k = \mathbb{E}(X^2) - \mathbb{E}(X).$$

$$\Rightarrow \mathrm{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = G''_X(1) + G'_X(1) - (G'_X(1))^2.$$

e) $G_{X+Y}(t) = \mathbb{E}(t^{X+Y}) = \mathbb{E}(t^X . t^Y) \underset{X,Y \text{ indép.}}{=} \mathbb{E}(t^X)\mathbb{E}(t^Y) = G_X(t) G_Y(t).$

2) a) $X \sim \mathcal{P}_o(\lambda) \Rightarrow p_k = \frac{\lambda^k}{k!} e^{-\lambda}, \quad \forall k \geq 0.$

$$\Rightarrow G_X(t) = \sum_{k=0}^{\infty} t^k . \frac{\lambda^k}{k!} e^{-\lambda} = \sum_{k=0}^{\infty} \frac{(t\lambda)^k}{k!} e^{-\lambda}$$
$$= e^{t\lambda} . e^{-\lambda} = \underline{e^{(t-1)\lambda}}, \quad \underline{\forall t \in \mathbb{R}}.$$

b) $\mathbb{E}(X) = G'_X(1).$

$G'_X(t) = \lambda e^{(t-1)\lambda} \Rightarrow G'_X(1) = \lambda \Rightarrow \underline{\mathbb{E}(X) = \lambda}.$

$G''_X(t) = \lambda^2 e^{(t-1)\lambda} \Rightarrow G''_X(1) = \lambda^2$

$$\Rightarrow \underline{\mathrm{Var}(X)} = G'_X(1) + G''_X(1) - (G'_X(1))^2$$
$$= \lambda + \lambda^2 - \lambda^2 = \underline{\lambda^2}$$

c) $G_X^{(k)}(t) = \lambda^k e^{(t-1)\lambda} \Rightarrow G_X^{(k)}(0) = \lambda^k e^{-\lambda}.$

$$\Rightarrow \mathbb{P}(X = k) = \frac{G_X^{(k)}(0)}{k!} = \underline{\underline{\frac{\lambda^k}{k!} e^{-\lambda}.}}$$

d) $G_{X+Y}(t) \underset{X,Y \text{ indép.}}{=} G_X(t) G_Y(t) = e^{(t-1)\lambda} . e^{(t-1)\mu} = e^{(t-1)(\lambda+\mu)}$

$$\Rightarrow \underline{X+Y \sim \mathcal{P}_o(\lambda+\mu)}$$

$\boxed{\text{Exc. 2}}$

1) $y \in [0,1[$, $\mathbb{P}(T \leq y \mid T < 1) = \dfrac{\mathbb{P}(T \leq y \cap T < 1)}{\mathbb{P}(T < 1)}$

$$= \frac{\mathbb{P}(T \leq y)}{\mathbb{P}(T<1)} = \frac{F_T(y)}{F_T(1)} = \frac{1-e^{-\lambda y}}{1-e^{-\lambda}}.$$

2) $\{X = k\} = \{[T] = k-1\} = \{k-1 \leq T < k\}$

$\Rightarrow \mathbb{P}(X=k) = \mathbb{P}(k-1 \leq T < k) = \displaystyle\int_{k-1}^{k} f_T(t)\,dt$

$$= \int_{k-1}^{k} \lambda e^{-\lambda t}\,dt = \left[-e^{-\lambda t}\right]_{k-1}^{k} = e^{-\lambda(k-1)} - e^{-\lambda k}.$$

(ou $\mathbb{P}(X=k) = \mathbb{P}(k-1 \leq T < k) = F_T(k) - F_T(k-1) \ldots$)

$\Rightarrow \mathbb{P}(X=k) = e^{-\lambda(k-1)}(1-e^{-\lambda}) = (1-p)^{k-1} \cdot p$, $\forall\ k = 1,2,\ldots$

avec $p = 1 - e^{-\lambda}$.

$\hookrightarrow$ $\underline{X \sim \text{Géométrique}\ (1-e^{-\lambda})}$.

