Correction CC2 de Probabilités et Statistique

2 IMACS, 2011–2012

Exc. 1

1) a) $X \sim \mathrm{Expo}(\mu)$, $Y = \mu X$; On a $F_X(x) = \begin{cases} 1 - \exp(-\mu x), & \text{si } x \geq 0 \\ 0, & \text{si } x < 0 \end{cases}$

b) $F_Y(y) = \mathbb{P}(Y \leq y) = \mathbb{P}(\mu X \leq y) = \mathbb{P}\left(X \leq \frac{y}{\mu}\right) = \begin{cases} 1 - \exp\left(-\mu \cdot \frac{y}{\mu}\right), & \text{si } \frac{y}{\mu} \geq 0 \\ 0, & \text{sinon} \end{cases}$

$$= \begin{cases} 1 - e^{-y}, & \text{si } y \geq 0 \\ 0, & \text{sinon} \end{cases}$$

$\hookrightarrow$ $\underline{Y \sim \mathrm{Expo}(1)}$.

2) $N_t \sim \mathcal{P}o(\lambda t)$

a) $\mathbb{P}(N_t = k) = \dfrac{(\lambda t)^k}{k!} e^{-\lambda t}$, $\forall\, k \geq 0$.

b) $\{T > t\} = \{N_t = 0\} \Rightarrow \underline{\mathbb{P}(T > t)} = \mathbb{P}(N_t = 0) = \dfrac{(\lambda t)^0}{0!} e^{-\lambda t}$
$= \underline{e^{-\lambda t}}$

c) $F_T(t) = \mathbb{P}(T \leq t) = 1 - \mathbb{P}(T > t)$
$$= \begin{cases} 1 - e^{-\lambda t}, & \text{si } t \geq 0 \\ 0, & \text{si } t \leq 0 \end{cases}$$

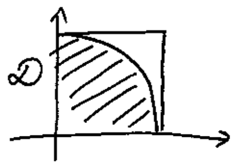
$f_T(t) = F_T'(t) = \begin{cases} \lambda e^{-\lambda t}, & \text{si } t \geq 0 \\ 0, & \text{sinon} \end{cases} \Rightarrow \underline{T \sim \mathrm{Expo}(\lambda)}$.

d) $\left.\begin{array}{l} T' = \lambda T \\ T \sim \mathrm{Expo}(\lambda) \end{array}\right\} \Rightarrow \underline{T' \sim \mathrm{Expo}(1)}$ (d'après la question 1))

Exc. 2 1) a) $f_{X,Y}(x,y) \overset{\text{car } X,Y \text{ indép.}}{=} f_X(x) \cdot f_Y(y) = \begin{cases} 1, & \text{si } (x,y) \in [0,1]^2 \\ 0, & \text{sinon} \end{cases}$

b) $\mathbb{P}((X,Y) \in \mathcal{D}) = \displaystyle\iint_{\mathcal{D}} f_{X,Y}(x,y)\, dx\, dy \underset{\text{car } \mathcal{D} \subset [0,1]^2}{=} \iint_{\mathcal{D}} 1\, dx\, dy$

$$= \text{Aire}(\mathcal{D}) = \boxed{\frac{\pi}{4}}.$$



2) $Z_n = \begin{cases} 1, & \text{si } X_n^2 + Y_n^2 \leq 1 \\ 0, & \text{sinon} \end{cases}$

a) $Z_n \sim Ber\left(\frac{\pi}{4}\right)$, car $\mathbb{P}(Z_n = 1) = \mathbb{P}(X_n^2 + Y_n^2 \leq 1) = \mathbb{P}((X_n, Y_n) \in \mathcal{D}) = \frac{\pi}{4}$.

b) $\frac{4(Z_1 + \ldots + Z_n)}{n} = \frac{(4Z_1) + \ldots + (4Z_n)}{n} \xrightarrow[n\to\infty]{\text{proba}} \mathbb{E}(4Z_1) = 4\mathbb{E}(Z_1) = 4 \cdot \frac{\pi}{4} = \boxed{\pi}$

(par la loi des grands nombres, car les variables $4Z_1, \ldots, 4Z_n$ sont indép. et de même loi

$$\left[\begin{array}{l} \text{ou } \frac{Z_1 + \ldots + Z_n}{n} \xrightarrow[n\to\infty]{\text{proba}} \mathbb{E}(Z_1) = \frac{\pi}{4} \\ \text{donc } \frac{4(Z_1 + \ldots + Z_n)}{n} \xrightarrow[n\to\infty]{\text{proba}} 4 \cdot \frac{\pi}{4} = \pi. \end{array}\right.$$

c) On peut utiliser ce résultat pour approximer par simulation la valeur de π.

