Reduction method for studying localized solutions of neural fields equations set on the Poincaré disk
Méthode de réduction pour l'étude de solutions localisées d'équations de champs neuronaux posées sur le disque de Poincaré

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Abstract

We present a reduction method to study localized solutions of an integrodifferential equation defined on the Poincaré disk. This equation arises in a problem of textures perception modeling in the visual cortex. We first derive a partial differential equation which is equivalent to the initial integrodifferential equation and then deduce that localized solutions which are further radially symmetric satisfy a fourth order ordinary differential equation.

Abstract

Dans cette note, nous présentons une méthode de réduction pour l'étude de solutions localisées d'une équation intégro-différentielle définie sur le disque de Poincaré. Ce genre d'équation est relié au problème de modélisation des textures par le cortex visuel. Nous dérivons tout d'abord une équation aux dérivées partielles équivalente à l'équation intégro-différentielle de départ et déduisons ensuite que les solutions qui sont en plus radialement symétriques satisfont une équation différentielle ordinaire d'ordre 4.

Keywords: Ordinary Differential Equation.

1 Version française abrégée

Dans cette note, on considère l'équation de champs neuronaux suivante:

$$\mathcal{L}\left(\frac{d}{dt}\right)V(\mathbf{z},t)=\int_{\mathbb{D}}W(d_{\mathbb{D}}(\mathbf{z},\mathbf{z}'))S(V(\mathbf{z}',t))\mathrm{dm}(\mathbf{z}'). \tag{1.1}$$

Cette équation, utilisée pour la modélisation de la perception de la texture dans le cortex visuel (voir [3, 5]), décrit l'évolution spatio-temporelle du potentiel de membrane moyen $V(\mathbf{z},t)$ au sein d'une hypercolonne d'orientations [8], où $\mathbf{z}$ est un point du disque de Poincaré $\mathbb{D}=\{\mathbf{z}\in\mathbb{C}\mid 0\leq|\mathbf{z}|<1\}$, $\mathcal{L}$ un opérateur polynômial d'évolution avec $\mathcal{L}(0)=1$. La fonction de connectivité W décrit les interactions entre les différentes populations de neurones au sein de l'hypercolonne. La nonlinéarité S est une fonction analytique de type sigmoidal que l'on prendra centrée à l'origine.

Le but de cette note est de trouver une équation aux dérivées partielles équivalente à l'équation (1.1) pour un type de fonction de connectivité bien précis et pour la classe des solutions localisées. Par définition, une solution localisée sera une fonction infiniment différentiable à support compact. Dans le cas de solutions localisées radialement symétriques, cette équation aux dérivées partielles se réduit à une équation différentielle ordinaire d'ordre quatre. L'intérêt pour l'étude des solutions localisées dans les équations de champs neuronaux vient du fait que ces solutions ont une place importance dans l'analyse et la compréhension de certains méchanismes corticaux lors de l'accomplissement de taches nécessitant le recours à la mémoire.

Le résultat principal de cette note est le théorème suivant ainsi que son corollaire dont la démonstration repose principalement sur l'analyse harmonique développée par Helgason [7] dans le disque de Poincaré.

Définition 1.1. *On introduit les coordonées géodésiques polaires (τ,θ) via $\mathbf{z}=\tanh(\tau/2)e^{i\theta}\in\mathbb{D}$. Si l'on note $\Delta_{\mathbb{D}}$ l'opérateur de Laplace-Beltrami dans le disque de Poincaré alors sa partie radiale, en coordonées géodésiques polaires (τ,θ), est donnée par (voir Helgason [7] page 38):*

$$\mathcal{A}_\tau=\frac{1}{\sinh\tau}\frac{d}{d\tau}\left(\sinh\tau\frac{d}{d\tau}\right). \tag{1.2}$$

