# Anomalous invasion speed in a system of coupled reaction-diffusion equations

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**Abstract**

In this paper, we provide a complete description of the selected spreading speed of systems of reaction-diffusion equations with unilateral coupling and prove the existence of anomalous spreading speeds for systems with monostable nonlinearities. Our work extends known results for systems with linear and quadratic couplings, and Fisher-KPP type nonlinearities. Our proofs rely on the construction of appropriate sub- and super-solutions.

**Keywords:** anomalous spreading speed, linear spreading speed, sub- and super-solutions.

## 1 Introduction

In this article we study the spreading properties of the following system of coupled reaction diffusion equations,

$$\begin{cases} u_t = du_{xx} + f(u) + \beta v^p(1-u) & , t > 0,\ x \in \mathbb{R}, \\ v_t = v_{xx} + v(1-v) & , t > 0,\ x \in \mathbb{R}, \\ u(0,x) = u_0(x), \quad v(0,x) = v_0(x) & , x \in \mathbb{R}, \end{cases} \tag{1.1}$$

with $d, \beta, p > 0$ and $f \in C^2$ a monostable function, that is

$$\begin{cases} f(0) = f(1) = 0, \\ 0 < f(u) & , 0 < u < 1, \\ f'(0) > 0 > f'(1). \end{cases} \tag{1.2}$$

Given initial conditions $0 \leq u_0, v_0 \leq 1$ being compactly supported perturbations of the Heaviside step function $\mathbf{1}_{x\leq 0}$, such systems typically present front-like solutions that propagates to the right with a certain spreading speed. In this study, we are interested in the asymptotic speed of propagation for the $u$ component. That is, if we define the invasion point

$$\kappa(t) = \sup_{x\in\mathbb{R}} \left\{ x \,|\, u(t,x) \geq \frac{1}{2} \right\},$$

we want to know the expression of the so-called selected speed or spreading speed,

$$s_{\mathrm{sel}} = \lim_{t\to\infty} \frac{\kappa(t)}{t},$$

with respect to $f$ and the parameters. Note that, given our initial conditions, an application of the comparison principle gives that the solutions of (1.1) satisfy $0 \leq u(t,x), v(t,x) \leq 1$ for all $t > 0$ and $x \in \mathbb{R}$, so that the threshold $1/2$ in the definition of $\kappa(t)$ is arbitrary and we shall see that the selected speed is identical for any threshold in $(0,1)$.

**The $v$ component.** Observe that the coupling only occurs in the $u$ equation, thus we can first look at the $v$ equation in isolation :

$$v_t = v_{xx} + v(1-v). \tag{1.3}$$

This is the well-known scalar Fisher-KPP equation, and has been the object of numerous studies, see [1, 2, 5, 9] among others. Given our compactly supported, positive initial condition, a classical result established in [1] proves that this component will spread at asymptotical speed $s_* = 2$ in the following sense :

$$\inf_{x\leq st} v(t,x) \underset{t\to\infty}{\longrightarrow} 1, \quad \text{for all } s < 2,$$

and

$$\sup_{x\geq st} v(t,x) \underset{t\to\infty}{\longrightarrow} 0, \quad \text{for all } s > 2.$$

Here $s_* = 2$ is also the minimal speed of monotone traveling wave solutions having the form $v(t,x) = \varphi(x - st)$ and satisfying

$$\varphi'' + s\varphi' + \varphi(1-\varphi) = 0 \text{ in } \mathbb{R}, \quad \varphi(-\infty) = 1, \quad \varphi(+\infty) = 0. \tag{1.4}$$

In fact, one can show [2, 10] that there exists a constant $x_\infty$ depending on the initial condition only such that $\kappa(t)$ has the following asymptotics

$$\kappa(t) = 2t - \frac{3}{2}\ln t + x_\infty + \varphi_*^{-1}(1/2) - \frac{2\sqrt{\pi}}{\sqrt{t}} + \mathrm{O}\left(\frac{1}{t^{1-\gamma}}\right), \tag{1.5}$$

as $t \longrightarrow +\infty$, for any $\gamma > 0$ and $\varphi_*$ denotes the critical traveling front solution of (1.4) at $s_* = 2$.

**The uncoupled case.** We now turn our attention to the $u$ component. We first look at the decoupled case $\beta = 0$. The $u$ equation in isolation shares many properties with the Fisher-KPP equation (1.3). More specifically, from [1], we know the existence of a spreading speed $s_0 \geq 2\sqrt{df'(0)}$ in the sense explained above:

$$\inf_{x\leq st} u(t,x) \underset{t\to\infty}{\longrightarrow} 1, \quad \text{for all } s < s_0,$$
$$\sup_{x\geq st} u(t,x) \underset{t\to\infty}{\longrightarrow} 0, \quad \text{for all } s > s_0.$$

In general, there is no explicit expression for the spreading speed $s_0$ with the hypothesis that $f$ is monostable only. However, this speed is linearly determined if $f$ is of Fisher-KPP type, that is $f$ monostable (see (1.2)) and

$$f(u) \leq f'(0)u, \text{ for all } 0 \leq u \leq 1. \tag{1.6}$$

In that case, given our initial condition, the component $u$ will spread at speed $s_0 = 2\sqrt{df'(0)}$ as it is the case for equation (1.3) where $d = f'(0) = 1$. Thus, for Fisher-KPP type nonlinearity, the selected speed depends on $f$ only through its derivative at $u = 0$. From now on we denote $\alpha := f'(0)$.

**The coupled case – Main result.** We now consider (1.1) in the coupled case $\beta > 0$, with $f$ of Fisher-KPP type satisfying (1.2) and (1.6). At first, one could expect that the selected speed for the $u$ component is given by $\max(2, 2\sqrt{d\alpha})$. For instance, if we consider values of $(d, \alpha)$ such that $2\sqrt{d\alpha} > 2$, and place ourselves in a window $y = x - 2\sqrt{d\alpha}t$, the $v$ component will converge to zero as $t \to \infty$, locally uniformly in $y$. Thus one could think that replacing $v$ by zero in the $u$ equation would give the spreading speed. This turns out to not always be the case, and there exists a domain for our parameters that leads to a selected speed strictly superior than $\max(2, 2\sqrt{d\alpha})$. This phenomenon was first observed in [11] and given the label of anomalous spreading, and rigorously studied in [7, 8]. In fact, we are going to prove the following theorem.

**Theorem 1.** *Consider (1.1) with $f$ of Fisher-KPP type satisfying (1.2) & (1.6) and with $d, \beta, p > 0$ and $\alpha = f'(0) > 0$. Fix initial data $0 \le u(0, x) \le 1$ and $0 \le v(0, x) \le 1$, each consisting of a compactly supported perturbation of the Heaviside step function $\mathbf{1}_{x \le 0}$. Then, there exist domains $\mathrm{I}, \mathrm{II}, \mathrm{III}$, depending on $p$, so that the selected speed $s_{\mathrm{sel}}(p)$ is given by*

$$s_{\mathrm{sel}}(p) = \begin{cases} 2 & , (d, \alpha) \in \mathrm{I}, \\ 2\sqrt{d\alpha} & , (d, \alpha) \in \mathrm{II}, \\ s_{\mathrm{anom}}(d, \alpha, p) & , (d, \alpha) \in \mathrm{III}, \end{cases}$$

*with*

$$s_{\mathrm{anom}}(d, \alpha, p) = \sqrt{\frac{\alpha - p}{p - dp^2}} + \sqrt{\frac{p - dp^2}{\alpha - p}}, \tag{1.7}$$