$\boxed{\text{Exc. 3}}$ 1) a) $\mathbb{P}(X > k) = (1-p)^k$, $\forall k \in \mathbb{N}$.

b) $F_X(x) = P(X \leq x) = 1 - \mathbb{P}(X > x) = \begin{cases} 1 - \mathbb{P}(X > [x]), & \text{si } x \geq 0 \\ 0, & \text{si } x < 0 \end{cases}$

$= \begin{cases} 1 - (1-p)^{[x]}, & \text{si } x \geq 0 \\ 0, & \text{si } x < 0 \end{cases}$

2) a) $F_{Y_n}(y) = \mathbb{P}(Y_n \leq y) = \mathbb{P}\left(\frac{X_n}{n} \leq y\right) = \mathbb{P}(X_n \leq ny)$

$= F_{X_n}(ny) = \begin{cases} 1 - \left(1 - \frac{1}{n}\right)^{[ny]}, & \text{si } y \geq 0 \\ 0, & \text{si } y < 0 \end{cases}$

b) On étudie la convergence de la suite $(F_{Y_n})_n$.

$$\lim_{n\to\infty} \left(1 - \frac{1}{n}\right)^{[ny]} = \lim_{n\to\infty} \left[\left(1 - \frac{1}{n}\right)^{-n}\right]^{-\frac{[ny]}{n}}$$

$$= e^{\lim_{n\to\infty}\left(\frac{-[ny]}{n}\right)}$$

car $\lim_{n\to\infty} \left(1 - \frac{1}{n}\right)^{-n} = e$

On $\frac{ny-1}{n} < \frac{[ny]}{n} \leq \frac{ny}{n}$ $\Rightarrow \lim_{n\to\infty} \left(1 - \frac{1}{n}\right)^{[ny]} = e^{-y}$, si $y \geq 0$

$\searrow y \swarrow$

$\Rightarrow F_{Y_n}(y) \xrightarrow[n\to\infty]{} \begin{cases} 1 - e^{-y}, & \text{si } y \geq 0 \\ 0, & \text{sinon} \end{cases}$ la fct. de répart. de la loi Expo(1)

$\Rightarrow \boxed{Y_n \xrightarrow{\text{loi}} Expo(1)}$

Exc. 4

1) $\hat{p}_n = \frac{X_1 + \dots + X_n}{n} \xrightarrow[n\to\infty]{\text{proba}} \mathbb{E}(X_1) = p$ par la loi des grands nombres

car les variables sont indép. et de même loi,
et $\mathbb{E}(X_1) = p$ car $X_i \sim \mathrm{Ber}(p)$.

↳ $\hat{p}_n$ est un estimateur consistant de p

$\mathbb{E}(\hat{p}_n) = \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}(X_i) = \frac{1}{n} \times np = p \Rightarrow$ estimateur non-biaisé.

↑ linéarité de l'espérance

2) Par le $\boxed{\text{TCL}}$ on a que $\hat{p}_n \underset{n \text{ grand}}{\overset{\text{loi}}{\approx}} \mathcal{N}(\mathbb{E}(\hat{p}_n), \mathrm{Var}(\hat{p}_n))$

or $\mathbb{E}(\hat{p}_n) = p$

$\mathrm{Var}(\hat{p}_n) = \underset{\uparrow \text{ indép.}}{\frac{1}{n^2}} \sum \mathrm{Var}(X_i) = \frac{1}{n^2} \times np(1-p) = \frac{p(1-p)}{n}$

↳ $\underline{\hat{p}_n \underset{n \text{ grand}}{\overset{\text{loi}}{\approx}} \mathcal{N}\left(p, \frac{p(1-p)}{n}\right)}$.