Théorème 1.1. *Considérons une solution $V\in\mathcal{D}(\mathbb{D})=\mathcal{C}_c(\mathbb{D})\cap\mathcal{C}^\infty(\mathbb{D})$ de l'équation (1.1) et écrivons la connectivité sous la forme*

$$W(x)=\alpha_1\mathcal{Q}_{a_1-\frac{1}{2}}(\cosh 2x)-\alpha_2\mathcal{Q}_{a_2-\frac{1}{2}}(\cosh 2x), \tag{1.3}$$

où $\mathcal{Q}_{a-\frac{1}{2}}$ est la fonction de Legendre associée de seconde espèce (voir [4]) et $\alpha_1,\alpha_2,a_1,a_2$ sont des constantes positives réelles. Si l'on pose

$$\alpha=a_1^2+a_2^2-\frac{1}{2},\quad \beta=a_1^2a_2^2-\frac{1}{4}(a_1^2+a_2^2)+\frac{1}{16},\quad \gamma=\alpha_1a_2^2-\alpha_2a_1^2-\frac{1}{4}(\alpha_1-\alpha_2)$$

alors V satisfait l'équation aux dérivées partielles suivantes

$$\mathcal{L}\left(\frac{d}{dt}\right)\left[\Delta_{\mathbb{D}}^2V(\mathbf{z},t)-\alpha\Delta_{\mathbb{D}}V(\mathbf{z},t)+\beta V(\mathbf{z},t)\right]=\gamma S(V(\mathbf{z},t))-(\alpha_1-\alpha_2)\Delta_{\mathbb{D}}S(V(\mathbf{z},t)). \tag{1.4}$$

Corollaire 1.1. *Avec la même fonction de connectivité W définie à l'équation (1.3), une solution stationaire radialement symétrique $U\in\mathcal{D}(\mathbb{D})$ de (1.1) satisfait l'équation différentielle ordinaire suivante*

$$\mathcal{A}_\tau^2U(\tau)-\alpha\mathcal{A}_\tau U(\tau)+\beta U(\tau)=\gamma S(U(\tau))-(\alpha_1-\alpha_2)\mathcal{A}_\tau S(U(\tau)). \tag{1.5}$$

Cette méthode de réduction d'un système dynamique en dimension infinie à une équation différentielle ordinaire d'ordre quatre, pour le cas des solutions localisées radialement symétriques, a deux avantages importants. Le premier est numérique: il est beaucoup plus rapide de résoudre une équation différentielle ordinaire qu'une équation intégro-différentielle et permet ainsi une plus grande exploration des différents régimes de l'équation en fonction de tous les paramètres du système. Le second est théorique: l'équation différentielle ordinaire ainsi obtenue peut être analysée en suivant les méthodes développées pour le cas euclidien dans [10, 11]. Ceci constituera le cadre de nos prochaines investigations.

2 Introduction

In this note, we consider neural fields (see [13, 1, 2]), which model the large-scale dynamics of spatially structured biological neural networks. In a recent study on the perception of textures in the visual cortex [3, 5], neural fields describe the mean voltage potential $V(\mathbf{z},t)$ within a Hubel and Wiesel hypercolumn of orientations [8], where $\mathbf{z}$ belongs to the Poincaré disk $\mathbb{D}=\{\mathbf{z}\in\mathbb{C}\mid 0\le|\mathbf{z}|<1\}$, by the following nonlinear integrodifferential equation set on $\mathbb{D}$:

$$\mathcal{L}\left(\frac{d}{dt}\right)V(\mathbf{z},t)=\int_{\mathbb{D}}W(d_{\mathbb{D}}(\mathbf{z},\mathbf{z}'))S(V(\mathbf{z}',t))\mathrm{dm}(\mathbf{z}'). \tag{2.1}$$