*and*

$$\begin{aligned}
\mathrm{I} &= \left\{ \alpha \le p(2 - dp) \mid d \le \frac{1}{p} \right\} \cup \left\{ \alpha \le \frac{1}{d} \mid d > \frac{1}{p} \right\}, \\
\mathrm{II} &= \left\{ \alpha \ge \frac{dp^2}{2dp - 1} \mid \frac{1}{2p} < d \le \frac{1}{p} \right\} \cup \left\{ \alpha \ge \frac{1}{d} \mid d > \frac{1}{p} \right\}, \\
\mathrm{III} &= \left\{ \alpha > p(2 - dp) \mid d < \frac{1}{2p} \right\} \cup \left\{ p(2 - dp) < \alpha < \frac{dp^2}{2dp - 1} \mid \frac{1}{2p} < d \le \frac{1}{p} \right\}.
\end{aligned}$$

Let first note that the system (1.1) has already been studied in the case $f(u) = \alpha u(1 - u)$, with $p = 1$ [7, 8] or $p = 2$ [4] and Theorem 1 is a natural generalization of those studies to $p > 0$. It is also important to remark that for $(d, \alpha) \in \mathrm{III}$ we have $s_{\mathrm{anom}}(d, \alpha, p) > \max(2, 2\sqrt{d\alpha})$ and in that respect $s_{\mathrm{anom}}$ is referred to as an anomalous spreading speed. We will see that it is the coupling $\beta v^p$ into the $u$ component of system (1.1) that induces a resonance in the dynamics leading to this anomalous spreading speed. We refer to Figure 2.2 for an illustration of the different domains defined in Theorem 1. It is interesting to note that as $p \to +\infty$, the domain III of existence of the anomalous speed shrinks as it is shifted close to axis $d = 0$ where it imposes large values for $\alpha$ as we have $\alpha \ge p(2 - dp)$ in that region. On the other hand, when $p \to 0^+$, the domain of existence of the anomalous speed becomes larger and eventually covers the whole quadrant $\alpha > 0$ and $d > 0$. In that respect, small values of $p$ enhance anomalous spreading.

**Strategy of the proof.** Contrary to the case $p = 1$ [7, 8], when $p \neq 1$ a linearization around the equilibrium state $(u, v) = (0, 0)$ is either impossible, or decouples the system. However, using the fact that $f$ is of Fisher-KPP type and that $\beta v^p(1-u) \leq \beta v^p$ whenever $v \geq 0$ and $u \geq 0$, we obtain the following system

$$\begin{cases} u_t = du_{xx} + \alpha u + \beta v^p & , t > 0,\, x \in \mathbb{R}, \\ v_t = v_{xx} + v & , t > 0,\, x \in \mathbb{R}, \end{cases} \tag{1.8}$$

which will serve as a natural super-system for (1.1). It turns out that a thorough study of (1.8) will help us:

- predict the selected speed for system (1.1) in the spirit of the approach presented for the case $p = 2$ in [4];
- devise elementary exponential solutions which will serve in the construction of sub- and super-solutions for (1.1).

The proof of Theorem 1 relies on the fact that each component of (1.1) satisfies the comparison principle, allowing us to apply the theory of sub- and super-solutions [3]. In fact, for each domain, we will explicitly construct sub- and super-solutions from which we will deduce Theorem 1.

**Application – Monostable nonlinearities.** As it will be clear in the proof of Theorem 1, the fact that the nonlinearity $f$ is of Fisher-KPP type plays a crucial role in our analysis. However, it turns out that Theorem 1 still provides valuable insights when considering nonlinearities that are only monostable (see (1.2)). More specifically, we will see that anomalous spreading speeds do occur for this type of nonlinearities and we will apply our results to a specific example. This is a new development compared to the original studies of [7, 8].

**Outline.** The outline of this paper is as follows. In Section 2 we determine the expression of the selected speed $s_{\text{sel}}(p)$ for any $p > 0$. By doing so, we highlight the existence of an anomalous speed depending on parameters $d, \alpha, p$. Section 3 is devoted to the proof of Theorem 1. Finally, in Section 4 we relax the Fisher-KPP condition and we consider a particular example of (1.1) for which we establish the existence of an anomalous speed on a particular domain of parameters.

## 2 Spreading speeds for system (1.1)

This section is devoted to the existence of a possible anomalous spreading speed for system (1.1). More specifically, we will explain how such a speed can be computed. We first start by explaining the case $p = 1$ for which a linearization around the equilibrium state $(u, v) = (0, 0)$ makes sense. Then, following the approach presented in [4], we derive the formula (1.7) for the anomalous spreading speed together with the domains I, II and III appearing in Theorem 1.

### 2.1 Study of (1.8) when $p = 1$

We place ourselves at $p = 1$. In that case system (1.8) is precisely the linearized form of (1.1) around the equilibrium state $(u = 0, v = 0)$, and in a moving frame $y = x - st$, it reads

$$\begin{cases} u_t = du_{yy} + su_y + \alpha u + \beta v & , t > 0, \, y \in \mathbb{R}, \\ v_t = v_{yy} + sv_y + v & , t > 0, \, y \in \mathbb{R}. \end{cases} \tag{2.1}$$

In the decoupled case $\beta = 0$, elementary solutions of (2.1) are exponentials of the form given by

$$u(t, y) = e^{\lambda t} \left( C_1 e^{\nu_u^+(s,\lambda) y} + C_2 e^{\nu_u^-(s,\lambda) y} \right),$$
$$v(t, y) = e^{\lambda t} \left( C_1 e^{\nu_v^+(s,\lambda) y} + C_2 e^{\nu_v^-(s,\lambda) y} \right),$$

with $\nu_u^\pm(s, \lambda)$ and $\nu_v^\pm(s, \lambda)$ roots of the dispersion relation for the $u$ and $v$ equations respectively :

$$D_u(\nu) := d\nu^2 + s\nu + \alpha - \lambda = 0,$$
$$D_v(\nu) := \nu^2 + s\nu + 1 - \lambda = 0.$$

When looking at the coupled case $\beta \neq 0$ with initial conditions $\mathbf{1}_{x \le 0}(x)$, it has been shown in [7] that the spreading speed of (1.8) can be inferred from the analyticity, or lack thereof, of the pointwise Green's function associated to system (2.1). It is obtained after a Laplace transform in time with parameter $\lambda \in \mathbb{C}$ and when considering delta Dirac initial conditions. The skew-product nature of the coupling implies that the dispersion relation of the full system (2.1) is the product of the dispersion relations $D_u$ and $D_v$. Non-removable singularities of the Green function appear for values of $s, \lambda$ such that the full dispersion relation admits pinched double roots, that is when one of these relations holds

$$\nu_u^+(s, \lambda) = \nu_u^-(s, \lambda), \tag{2.2a}$$
$$\nu_v^+(s, \lambda) = \nu_v^-(s, \lambda), \tag{2.2b}$$
$$\nu_u^\pm(s, \lambda) = \nu_v^\mp(s, \lambda). \tag{2.2c}$$

Then the spreading speed of (2.1) is exactly the minimal value of $s$ such that those singularities are all located in the stable half plane $\mathrm{Re}(\lambda) \le 0$, that is :

$$s_{\text{lin}} := \sup \left\{ s > 0 \,|\, \text{all couples } (s, \lambda) \text{ solutions of (2.2) satisfy } \mathrm{Re}(\lambda) > 0 \right\}.$$

We refer to it as the linear spreading speed. In fact, solving (2.2) one obtains the following expression for the linear spreading speed [7]:

$$s_{\text{lin}} = \begin{cases} 2 & , \alpha \le 2 - d, \\ 2\sqrt{d\alpha} & , d > \frac{1}{2} \text{ and } \alpha \ge \frac{d}{2d-1}, \\ s_{\text{anom}} = \sqrt{\frac{\alpha-1}{1-d}} + \sqrt{\frac{1-d}{\alpha-1}} & , \text{otherwise}. \end{cases}$$

Due to the fact that $f$ is of Fisher-KPP type, this linear spreading speed usually provides a good predictor for the selected speed of the nonlinear system. A more thorough study [8] allows one to show that the selected speed for the nonlinear system (1.1) is equal to the linear spreading speed except on the domain $\mathrm{IV} = \left\{ d > 1 \, , \, \max(2 - d, 0) < \alpha < \frac{d}{2d-1} \right\}$, where the selected speed is equal to $\max(2, 2\sqrt{d\alpha})$, and one recovers precisely the statement of Theorem 1 with $p = 1$. What is crucial here is that all values taken by the selected speed of (1.1) are captured by the linear speeds of (2.1). In that respect, the system (1.1) when $p = 1$ is said to be linearly determined.

