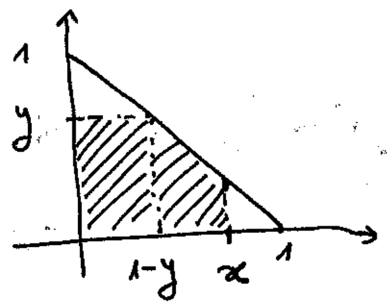
Calcul de la fonction de répartition de l'Exemple du poly

$$f(x,y) = \begin{cases} 2, & \text{si } (x,y) \in \mathcal{D} \\ 0, & \text{sinon} \end{cases}$$

avec $\mathcal{D} = \{ (x,y) \in \mathbb{R}^2 \mid x \geq 0,\ y \geq 0,\ x+y \leq 1 \}$.

$$F_{X,Y}(x,y) = \mathbb{P}(X \leq x, Y \leq y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f(u,v)\, du\, dv.$$

- **1^er^ cas :** si $x, y \in [0,1]$ et $x+y > 1 \Leftrightarrow 1-y < x$

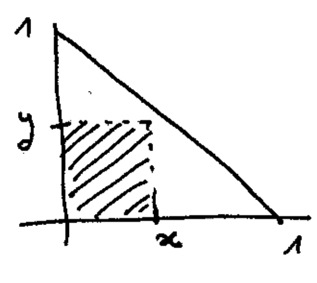


$$\Rightarrow F_{X,Y}(x,y) = \int_0^{1-y} \int_0^{y} 2\, du\, dv + \int_{1-y}^{x} \left(\int_0^{1-u} 2\, dv \right) du$$

$$= 2y(1-y) + \int_{1-y}^{x} 2(1-u)\, du$$

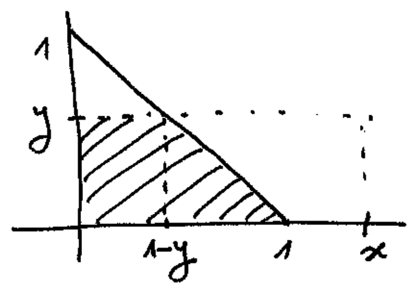
$$= 2y(1-y) - \left[(1-u)^2 \right]_{1-y}^{x} = \underline{2y - y^2 - (1-x)^2}$$

- **2^ème^ cas :** si $x, y \in [0,1]$ et $x+y \leq 1$ $\quad \left((x,y) \in \mathcal{D} \right)$



$$\Rightarrow F_{X,Y}(x,y) = \int_0^{x} \int_0^{y} 2\, du\, dv = \boxed{2xy}$$

- **3^ème^ cas :** si $x > 1$, $y \in [0,1]$



$$F_{X,Y}(x,y) = \iint_{\mathcal{D} \cap (-\infty, x] \times (-\infty, y]} 2\, du\, dv$$

$$= 2 \times \text{"Aire hachurée"} = 2\left(y(1-y) + \frac{y^2}{2} \right)$$

(calcul géométrique de l'aire)

$= 2y\left(1-\frac{y}{2}\right) = \underline{4y(2-y).}$

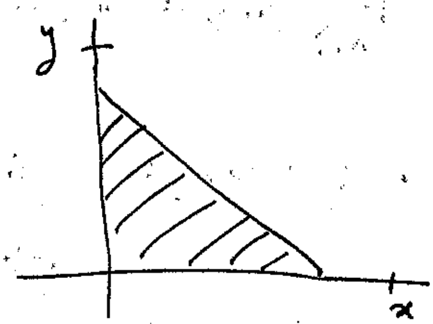
- $\underline{\text{4ème cas}}$: $\underline{\text{si } y > 1 \text{ et } x \in [0,1]}$

(par symétrie)

$F_{X,Y}(x,y) = 4x(2-x).$

- $\underline{\text{5ème cas}}$: $\underline{\text{si } x > 1 \text{ et } y > 1}$

$F_{X,Y}(x,y) = \iint_{\mathcal{D}} 2\, du\, dv = 1.$



- $\underline{\text{6ème cas}}$: si $x < 0$ ou $y < 0$

$\rightarrow F_{X,Y}(x,y) = 0.$

le 2ème cas ci-dessus

Donc $f(x,y) = \dfrac{\partial^2 F_{X,Y}}{\partial x \partial y}(x,y) = \begin{cases} 2, & \text{si } (x,y) \in \mathcal{D} \\ 0, & \text{sinon (tous les autres cas)} \end{cases}$