3) On normalise : $\frac{\hat{p}_n - p}{\sqrt{\frac{p(1-p)}{n}}} \underset{n \text{ grand}}{\overset{\text{loi}}{\approx}} \mathcal{N}(0,1)$.

$\Leftrightarrow \frac{\sqrt{n}(\hat{p}_n - p)}{\sqrt{p(1-p)}} \xrightarrow[n \text{ grand}]{\text{loi}} \mathcal{N}(0,1)$.

Comme $\hat{p}_n \xrightarrow[n\to\infty]{\text{proba}} p$ on a aussi $\frac{\sqrt{n}(\hat{p}_n - p)}{\sqrt{\hat{p}_n(1-\hat{p}_n)}} \xrightarrow[n\to\infty]{\text{loi}} \mathcal{N}(0,1)$

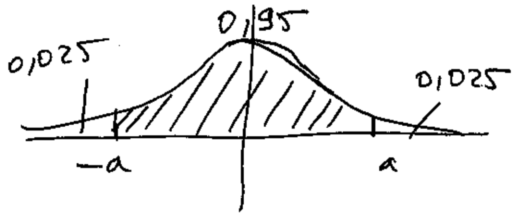
On cherche $\underline{a}$ t.q.

$\mathbb{P}(-a \leq Z \leq a) = 0{,}95$ pour $Z \sim \mathcal{N}(0,1)$

$\Rightarrow F_Z(a) = \mathbb{P}(Z < -a) + \mathbb{P}(-a \leq Z \leq a)$

$= 0{,}025 + 0{,}95 = 0{,}975$

par symétrie de la loi $\mathcal{N}(0,1)$



$\Rightarrow \underline{a = 1{,}96}$

$\Rightarrow \mathbb{P}\left(-1{,}96 \leq \frac{\sqrt{n}(\hat{p}_n - p)}{\sqrt{\hat{p}_n(1-\hat{p}_n)}} \leq 1{,}96\right) \xrightarrow[n\to\infty]{} \mathbb{P}(-1{,}96 \leq Z \leq 1{,}96) = 0{,}95$

$\Leftrightarrow \mathbb{P}\left(\hat{p}_n - 1{,}96\sqrt{\frac{\hat{p}_n(1-\hat{p}_n)}{n}} \leq p \leq \hat{p}_n + 1{,}96 \cdot \sqrt{\frac{\hat{p}_n(1-\hat{p}_n)}{n}}\right) \xrightarrow[n\to\infty]{} 0{,}95$

$\Rightarrow \underline{IC_{0,95}(p) = \left[\hat{p}_n \pm 1{,}96 \cdot \sqrt{\frac{\hat{p}_n(1-\hat{p}_n)}{n}}\right]}$.

4) $\hat{p}_n = \frac{20}{100} = 0,2$

$$IC_{0,95}(p) = \left[0,2 \pm 1,96 \times \sqrt{\frac{0,2 \times 0,8}{100}}\right] = \left[0,2 \pm \frac{1,96 \cdot 4}{100}\right]$$

$$= \left[0,2 \pm 0,0784\right] = \left[0,1216 ; 0,2784\right]$$

5) $N \sim \mathcal{B}(400 ; 0,2)$

$\mathbb{E}(N) = 400 \times 0,2 = 80$; $\quad Var(N) = 400 \times 0,2 \times (1-0,2)$
$\qquad = 80 \times 0,8 = 64$

6) Par le TCL on a

$$N \overset{loi}{\approx} \mathcal{N}(\mathbb{E}(N), Var(N)) = \mathcal{N}(80,64)$$

car $\begin{cases} 400 \geq 30 \\ 400 \times 0,2 \geq 5 \\ 400 \times (1-0,2) \geq 5 \end{cases}$ (les conditions pratiques pour assurer que l'approx. est bonne)

7) $\mathbb{P}(N > 72) = \mathbb{P}\left(\frac{N-80}{8} > \frac{72-80}{8}\right)$

$\approx \mathbb{P}(Z > -1)$, avec $Z \sim \mathcal{N}(0,1)$

car $\frac{N-80}{4} \overset{loi}{\approx} \mathcal{N}(0,1)$

$= \mathbb{P}(Z < 1)$
par symétrie

$= F_Z(1)$

$= \boxed{0,841}$