$\mathcal{L}$ is polynomial in $\dfrac{d}{dt}$ with $\mathcal{L}(0)=\ell>0$. After a rescaling, we suppose from nom on that $\ell=1$. The distance $d_{\mathbb{D}}$ is the usual hyperbolic distance on $\mathbb{D}$ defined by

$$d_{\mathbb{D}}(\mathbf{z},\mathbf{z}')=\tanh^{-1}\left(\frac{|\mathbf{z}-\mathbf{z}'|}{|1-\bar{\mathbf{z}}\mathbf{z}'|}\right)\quad\forall(\mathbf{z},\mathbf{z}')\in\mathbb{D}^2, \tag{2.2}$$

and the measure element $\mathrm{dm}(\mathbf{z})$ is given by

$$\mathrm{dm}(\mathbf{z}) = \frac{dz_1 dz_2}{(1-|\mathbf{z}|^2)^2}, \text{ with } \mathbf{z} = z_1 + iz_2. \tag{2.3}$$

The connectivity function W and the nonlinearity S, an analytic function of sigmoidal shape centered at the origin, describe the relationship bewteen firing rate and membrane potential. Stationary states U of equation (2.1) are solutions of:

$$U(\mathbf{z}) = \int_{\mathbb{D}} W(d_{\mathbb{D}}(\mathbf{z},\mathbf{z}'))S(U(\mathbf{z}'))\mathrm{dm}(\mathbf{z}') \quad \forall \mathbf{z} \in \mathbb{D}. \tag{2.4}$$

The aim of this note is to derive a partial differential equation (see equation (3.7) in Theorem 3.1) which is equivalent to the integrodifferential equation (2.1) for a given connectivity function and for localized solutions. Then, we deduce that when the stationary state is further supposed to be radially symmetric it satisfies a fourth order ordinary equation (see equation (3.8) in Corollary 3.1). A localized solution V of equation (2.1) is defined to be a differentiable function of compact support. One of the reasons why this particular class of solutions (localized solutions) is of interest is that they are thought to arise in cortical circuits performing certain working memory tasks.

Direct numerical computations of solutions of (2.4) require an accurate approximation of the integral and is very time consuming, therefore, it is difficult to experiment on how the connectivity shapes the stationary solutions in conjunction with the nonlinearity S. This is why we believe that our reduction method is a major step toward a rigorous study of localized solutions set on the Poincaré disk, as it has been done for the Euclidean case in [10, 11]. This will be the subject of a forthcoming paper. Note that this method which reduces an integrodifferential equation into a partial differential equation has been introduced in [9] for neural fields set on the Euclidean plane.

3 Main result

Let O denote the center of the Poincaré disk that is the point represented by $\mathbf{z} = 0$ and dg denote the Haar measure on the group $G = \mathrm{SU}(1,1)$ (see [7]), normalized by:

$$\int_G F(g\cdot O)dg \stackrel{def}{=} \int_{\mathbb{D}} F(\mathbf{z})\mathrm{dm}(\mathbf{z}), \text{ for all } F \in \mathrm{L}^1(\mathbb{D},\mathrm{dm}). \tag{3.1}$$

Given two function $F_1 \in \mathrm{L}^1(\mathbb{D},\mathrm{dm})$ and $F_2 \in \mathrm{L}^p(\mathbb{D},\mathrm{dm})$, $1 \leq p \leq \infty$, we define the convolution $*$ by:

$$(F_1 * F_2)(\mathbf{z}) = \int_G F_1(g^{-1}\cdot\mathbf{z})F_2(g\cdot O)\,dg. \tag{3.2}$$

Then if we denote $\mathbf{W}(\mathbf{z}) = W(d_{\mathbb{D}}(\mathbf{z},0))$, the right-hand side of equation (2.1) is a convolution in $\mathbb{D}$ provided that $\mathbf{W} \in \mathrm{L}^1(\mathbb{D},\mathrm{dm})$ (note that we used the fact that S is a bounded function).