## 2.2 An heuristic approach

When $p \neq 1$, we have already explained that either a linearization is impossible or it decouples the system. Nevertheless, we would like to proceed along similar lines as in the case $p = 1$, and it turns out that system (1.8) is a natural "generalization" of the linearized system. Therefore, we rewrite (1.8) in a moving frame $y = x - st$ and obtain

$$\begin{cases} u_t = du_{yy} + su_y + \alpha u + \beta v^p & , t > 0,\ y \in \mathbb{R}, \\ v_t = v_{yy} + sv_y + v & , t > 0,\ y \in \mathbb{R}. \end{cases} \tag{2.3}$$

Our goal here is to mimic the step that leads us to compute the linear spreading speed $s_{\text{lin}}$ of system (2.1). This heuristic approach on system (2.3) will give us an educated guess on the expressions of domains I, II, III and the expression of $s_{\text{anom}}$ in the general case $p > 0$. And then, in the next section we will provide a theoretical proof of our guess using techniques of sub- and super-solutions.

First, recall that when $p = 1$, the spreading speed can be computed by looking at the resonance between decay rates $\nu^{\pm}_{u,v}(s, \lambda)$ corresponding to equations $u$ and $v$ isolated. If one considers exponential solutions of the form

$$u(t, y) = e^{\Lambda t} e^{\nu_u(s,\Lambda) y},$$
$$v(t, y) = e^{\lambda t} e^{\nu_v(s,\lambda) y},$$

then in order for those functions to satisfy (2.3) we necessarily need

$$\Lambda = p\lambda,$$
$$\nu_u(s, \Lambda) = p\nu_v(s, \lambda).$$

The heuristic is the following : for fixed values of $(d, \alpha, p)$, we seek the couples $(s, \lambda)$ solutions of any of the four equations

$$\nu^+_u(s, \lambda) = \nu^-_u(s, \lambda), \tag{2.4a}$$
$$\nu^+_v(s, \lambda) = \nu^-_v(s, \lambda), \tag{2.4b}$$
$$\nu^{\pm}_u(s, p\lambda) = p\nu^{\mp}_v(s, \lambda), \tag{2.4c}$$

and we want to find the value of the speed

$$s_{\text{lin}}(p) = \sup \{ s > 0 \,|\, \text{all couples } (s, \lambda) \text{ solutions of (2.4) satisfy } \text{Re}(\lambda) > 0 \}. \tag{2.5}$$

We will call that speed the linear speed despite (2.3) not being linear. That is because it will play a similar role of predictor for the selected speed of the nonlinear system, just like the case $p = 1$. Obviously, $s_{\text{lin}}(1) = s_{\text{lin}}$.

**Remark 2.1.** *It is important to note that when $p = 2$, we recover the "2 : 1- resonant spreading speed" from [4]. Actually, for any $p \geq 1$ being an integer, the spreading speed (2.5) can be interpreted as a "$p$ : 1- resonant spreading speed". However, for general $p > 0$ we could not use the definition [4, Definition 2.1] and this is why we proposed the natural generalization (2.5).*

### 2.3 Expression of the spreading speed $s_{\text{lin}}(p)$

In this section, we prove the following result which is a first step in proving our main Theorem 1.

**Proposition 2.2.** *The linear speed defined by (2.5) is given by*

$$s_{\text{lin}}(p) = \begin{cases} 2 & , \alpha \leq 2p - dp^2, \\ 2\sqrt{d\alpha} & , d > \frac{1}{2p} \text{ and } \alpha \geq \frac{dp^2}{2dp-1} , \\ s_{\text{anom}}(d, \alpha, p) & , \text{otherwise}, \end{cases} \tag{2.6}$$

*with*

$$s_{\text{anom}}(d, \alpha, p) = \sqrt{\frac{\alpha - p}{p - dp^2}} + \sqrt{\frac{p - dp^2}{\alpha - p}}.$$

**Proof.** We solve explicitly each equation. Solutions of (2.4a) and (2.4b) are respectively

$$\left\{ \left( s, \lambda = \alpha - \frac{s^2}{4d} \right) \mid s > 0 \right\}, \text{ and } \left\{ \left( s, \lambda = 1 - \frac{s^2}{4} \right) \mid s > 0 \right\}.$$

Thus it requires $s_{\text{lin}}(p) \geq \max(2, 2\sqrt{d\alpha})$. Equations (2.4c) involve more work. We first consider the particular case $d = 1/p$. One checks it implies that $\alpha = p$ and the solutions are

$$\left\{ \left( s, \lambda = 1 - \frac{s^2}{4} \right) \mid s > 0 \right\},$$

meaning that the case $d = 1/p$ does not impose more restrictive conditions for $s_{\text{lin}}(p)$ than the ones previously derived. From now on, we suppose that $d \neq 1/p$, and we define

$$X := \frac{\alpha - dp^2}{p - dp^2}, \qquad Y := \frac{\alpha - p}{p - dp^2}.$$

Using these notations, we obtain a parametrization of the solutions of (2.4c) as a function of the speed $s$, namely two curves in the complex plane

$$\lambda_{\pm}(s) = X \pm s\sqrt{Y}. \tag{2.7}$$

More precisely, couples $(s, \lambda_{\pm}(s))$ are exactly the solutions of one of the four equations

$$\begin{aligned} \nu_u^{\pm}(s, p\lambda) &= p\nu_v^{\pm}(s, \lambda), \\ \nu_u^{\pm}(s, p\lambda) &= p\nu_v^{\mp}(s, \lambda). \end{aligned}$$

Recall that only (2.4c) leads to an anomalous speed, from an heuristical point of view. From there, one can in fact compute explicitly the values $\nu_{u,v}^{\pm}(s, \lambda)$ when $\lambda$ satisfies (2.7),

$$\begin{aligned} \nu_u^{+}(s, p\lambda_{\pm}(s)) &= -\frac{s}{2d} + \frac{1}{2d}\sqrt{s^2 - 4d\alpha + 4dp\left(\frac{\alpha - dp^2}{p - dp^2} \pm s\sqrt{Y}\right)}, \\ &= -\frac{s}{2d} + \frac{1}{2d}\sqrt{s^2 \pm 4dp\sqrt{Y}s + 4d^2p^2\left(\frac{\alpha - p}{p - dp^2}\right)}, \\ &= -\frac{s}{2d} + \sqrt{\left(\frac{s}{2d}\right)^2 \pm \frac{p}{d}\sqrt{Y}s + p^2Y}, \\ &= -\frac{s}{2d} + \sqrt{\left(\frac{s}{2d} \pm p\sqrt{Y}\right)^2}, \end{aligned}$$

and as a consequence, we have

$$\nu_u^+(s,p\lambda_\pm(s)) = \begin{cases} \pm p\sqrt{Y} & \text{, if } \mathrm{Re}\left(\frac{s}{2d} \pm p\sqrt{Y}\right) \geq 0, \\ -\frac{s}{d} \mp p\sqrt{Y} & \text{, if } \mathrm{Re}\left(\frac{s}{2d} \pm p\sqrt{Y}\right) \leq 0. \end{cases}$$