3) $\mathbb{P}(Y \leq y \cap X = k) = \mathbb{P}(\underbrace{T-[T]}_{\substack{\text{la partie}\\ \text{fractionnaire}\\ \text{de } T}} \leq y \cap [T] = k-1)$

$= \mathbb{P}(T \leq k-1+y \cap [T] = k-1)$

$= \mathbb{P}(k-1 \leq T \leq k-1+y)$ car $y \in [0,1[$

$= e^{-\lambda(k-1)} - e^{-\lambda(k-1+y)} = e^{-\lambda(k-1)}(1-e^{-\lambda y}).$

4) $F_Y(y) = \mathbb{P}(Y \leq y) = \displaystyle\sum_{k=1}^{\infty} \mathbb{P}(Y \leq y \cap X = k)$

$= \underbrace{\displaystyle\sum_{k=1}^{\infty} e^{-\lambda(k-1)}}_{\text{série géométrique de raison } e^{-\lambda}} (1-e^{-\lambda y}) = \dfrac{1}{1-e^{-\lambda}}(1-e^{-\lambda y})$. $\forall\ y \in [0,1[$ (la même proba que $\mathbb{P}(T \leq y \mid T<1)$).

↳ la partie fractionnaire $T - [T]$
a la même loi que
la loi de T sachant $\{T < 1\}$.
$\overbrace{\text{conditionnelle}}$

5) $$f_Y(y) = F_Y'(y) = \frac{\lambda}{1-e^{-\lambda}} e^{-\lambda y}, \quad \forall y \in [0,1].$$

$$\mathbb{E}(Y) = \int_0^1 y \cdot \frac{\lambda}{1-e^{-\lambda}} e^{-\lambda y} = \frac{1}{1-e^{-\lambda}} \cdot \int_0^1 y \left(-e^{-\lambda y}\right)' dy$$

$$\stackrel{(IPP)}{=} \frac{1}{1-e^{-\lambda}} \left(\left[-y e^{-\lambda y}\right]_0^1 + \int_0^1 e^{-\lambda y} dy \right)$$

$$= \frac{1}{1-e^{-\lambda}} \left(-e^{-\lambda} - \left[\frac{e^{-\lambda y}}{\lambda}\right]_0^1 \right)$$

$$= \frac{1}{1-e^{-\lambda}} \left(-e^{-\lambda} - \frac{e^{-\lambda}}{\lambda} + \frac{1}{\lambda} \right)$$

$$= \underbrace{\frac{1}{\lambda} - \frac{e^{-\lambda}}{1-e^{-\lambda}}}$$

• Une autre méthode : $Y = T - [T] = T - X + 1$

$$\Rightarrow \mathbb{E}(Y) = \mathbb{E}(T) - \mathbb{E}(X) + 1$$
$$= \frac{1}{\lambda} - \underbrace{\frac{1}{1-e^{-\lambda}}}_{\text{car } X \sim \mathrm{Geom}(1-e^{-\lambda})} + 1 = \frac{1}{\lambda} - \frac{e^{-\lambda}}{1-e^{-\lambda}}.$$

$$\mathrm{Var}(Y) = \mathbb{E}(Y^2) - (\mathbb{E}(Y))^2 \stackrel{!}{=} \frac{2}{\lambda^2} - \frac{e^{-\lambda}}{1-e^{-\lambda}}\left(\frac{2}{\lambda}+1\right) - \left(\frac{1}{\lambda} - \frac{e^{-\lambda}}{1-e^{-\lambda}}\right)^2 = \boxed{\frac{1}{\lambda^2} - \frac{e^{-\lambda}}{(1-e^{-\lambda})^2}}$$

car $$\mathbb{E}(Y^2) = \int_0^1 y^2 \cdot \frac{\lambda}{1-e^{-\lambda}} e^{-\lambda y} dy = \frac{1}{1-e^{-\lambda}} \cdot \int_0^1 y^2 \left(-e^{-\lambda y}\right)' dy$$

$$\stackrel{(IPP)}{=} \frac{1}{1-e^{-\lambda}} \left(\left[-y^2 e^{-\lambda y}\right]_0^1 + 2\int_0^1 y e^{-\lambda y} dy \right) = \frac{-e^{-\lambda}}{1-e^{-\lambda}} + \frac{2}{\lambda} \mathbb{E}(Y)$$
$$= \frac{2}{\lambda^2} - \frac{e^{-\lambda}}{1-e^{-\lambda}}\left(\frac{2}{\lambda}+1\right).$$