Let b be a point on the circle $\partial\mathbb{D}$. For $\mathbf{z} \in \mathbb{D}$, we define the "inner product" $<\mathbf{z}, b>$ to be the algebraic distance to the origin of the (unique) horocycle based at b through $\mathbf{z}$ (see [7]). The Helgason-Fourier transform in $\mathbb{D}$ is defined as (see [7]):

$$\widetilde{F}(\rho,b) = \frac{1}{2\pi}\int_{\mathbb{D}} F(\mathbf{z})e^{\left(\frac{1}{2}-i\rho\right)<\mathbf{z},b>}\mathrm{dm}(\mathbf{z}) \quad \forall(\rho,b) \in \mathbb{C}\times\partial\mathbb{D} \tag{3.3}$$

for a function $F : \mathbb{D} \to \mathbb{C}$ such that $\mathbf{z} \to F(\mathbf{z})e^{\left(\frac{1}{2}-i\rho\right)<\mathbf{z},b>} \in \mathrm{L}^1(\mathbb{D},\mathrm{dm})$. If $F_1, F_2 : \mathbb{D} \to \mathbb{C}$ are two functions which both satisfy $F_j(\mathbf{z})e^{\left(\frac{1}{2}-i\rho\right)<\mathbf{z},b>} \in \mathrm{L}^1(\mathbb{D},\mathrm{dm})$ for $j = 1,2$ then the convolution $F_1 * F_2$ has for Helgason-Fourier transform (see [7]):

$$\widetilde{F_1 * F_2}(\rho,b) = \widetilde{F_1}(\rho,b)\widetilde{F_2}(\rho,b) \quad \forall(\rho,b) \in \mathbb{C}\times\partial\mathbb{D}.$$

A radial function on $\mathbb{D}$ is by defintion a function which only depends upon $d_{\mathbb{D}}(\mathbf{z},0)$. From now on we write $\mathbf{z} = \tanh(\tau/2)e^{i\theta} \in \mathbb{D}$. If F is a radial function which satisfies $F(\tau)e^{\tau/2} \in \mathrm{L}^1(\mathbb{R}_*^+)$, then for all $\rho \in \mathbb{R}$ and all $\tau > 0$ the Helgason-Fourier transform of F reduces to its Mehler-Fock transform [6, 7]:

$$\widetilde{F}(\rho) = \int_0^{+\infty} F(\tau)\mathcal{P}_{-\frac{1}{2}+i\rho}(\cosh\tau)\sinh\tau\, d\tau, \tag{3.4}$$

where $\mathcal{P}_{-\frac{1}{2}+i\rho}$ is the associated Legendre function of the first kind [4].

Finally if we denote $\Delta_{\mathbb{D}}$, the Laplace-Beltrami operator on $\mathbb{D}$, then its radial part is given in geodesic polar coordinates (τ, θ) by (see Helgason [7] pp. 38)

$$\mathcal{A}_\tau = \frac{1}{\sinh\tau}\frac{d}{d\tau}\left(\sinh\tau\frac{d}{d\tau}\right). \tag{3.5}$$

We can now state the main result of this note.

Theorem 3.1. *Let us consider a solution $V \in \mathcal{D}(\mathbb{D}) = \mathcal{C}_c(\mathbb{D}) \cap \mathcal{C}^\infty(\mathbb{D})$ of the equation (2.1) and write the integral operator kernel*

$$\mathbf{W}(\mathbf{z}) = \alpha_1 \mathcal{Q}_{a_1-\frac{1}{2}}(\cosh 2d(\mathbf{z},0)) - \alpha_2 \mathcal{Q}_{a_2-\frac{1}{2}}(\cosh 2d(\mathbf{z},0)), \tag{3.6}$$

where $\mathcal{Q}_{a-\frac{1}{2}}$ is the associated Legendre function of the second kind and $\alpha_1, \alpha_2, a_1, a_2$ are positive real constants. If we denote

$$\alpha = a_1^2 + a_2^2 - \frac{1}{2}, \quad \beta = a_1^2 a_2^2 - \frac{1}{4}(a_1^2 + a_2^2) + \frac{1}{16}, \quad \gamma = \alpha_1 a_2^2 - \alpha_2 a_1^2 - \frac{1}{4}(\alpha_1 - \alpha_2)$$

then V satisfies the following partial differential equation

$$\mathcal{L}\left(\frac{d}{dt}\right)\left[\Delta_{\mathbb{D}}^2 V(\mathbf{z},t) - \alpha\Delta_{\mathbb{D}} V(\mathbf{z},t) + \beta V(\mathbf{z},t)\right] = \gamma S(V(\mathbf{z},t)) - (\alpha_1 - \alpha_2)\Delta_{\mathbb{D}} S(V(\mathbf{z},t)). \tag{3.7}$$