Note that we used the convention that a square root of a complex number $z$ is the only complex number $\zeta$ which satisfies $\zeta^2 = z$ and $\arg(\zeta) \in (-\frac{\pi}{2}, \frac{\pi}{2}]$ if $\zeta \neq 0$. One can do the same calculation for all the roots. We obtain the following expressions, depending on $Y' := \mathrm{Re}(\sqrt{Y}) \in \mathbb{R}_+$,

$$\nu_u^+(s,p\lambda_\pm(s)) = \begin{cases} -\frac{s}{d} \mp p\sqrt{Y} & \text{, if } s \leq \mp 2dpY', \\ \pm p\sqrt{Y} & \text{, if } s \geq \mp 2dpY', \end{cases} \quad \nu_u^-(s,p\lambda_\pm(s)) = \begin{cases} \pm p\sqrt{Y} & \text{, if } s \leq \mp 2dpY', \\ -\frac{s}{d} \mp p\sqrt{Y} & \text{, if } s \geq \mp 2dpY', \end{cases}$$

and

$$p\nu_v^+(s,\lambda_\pm(s)) = \begin{cases} -sp \mp p\sqrt{Y} & \text{, if } s \leq \mp 2Y', \\ \pm p\sqrt{Y} & \text{, if } s \geq \mp 2Y', \end{cases} \quad p\nu_v^-(s,\lambda_\pm(s)) = \begin{cases} \pm p\sqrt{Y} & \text{, if } s \leq \mp 2Y', \\ -sp \mp p\sqrt{Y} & \text{, if } s \geq \mp 2Y'. \end{cases}$$

From there, one can directly solve (2.4c) for each $\lambda(s) = \lambda_\pm(s)$. The case $Y \leq 0$ implies $Y' = 0$, and leads to only one solution, $(s,\lambda) = (0, X)$. As we already have $s_{\mathrm{lin}}(p) \geq \max(2, 2\sqrt{d\alpha}) > 0$, there is no additional condition on $s_{\mathrm{lin}}(p)$ if $Y \leq 0$. Thus $s_{\mathrm{lin}}(p) = \max(2, 2\sqrt{d\alpha})$ on $\{Y \leq 0\}$.

We now restrict ourselves to the domain

$$\{Y > 0\} = \{\alpha > p\,,\, d < 1/p\} \cup \{\alpha < p\,,\, d > 1/p\}.$$

Note that it implies $Y' > 0$. The couples $(s, \lambda_\pm(s))$ play symmetric roles regarding the sign of $s$. In fact, a necessary condition for $(s, \lambda_+(s))$ to be solution of (2.4c) is $s < 0$. Thus it has no impact on $s_{\mathrm{lin}}(p)$. However, note that if we took initial conditions of the form $\mathbf{1}_{x\geq 0}$, we would propagate to the left at nonpositive speed and we would exclude $(s, \lambda_-(s))$ instead. Consider now the couple $(s, \lambda_-(s))$. We find the following solutions,

$$\left\{(s, X - s\sqrt{Y}) \,|\, s \in [2dpY'; 2Y']\right\} \quad \text{, if } Y > 0 \text{ and } d < 1/p, \tag{2.8}$$

$$\left\{(s, X - s\sqrt{Y}) \,|\, s \in [2Y'; 2dpY']\right\} \quad \text{, if } Y > 0 \text{ and } d > 1/p, \tag{2.9}$$

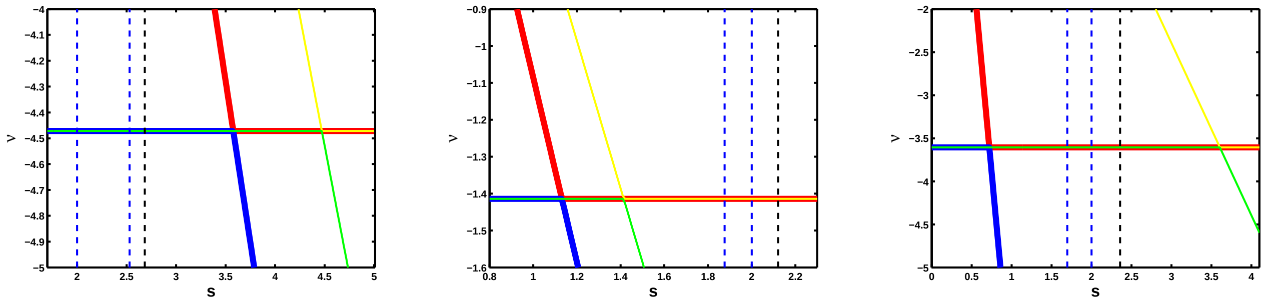
with (2.8) being solution of $\nu_u^+(s, p\lambda_-(s)) = p\nu_v^-(s, p\lambda_-(s))$ and (2.9) being solution of $\nu_u^-(s, p\lambda_-(s)) = p\nu_v^+(s, p\lambda_-(s))$.

Note that

$$\mathrm{Re}(\lambda_-(s)) \leq 0 \Leftrightarrow s \geq s_{\mathrm{anom}} := \frac{X}{\sqrt{Y}} = \sqrt{Y} + \frac{1}{\sqrt{Y}}.$$

At this point, we cannot conclude yet that $s_{\mathrm{lin}}(p) \geq s_{\mathrm{anom}}$ whenever $Y > 0$. Indeed, for either (2.8) or (2.9) to be satisfied there are three possibilities, depending on the values $(d, \alpha, p)$:

- If we have $s_{\mathrm{anom}} \leq \min(2dpY', 2Y')$, then the couples $(s, \lambda_-(s))$ already satisfy $\mathrm{Re}(\lambda_-(s)) \leq 0$. That is, there is no additional condition for $s_{\mathrm{lin}}(p)$ (see Figure 1(a)). Hence $s_{\mathrm{lin}}(p) = \max(2, 2\sqrt{d\alpha})$.




(a) $(d, \alpha, p) = (0.4, 4, 2)$ (b) $(d, \alpha, p) = (0.4, 2.2, 2)$ (c) $(d, \alpha, p) = (0.1, 7.2, 2)$

Figure 2.1: *Variations of the roots* $\nu^{\pm}_{u,v}$ *with respect to* $s$*. In red,* $\nu_u^+(s, p\lambda_-(s))$*, in blue,* $\nu_u^-(s, p\lambda_-(s))$*, in green,* $p\nu_v^-(s, \lambda_-(s))$*, in yellow* $p\nu_v^+(s, \lambda_-(s))$*. The two blue dotted lines represent* $s = 2$ *and* $s = 2\sqrt{d\alpha}$*. The black one represents* $s = s_{\text{anom}}$*. We have* $\nu_u^+ = p\nu_v^-$ *or* $\nu_u^- = p\nu_v^+$ *only if* $s \in \mathcal{I} := [\min(2dpY', 2Y'), \max(2dpY', 2Y')]$*. On each figure,* $\mathcal{I}$ *is precisely the intersection of the red and the green curves. In (a) and (b), we have that* $s = s_{\text{anom}}$ *does not intersect the segment* $\mathcal{I}$*, so that no we do not have additional conditions on* $s_{\text{lin}}$*. In (c),* $s = s_{\text{anom}}$ *intersects* $\mathcal{I}$ *and we need to have* $s_{\text{lin}} \geq s_{\text{anom}}$ *and an anomalous speed can appear.*

- If we have $s_{\text{anom}} \geq \max(2dpY', 2Y')$, then we are then forced to have $s_{\text{lin}}(p) \geq \max(2dpY', 2Y')$, but a stronger condition is not needed, since beyond that point $(s, \lambda_-(s))$ is not a solution anymore. Using the fact that
$$s_{\text{anom}} = \sqrt{Y} + \frac{1}{\sqrt{Y}} = dp\sqrt{Y} + \frac{\alpha}{p}\sqrt{\frac{1}{Y}},$$
the condition $s_{\text{anom}} \geq \max(2dpY', 2Y')$ can be rewritten as $Y \leq \min(1, \alpha/dp^2)$. Thus $2Y' \leq 1$ and $2dpY' \leq 2\sqrt{d\alpha}$. This leads to the condition $s_{\text{lin}}(p) \geq \max(2, 2\sqrt{d\alpha})$, thus does not imply an anomalous speed (Figure 1(b)).