Corollary 3.1. *With the connectivity function $\mathbf{W}$ defined in (3.6), radially symmetric solution $U \in \mathcal{D}(\mathbb{D})$ of (2.4) satisfies the following fourth order ordinary differential equation*

$$\mathcal{A}_\tau^2 U(\tau) - \alpha\mathcal{A}_\tau U(\tau) + \beta U(\tau) = \gamma S(U(\tau)) - (\alpha_1 - \alpha_2)\mathcal{A}_\tau S(U(\tau)). \tag{3.8}$$

Proof. For $a > 0$ the following formula holds (pp. 788 [12]):

$$\int_0^{+\infty} \mathcal{Q}_{a-\frac{1}{2}}(\cosh\tau)\mathcal{P}_{-\frac{1}{2}+i\rho}(\cosh\tau)\sinh\tau d\tau = \frac{1}{a^2+\rho^2},$$

such that $\mathbf{W}$ has the following Mehler-Fock transform

$$\widetilde{\mathbf{W}}(\rho) = \frac{\alpha_1}{a_1^2+\rho^2} - \frac{\alpha_2}{a_2^2+\rho^2}. \tag{3.9}$$

Since S is analytic and $S(0) = 0$, it is easy to see that if $V \in \mathcal{D}(\mathbb{D})$ then $S(V) \in \mathcal{D}(\mathbb{D})$. We can now take the Helgason-Fourier transform of equation (2.1) and using property of the convolution this yields

$$\mathcal{L}\left(\frac{d}{dt}\right)\widetilde{V}(\rho,b,t) = \widetilde{\mathbf{W}}(\rho)\widetilde{S(V)}(\rho,b,t).$$

Combining with equation (3.9) we obtain

$$\mathcal{L}\left(\frac{d}{dt}\right)\left[(a_1^2+\rho^2)(a_2^2+\rho^2)\widetilde{V}(\rho,b,t)\right] = \left(\alpha_1 a_2^2 - \alpha_2 a_1^2 + \rho^2(\alpha_1-\alpha_2)\right)\widetilde{S(V)}(\rho,b,t).$$

We notice that

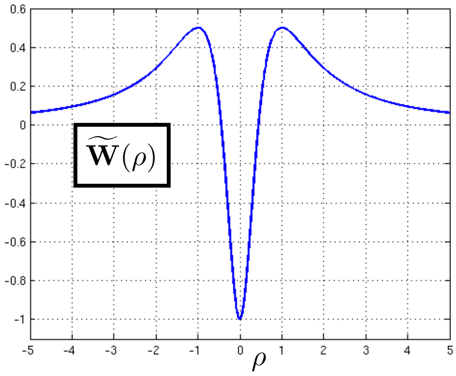
$$\begin{aligned}(a_1^2+\rho^2)(a_2^2+\rho^2) &= \left(\frac{1}{4}+\rho^2\right)^2 + \alpha\left(\frac{1}{4}+\rho^2\right) + \beta \\ \alpha_1 a_2^2 - \alpha_2 a_1^2 + \rho^2(\alpha_1-\alpha_2) &= \gamma + (\alpha_1-\alpha_2)\left(\frac{1}{4}+\rho^2\right).\end{aligned}$$

Finally applying the inverse Helgason-Fourier transform and equation (A.2) of appendix we recover equation (3.7). The proof of the corollary, based on the inverse formula (A.4), is straightforward.

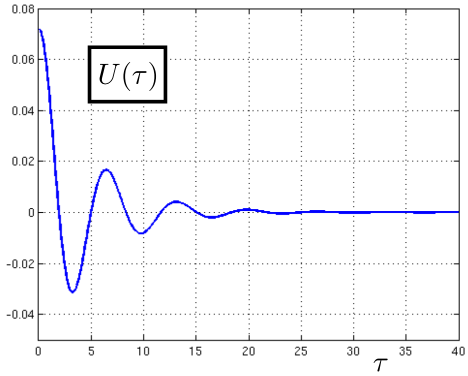
□

4 Numerical analysis

From corollary 3.1, if we want to compute radially symmetric stationary solutions of (2.4), we need to solve equation (3.8). We give a numerical example when $S(x) = \frac{1}{1+e^{-\mu x}} - \frac{1}{2}$ with $\mu = 7$. Constants in the connectivity function are set to $\alpha_1 = 3, \alpha_2 = 4/3, a_1 = 1$ and $a_2 = \sqrt{3}/3$ and the shape of $\widetilde{\mathbf{W}}$ is ploted in figure 1(a). We have solved (3.8) by implementing a finite difference scheme on $[0, 40]$ with an initial condition $U_0(\tau) = 0.1\cos(\tau)/\cosh(\tau)$ and the solution $U(\tau)$ is ploted in figure 1(b).



(a) Meher-Fock transform $\widetilde{\mathbf{W}}$.



(b) Stationary radially symmetric solution of (2.4).

Figure 1: Left. Plot of $\widetilde{\mathbf{W}}$ as a function of ρ, the value of the parameters are $\alpha_1 = 3, \alpha_2 = 4/3, a_1 = 1$ and $a_2 = \sqrt{3}/3$. Right. Plot of a stationary radially symmetric solution of (2.4) obtained by solving (3.8).

A Technical proposition

Proposition A.1. *For $F \in \mathcal{D}(\mathbb{D})$ and all $k \geq 1$,*

$$(-1)^k \left(\frac{1}{4} + \rho^2\right)^k \widetilde{F}(\rho, b) = \frac{1}{2\pi} \int_{\mathbb{D}} \Delta_{\mathbb{D}}^k F(\mathbf{z}) e^{\left(\frac{1}{2} - i\rho\right)<\mathbf{z},b>} dm(\mathbf{z}) \quad \forall (\rho, b) \in \mathbb{C} \times \partial\mathbb{D} \tag{A.1}$$

and

$$\Delta_{\mathbb{D}}^k F(\mathbf{z}) = (-1)^k \int_0^{+\infty} \int_{\partial\mathbb{D}} \left(\frac{1}{4} + \rho^2\right)^k \widetilde{F}(\rho, b) e^{\left(\frac{1}{2} + i\rho\right)<\mathbf{z},b>} \rho \tanh(\pi\rho) d\rho db \tag{A.2}$$

where db is the circular measure on $\partial\mathbb{D}$ normalized by $\int db = 1$. Let $\mathcal{D}^\natural(\mathbb{D})$ denote the space of radial functions in $\mathcal{D}(\mathbb{D})$, then equations (A.1) and (A.2) reduce to the following equations in the case of radial function $F \in \mathcal{D}^\natural(\mathbb{D})$:

$$(-1)^k \left(\frac{1}{4} + \rho^2\right)^k \widetilde{F}(\rho) = \int_0^{+\infty} \mathcal{A}_\tau^k F(\tau) \mathcal{P}_{-\frac{1}{2}+i\rho}(\cosh\tau) \sinh\tau d\tau \tag{A.3}$$

and

$$\mathcal{A}_\tau^k F(\tau) = (-1)^k \int_0^{+\infty} \left(\frac{1}{4} + \rho^2\right)^k \widetilde{F}(\rho) \mathcal{P}_{-\frac{1}{2}+i\rho}(\cosh\tau) \rho \tanh(\pi\rho) d\rho. \tag{A.4}$$

Proof. See [6, 7]. □

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