- If we have $\min(2dpY', 2Y') < s_{\text{anom}} < \max(2dpY', 2Y')$, then we are forced to have $s_{\text{lin}}(p) \geq s_{\text{anom}}$ (see Figure 1(c)). This is the only case where an anomalous speed is susceptible to appear.

As a consequence, we place ourselves in the last case. First, we check that $s_{\text{anom}}$ is indeed an anomalous speed. One can check that for every $(d, \alpha) \in \{Y > 0\}$ we have $s_{\text{anom}} \geq \max(2, 2\sqrt{d\alpha})$. Moreover, we can prove the following facts :

$$s_{\text{anom}} = 2 \Leftrightarrow Y = 1 \Leftrightarrow \alpha = 2p - dp^2 \Leftrightarrow s_{\text{anom}} = 2Y', \tag{2.10}$$

$$s_{\text{anom}} = 2\sqrt{d\alpha} \Leftrightarrow Y = \frac{\alpha}{dp^2} \Leftrightarrow \left(d > \frac{1}{2p} \text{ and } \alpha = \frac{dp^2}{2dp - 1}\right) \Leftrightarrow s_{\text{anom}} = 2dpY'. \tag{2.11}$$

This means that in the last case, we have $s_{\text{anom}} > \max(2, 2\sqrt{d\alpha})$, thus an anomalous speed. This also proves that $s_{\text{lin}}(p)$ given by (2.6) is a continuous expression with respect to parameters $d, \alpha, p$.

Now we want to know for which values of $(d, \alpha)$ we are in the last case. Note that it happens if and only if $(s_{\text{anom}}, \lambda_-(s_{\text{anom}}))$ is a solution of (2.8) or (2.9), and $s_{\text{anom}} \notin \{2Y', 2dpY'\}$. Thus it is equivalent to find the couples $(d, \alpha)$ for which

$$\nu_u^+(s_{\text{anom}}, p\lambda_-(s_{\text{anom}})) = p\nu_v^-(s_{\text{anom}}, \lambda_-(s_{\text{anom}})),$$

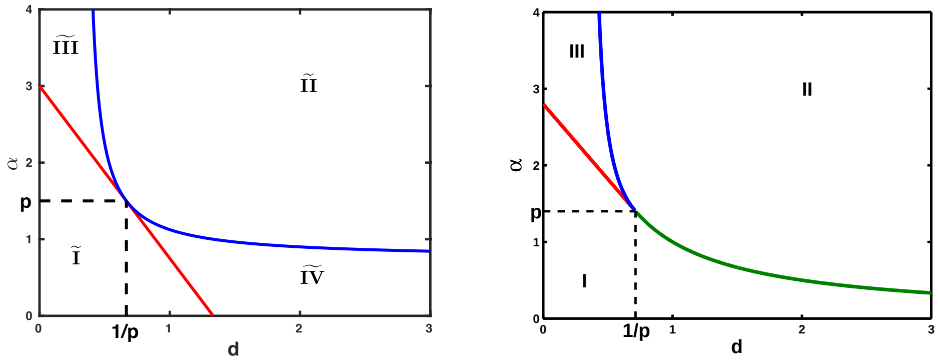



Figure 2.2: *Differences between the speeds $s_{\text{lin}}(p)$ (left) and $s_{\text{sel}}^{\text{KPP}}(p)$ (right) with $p = 1.5$. On the domains I and $\widetilde{I}$ both speeds are equal to 2. On the domains II and $\widetilde{II}$ both speeds are equal to $2\sqrt{d\alpha}$. On domains $\widetilde{III}$, $\widetilde{IV}$ and III both speeds are equal to $s_{\text{anom}} > \max(2, 2\sqrt{d\alpha})$. Notice that when considering the selected speed $s_{\text{sel}}^{\text{KPP}}(p)$ for the full system (1.1), the domain $\widetilde{IV}$ disappears, and in that region of parameters the selected speed is simply equal to* $\max(2, 2\sqrt{d\alpha})$.

$$\nu_u^-(s_{\text{anom}}, p\lambda_-(s_{\text{anom}})) = p\nu_v^+(s_{\text{anom}}, \lambda_-(s_{\text{anom}})),$$

and remove from our set of solutions the points which satisfy $s_{\text{anom}} \in \{2Y', 2dpY'\}$, that is the two curves $(d, 2p - dp^2)$ and $(d, \frac{dp^2}{2dp-1})$ in the first quadrant of the $(d, \alpha)$ plane, according to (2.10) & (2.11). As a consequence, if we set

$$\widetilde{\text{I}} := \left\{\alpha \leq 2p - dp^2\right\}, \quad \widetilde{\text{II}} := \left\{d > \frac{1}{2p}\,,\, \alpha \geq \frac{dp^2}{2dp-1}\right\},$$

together with

$$\widetilde{\text{III}} := \left\{d < \frac{1}{p}\,,\, 2p - dp^2 \leq \alpha\right\} \cap \left\{\frac{1}{2p} < d < \frac{1}{p}\,,\, \alpha \leq \frac{dp^2}{2dp-1}\right\},$$
$$\widetilde{\text{IV}} := \left\{\frac{1}{p} < d\,,\, 2p - dp^2 \leq \alpha \leq \frac{dp^2}{2dp-1}\right\}.$$

then $s_{\text{lin}}(p) = s_{\text{anom}}$ precisely on domains $\widetilde{\text{III}}$ and $\widetilde{\text{IV}}$, while we have $s_{\text{lin}}(p) = 2$ on domain $\widetilde{\text{I}}$ and $s_{\text{lin}}(p) = 2\sqrt{d\alpha}$ on domain $\widetilde{\text{II}}$. This ends the proof of Proposition 2.2. ■

As we have already mentioned, the computation of the linear spreading speed given by Proposition 2.2 will help us get an estimate of the selected speed $s_{\text{sel}}^{\text{KPP}}(p)$ for system (1.1). In the following section, we will prove that the spreading speed of the nonlinear system $s_{\text{sel}}^{\text{KPP}}(p)$ is equal to the linear spreading speed $s_{\text{lin}}(p)$ for $(d, \alpha)$ in domains $\widetilde{\text{I}}, \widetilde{\text{II}}, \widetilde{\text{III}}$, but is equal to $\max(2, 2\sqrt{d\alpha})$ for $(d, \alpha)$ in $\widetilde{\text{IV}}$. For this reason, we shall call $\widetilde{\text{III}}$ the relevant domain, with $\nu_u^+, \nu_v^-$ associated relevant double roots, and $\widetilde{\text{IV}}$ the irrelevant domain, with $\nu_u^-, \nu_v^+$ associated irrelevant double roots. We illustrate the differences between $s_{\text{lin}}(p)$ and $s_{\text{sel}}^{\text{KPP}}(p)$ in Figure 2.2.

# 3 Proof of Theorem 1

In order to prove Theorem 1, we will construct explicit sub- and super-solutions in every domain $\widetilde{\text{I}}$-$\widetilde{\text{IV}}$. Some of these have already been done in [7, 8] in the case $p = 1$ and $f(u) = \alpha u(1 - u)$. In that event, we will only explain how to adapt the proof in the general case. For the next two sections, given $v$, we define for any function $u$

$$N(u) := u_t - du_{xx} - f(u) - \beta v^p(1-u).$$

Let us already recall that whenever $v \geq 0$ and $u \geq 0$, we have

$$N(u) \geq u_t - du_{xx} - \alpha u - \beta v^p.$$

Finally, throughout the sequel we will simply denote $s_{\text{sel}}$ instead of $s_{\text{sel}}(p)$.

### 3.1 Super-solutions

For the sake of readability, we will write $\nu_{u,v}(s)$, and if not confusing only $\nu_{u,v}$, instead of $\nu_{u,v}(s,0)$. Furthermore, for this section, we define slightly different domains in the first quadrant of the $(d,\alpha)$ plane

$$\widehat{\text{I}} := \left\{\alpha < 2p - dp^2\right\},\ \widehat{\text{II}} := \left\{d > \frac{1}{2p},\ \alpha > \frac{dp^2}{2dp-1}\right\},\ \widehat{\text{III}} := \left\{d < \frac{1}{p}\right\} \backslash (\widehat{\text{I}} \cup \widehat{\text{II}}),\ \widehat{\text{IV}} := \left\{d > \frac{1}{p}\right\} \backslash (\widehat{\text{I}} \cup \widehat{\text{II}}),$$

and the point $\widehat{\text{V}} := \left\{d = \frac{1}{p}, \alpha = p\right\}$.

#### 3.1.1 Domain $\widehat{\text{IV}}$

**Lemma 3.1.** *For any $(d,\alpha) \in \widehat{\text{IV}}$, we have $s_{\text{sel}} \leq \max(2, 2\sqrt{d\alpha})$.*

**Proof.** A similar proof was done in [7] in the case $p = 1$. However, to give the reader more insight about how we construct those super-solutions which will be used later on, we write it all here.

We consider $s > \max(2, 2\sqrt{d\alpha})$ and $(d,\alpha) \in \widehat{\text{IV}}$. Then for any $C_v > 0$, a super-solution for the $v$ equation is given by

$$\overline{v}(t,x) = \min\left(1, C_v e^{\nu_v^-(s)(x-st)}\right).$$

Notice that $\overline{v}$ changes its expression at the point $y_v = \frac{1}{\nu_v^-(s)} \log\left(\frac{1}{C_v}\right)$ in the moving frame $y = x - st$. It is also a sub-solution at $y = y_v$ since

$$0 = \lim_{y \to y_v^-} \partial_y \overline{v} > \lim_{y \to y_v^+} \partial_y \overline{v}.$$

We then take $C_v > 0$ large enough so that $\overline{v}(0,x) \geq v(0,x)$. Thus $v(t,x) \leq \overline{v}(t,x)$ for all $(t,x)$.

We now turn our attention to the $u$ component and construct a similar super-solution. We seek $\overline{u}(t,x)$ so that $N(\overline{u}) \geq 0$ for all $t > 0$ and $x \in \mathbb{R}$. We claim that for any $C_v > 0$, we can find $C_u^*(C_v) > 0$ and $\tau(C_u, C_v)$ so that

$$\overline{u}(t,x) = \begin{cases} 1 & , x - st \leq \tau, \\ C_u e^{\nu_u^-(x-st)} + C_v^p \kappa e^{p\nu_v^-(x-st)} & , x - st > \tau, \end{cases}$$

is a super-solution whenever $C_u > C_u^*$. One easily verifies that $\overline{u}$ is a super-solution when $x - st \leq \tau$. If $\tau$ is taken larger than $y_v$, then for $x - st > \tau$ we have the following inequality for $N(\overline{u})$, assuming $\overline{u} \in [0,1]$ :

$$N(\overline{u}) \geq \left[\overline{u}_t - d\overline{u}_{xx} - \alpha\overline{u} - \beta C_v^p e^{p\nu_v^-(x-st)}\right] + \beta(\overline{v}^p - v^p).$$

For $(d,\alpha) \in \widehat{\text{IV}}$, we have $p\nu_v^-(s) < \nu_u^-(s) < 0$, which implies that

$$D_u(p\nu_v^-) \;=\; d(p\nu_v^-)^2 + sp\nu_v^- + \alpha > 0.$$

If we now consider the equation

$$u_t = du_{xx} + \alpha u + \beta C_v^p e^{p\nu_v^-(x-st)},$$

we find a solution

$$\tilde{u}(t,x) = C_u e^{\nu_u^-(x-st)} + C_v^p \kappa e^{p\nu_v^-(x-st)},$$

with

$$\kappa = \frac{-\beta}{D_u(p\nu_v^-)} < 0.$$

Therefore, we have $N(\bar{u}) \geq \beta\,(\bar{v}^p - v^p)$ if $\tau > y_v$. On the other hand, the fact that $p\nu_v^- < \nu_u^-$ implies $\tilde{u}(t,x) > 0$ for $x$ sufficiently large. Moreover, $\tilde{u}(x - st)$ has a unique maximum at

$$\xi_{\max} = -\frac{1}{\nu_u^- - p\nu_v^-}\log\left(\frac{-C_u\nu_u^-}{C_v^p\kappa p\nu_v^-}\right).$$

From there, when $C_u \to +\infty$, we have $\xi_{\max} \to -\infty$. Thus there exists $C_u^*(C_v) > 0$ such that for any $C_u > C_u^*$, the following two conditions are satisfied: (i) $\xi_{\max} < y_v(C_v)$ and (ii) $\tilde{u}(t, y_v + st) > 1$. Then, as $\tilde{u}(t,x) \to 0^+$ as $x \to +\infty$, there exists $\tau(C_u, C_v) > y_v$ such that $\tilde{u}(t, \tau + st) = 1$. This guarantees the continuity of $\overline{u}$. Since $\tilde{u}_x < 0$ for $\xi > \xi_{\max}$, we have that $\tilde{u} < 1$ for $x - st > \tau$. Thus $\overline{u}$ propagates to the right with speed $s$.

In addition, for $\xi > \xi_{\max}$, we have $\tilde{u} > 0$. Indeed, if there exists $\xi_0 > \xi_{\max}$ such that $u(t, \xi_0 + st) = 0$, then it would be in contradiction with $\tilde{u}_x < 0$ and $\tilde{u}(t,x) \to 0^+$. Thus we also have $\overline{u} > 0$. As $\overline{v} > v$, we have that $N(\overline{u}) \geq 0$ for all $(t,x)$. It is also straightforward to check that $\overline{u}$ is also a super-solution on $x - st = \tau$.

Finally, given $u(0,x) = \mathbf{1}_{x\leq 0}$, we can take $C_u^*$ even larger to ensure that $\overline{u}(0,x) \geq u(0,x)$. Therefore we have $\overline{u}(t,x) \geq u(t,x)$ for any $t > 0$ and $x \in \mathbb{R}$, so the spreading speed of the $u$ component is bounded above by $s$. This construction holds for any $s > \max(2, 2\sqrt{d\alpha})$ and thus $s_{\text{sel}} \leq \max(2, 2\sqrt{d\alpha})$. ∎

### 3.1.2 Domain $\widehat{\text{II}}$

**Lemma 3.2.** *For any $(d,\alpha) \in \widehat{\text{II}}$, we have $s_{\text{sel}} \leq 2\sqrt{d\alpha}$.*

**Proof.** We can do the exact same proof as with $(d,\alpha) \in \widehat{\text{IV}}$, by considering $(d,\alpha) \in \widehat{\text{II}}$ and $s > 2\sqrt{d\alpha} > 2$ instead. We just have to make sure that in that domain $p\nu_v^-(s) < \nu_u^-(s)$, which will then imply that $D_u(p\nu_v^-) > 0$. Thus, in order to have the same proof, all that is needed is to have

$$p\nu_v^-(2\sqrt{d\alpha}) < \nu_u^-(2\sqrt{d\alpha}).$$

Indeed, if $s$ is taken close enough to $2\sqrt{d\alpha}$, we will have

$$p\nu_v^-(s) < \nu_u^-(s),$$

and the same proof can hold. We know that $\nu_u^-(2\sqrt{d\alpha}) = -\sqrt{\frac{\alpha}{d}}$. Thus we solve

$$\begin{aligned} p\nu_v^-(2\sqrt{d\alpha}) = -p\sqrt{d\alpha} - p\sqrt{d\alpha - 1} \quad &< \quad -\sqrt{\frac{\alpha}{d}}, \\ \sqrt{1 - \frac{1}{d\alpha}} \quad &> \quad \frac{1}{dp} - 1. \end{aligned}$$

One can check this is always true for $(d, \alpha) \in \widehat{\text{II}}$, and the proof is complete. ■

Note that the last inequality does not hold when $\alpha = 1/d = p$. This is partly the reason why we had to redefine our domains $\widehat{\text{I}}$-$\widehat{\text{V}}$.

### 3.1.3 Domain $\widehat{\text{III}}$

**Lemma 3.3.** *For any $(d, \alpha) \in \widehat{\text{III}}$, we have $s_{\text{sel}} \leq s_{\text{anom}}$.*

**Proof.** A similar proof was given in [8] with a coupling term $\beta v$ instead of $\beta v^p(1-u)$. Our super-solution will be similar to the one constructed for $(d, \alpha) \in \widehat{\text{IV}}$, and the proof is also simpler. For this reason we will briefly mention the different steps.

Consider $(d, \alpha) \in \widehat{\text{III}}$ and $s > s_{\text{anom}}$. Then we can choose $C_v > 0$ so that

$$\overline{v}(t, x) = \min\left(1, C_v e^{\nu_v^-(s)(x-st)}\right),$$

is a super-solution of the $v$ equation and satisfies $\overline{v}(t, x) \geq v(t, x)$ for all $t > 0$ and $x \in \mathbb{R}$. Notice that $\overline{v}$ changes its expression at the point $y_v = \frac{1}{\nu_v^-(s)} \log\left(\frac{1}{C_v}\right)$ in the moving frame $y = x - st$. Then, we can find $C_u^*(C_v)$ such that for all $C_u > C_u^*$ there exists a $\tau(C_u, C_v)$ for which

$$\overline{u}(t, x) = \begin{cases} 1 & , x - st \leq \tau, \\ C_u e^{\nu_u^+(s)(x-st)} + \kappa C_v^p e^{p\nu_v^-(s)(x-st)} & , x - st \geq \tau, \end{cases}$$

is a super-solution for the $u$ component, with

$$\kappa = \frac{-\beta}{D_u(p\nu_v^-)}.$$

Note that for $(d, \alpha) \in \widehat{\text{III}}$ we have $p\nu_v^-(s) < \nu_u^+(s)$ and $D_u(p\nu_v^-) < 0$, so that $\kappa > 0$. Then the sum of exponentials in $\overline{u}$ expression is a positive non increasing function that tends to infinity as $x - st \to -\infty$ and to zero as $x - st \to \infty$. Therefore we can choose $\tau$ as the unique value that makes $\overline{u}$ continuous. Then we can choose $C_u^*$ large enough so that for every $C_u > C_u^*$ we have both $\tau > y_v(C_v)$ and $\overline{u}(0, x) > u(0, x)$. In the end we have $u(t, x) \leq \overline{u}(t, x)$ for all $(t, x)$. As $\overline{u}$ propagates to the right at speed $s$ taken arbitrarily close to $s_{\text{anom}}$, we have $s_{sel} \leq s_{\text{anom}}$ for $(d, \alpha) \in \widehat{\text{III}}$. ■

### 3.1.4 Domain $\widehat{\text{I}}$

**Lemma 3.4.** *For any $(d,\alpha)\in\widehat{\text{I}}$, we have $s_{\text{sel}}\le 2$.*

**Proof.** We can do the exact same proof as for $(d,\alpha)\in\widehat{\text{III}}$, by considering $(d,\alpha)\in\widehat{\text{I}}$ and $s>2>2\sqrt{d\alpha}$ instead. We just have to make sure that $p\nu_v^-(s)<\nu_u^+(s)$ and $D_u(p\nu_v^-)<0$. All that is needed is to have

$$\nu_u^+(2)\ge p\nu_v^-(2)>\nu_u^-(2).$$

Indeed, using the fact that $\nu_u^+$ is increasing with $s$ and $\nu_v^-$ is decreasing with $s$, if $s$ is taken close enough to 2, we will have

$$\nu_u^+(s)>p\nu_v^-(s)>\nu_u^-(s),$$

and the same proof holds. We know that $p\nu_v^-(2)=-p$. Thus we solve

$$\begin{aligned}\nu_u^+(2)=\frac{1}{d}\left(-1+\sqrt{1-d\alpha}\right) &\ \ge\ -p,\\ \sqrt{1-d\alpha} &\ \ge\ 1-dp,\end{aligned}$$

$$\begin{aligned}\nu_u^-(2)=\frac{1}{d}\left(-1-\sqrt{1-d\alpha}\right) &\ <\ -p,\\ \sqrt{1-d\alpha} &\ >\ dp-1.\end{aligned}$$

One can check this is true when $(d,\alpha)\in\widehat{\text{I}}$, which ends the proof. ∎

**Remark 3.5.** *Note that a similar proof as the one for Lemma 3.4 does not hold for $(d,\alpha)\in\{d>1/p\,,\ \alpha=2p-dp^2\}$. Indeed, recall that for those values we have both $\nu_u^+(s_{\text{anom}})=p\nu_v^-(s_{\text{anom}})$ and $s_{\text{anom}}=2$. Thus we have $\nu_u^+(2)=p\nu_v^-(2)=\nu_u^-(2)$. Also, as $d>1/p$, we have $\alpha<1/d$ thus $2\sqrt{d\alpha}<2$. So we have*

$$\lim_{s\to2^+}\partial_s\nu_v^-(s)=\lim_{s\to2^+}\left(-\frac{1}{2}-\frac{1}{2}\frac{s}{\sqrt{s^2-4}}\right)=-\infty,$$

$$\partial_s\nu_u^-(2)=-\frac{1}{2d}-\frac{1}{2d}\frac{2}{\sqrt{4-4d\alpha}}>-\infty.$$

*As a consequence, we have $\nu_u^-(s)>p\nu_v^-(s)$ when $s\to2$. This is why we removed this curve from the domain $\widehat{I}$. However, note that this case was dealt within the proof of Lemma 3.1 on the domain $\widehat{IV}$.*

### 3.1.5 Point $\widehat{\text{V}}$

**Lemma 3.6.** *Let $d=\frac{1}{p}$ and $\alpha=p$, then we have $s_{\text{sel}}\le 2=2\sqrt{d\alpha}$.*

**Proof.** As $\alpha=1/d=p$, we have $\nu_u^\pm=p\nu_v^\pm$ for all $s$. Thus none of the previous proofs work because they rely on the fact that $D_u(p\nu_v)\neq0$. Yet, we can construct a similar super-solution for this case.

We consider $s>2$, and $\nu_v(s)=-\frac{s}{2}\in]\nu_v^-(s),\nu_v^+(s)[$. It is still true that

$$\overline{v}(t,x)=\min(1,C_ve^{\nu_v(s)(x-st)}),$$

is a super-solution of the $v$ equation for any $C_v > 0$. We also have

$$\nu_u^-(s) < p\nu_v(s) < \nu_u^+(s),$$

so that we have $D_u(p\nu_v) < 0$. Let

$$\overline{u}(t,x) = \begin{cases} 1 & , x - st \leq \tau, \\ C_u e^{\nu_u^+(s)(x-st)} + \kappa C_v^p e^{p\nu_v(s)(x-st)} & , x - st \geq \tau. \end{cases}$$

We can proceed the same way as when $(d,\alpha) \in \widehat{\text{III}}$ and show that $\overline{u}$ is a super-solution, concluding the proof of the lemma. ∎

### 3.2 Sub-solutions

We come back to our former definitions of the domains $\widetilde{\text{I}}$-$\widetilde{\text{IV}}$ given in Section 2. In order to have the lower bound of Theorem 1, we prove that $s_{\text{sel}} \geq 2$ and $s_{\text{sel}} \geq 2\sqrt{d\alpha}$ for any $(d,\alpha)$, then $s_{\text{sel}} \geq s_{\text{anom}}$ for $(d,\alpha) \in \widetilde{\text{III}}$.

#### 3.2.1 Lower bound $s_{\text{sel}} \geq 2\sqrt{d\alpha}$

**Lemma 3.7.** *For any $d > 0$ and $\alpha > 0$, we have $s_{\text{sel}} \geq 2\sqrt{d\alpha}$.*

**Proof.** Let's consider the $u$ equation in isolation, that is

$$u_t = du_{xx} + f(u). \tag{3.1}$$

A phase-plane analysis (see for example [1]) shows that this equation admits solutions of the form $U_s(x-st)$, unique up to a translation for every fixed $s \in \mathbb{R}$. If we consider $0 < s < 2\sqrt{d\alpha}$, then 1 is an unstable saddle-node point, while 0 is a stable focus. Thus those solutions tend to 1 as $x - st \to -\infty$ and converge to 0 by oscillating when $x - st \to \infty$. In particular, they attain 0 in finite time. For any $s \in (0, 2\sqrt{d\alpha})$ we define $U_{\text{osc}}(x - st)$ as a solution of (3.1), cut off and set equal to zero for all $x - st$ greater than its smallest zero. Then one can check that $U_{\text{osc}}$ is a sub-solution of (3.1). Since, for the full system (1.1), we only add the negative contribution $-\beta v^p(1-u)$ when considering $N(u)$, it is also a sub-solution for the coupled equation. Using the invariance by translation, we can ensure that $U_{\text{osc}}(x) \leq u(0,x)$, so that we have $U_{\text{osc}}(x - st) \leq u(t,x)$ for every $(t,x)$. Since we can choose $s$ arbitrarily close to $2\sqrt{d\alpha}$, we have $s_{\text{sel}} \geq 2\sqrt{d\alpha}$. ∎

#### 3.2.2 Lower bound $s_{\text{sel}} \geq 2$

**Lemma 3.8.** *For any $d > 0$ and $\alpha > 0$, we have $s_{\text{sel}} \geq 2$.*

While we obtained a sub-solution by removing the coupling in the proof of the previous lemma, this does not work here. We have to use the coupling term in the expression of our sub-solution, otherwise it would mean that the selected speed of (3.1) with compactly supported initial condition would also be greater than 2, which is obviously false. There are multiple ways to achieve it, and we present here a simple one.

**Proof.** Consider $s < 2$. We define the following function depending on $y = x - st$ :

$$\psi(y) = 1 - A\cosh(B(y - C)),$$

with $A \in (0,1)$, $B > 0$ and $C \in \mathbb{R}$. The maximum of $\psi$ is $\psi(C) = 1 - A < 1$, and there exists $y_+ > C$ such that $\psi(y_+) = 0$ and $\psi_y(y_+) = \psi_x(y_+) < 0$. In fact, we have

$$y_+ = \frac{\text{Argcosh}(1/A)}{B} + C.$$

Thus one can define the continuous function

$$\underline{u}(t,x) = \underline{u}(y) \quad = \quad \begin{cases} 1 - A & , y \leq C, \\ \psi(y) & , C \leq y \leq y_+, \\ 0 & , y \geq y_+. \end{cases}$$

It is obvious that $\underline{u}$ is a sub-solution when $y \notin (C, y_+)$. We also note that at the matching points $y = C$ and $y = y_+$, we have

$$\begin{aligned} \lim_{y \to C^-} \partial_y \underline{u} \quad &= \quad \lim_{y \to C^+} \partial_y \underline{u}, \\ \lim_{y \to y_+^-} \partial_y \underline{u} \quad &< \quad \lim_{y \to y_+^+} \partial_y \underline{u}. \end{aligned}$$

On the other hand, since $f(\underline{u}) \geq 0$, one can check that

$$\begin{aligned} N(\underline{u}) \quad &\leq \quad \underline{u}_t - d\underline{u}_{yy} - s\underline{u}_y - f(\underline{u}) - \beta v^p(1 - \underline{u}), \\ &\leq \quad \left[dB^2 - \beta v^p\right] A\cosh(B(y - C)) + sAB\sinh(B(y - C)), \\ &\leq \quad \left[dB^2 + sB - \beta \underline{v}^p\right] A\cosh(B(y - C)), \end{aligned}$$

with $\underline{v} \leq v$ a sub-solution of the $v$ equation. So we need $dB^2 + sB - \beta\underline{v}^p(t,y) \leq 0$ when $y \in [C, y_+]$.

As $s < 2$, let us consider $V_{\text{osc}}(x - st)$ the sub-solution of the $v$ equation, with $V_{\text{osc}}$ constructed the exact same way as $U_{\text{osc}}$ above. One can choose a translate of $V_{\text{osc}}$ such that its first zero occurs when $y = 0$. Then we have $V_{\text{osc}}(x - st) \leq v(t,x)$ for all $(t,x)$. This allows us to choose $\underline{v} = V_{\text{osc}}$.

As we have $V_{\text{osc}}(y) \underset{y \to -\infty}{\longrightarrow} 1$, there exists $y_0(s) < 0$ such that $V_{\text{osc}}(y) \geq \frac{1}{2}$ whenever $y \leq y_0$. The parameters $d, s, \beta, p$ being fixed, we can select a $B$ small enough so that

$$dB^2 + sB - \beta\left(\frac{1}{2}\right)^p \leq 0.$$

Thus we have $N(\underline{u}) \leq 0$ if we can ensure that $V_{\text{osc}}(y) \geq \frac{1}{2}$ for $y \in [C, y_+]$, that is if $y_+ \leq y_0$. Given the expression of $y_+$, and as $y_0$ depends only on $s$, one can choose $C$ very negative so that $y_+ \leq y_0$. Thus $\underline{u}$ is a sub-solution.

Finally, we have to verify if $\underline{u}(0,x) \leq u_0(x)$. We have $\underline{u} \leq 1 - A$, $y_+ < 0$, and $\underline{u} = 0$ outside of $(-\infty, y_+)$. By choosing $C$ possibly even more negative, we can ensure that $\underline{u}(0,x) \leq u_0(x)$. So that eventually we have $\underline{u}(x - st) \leq u(t,x)$ for every $(t,x)$.

In the end, any threshold $h \in (0, 1 - A)$ travels at speed at least equal to $s$. As we can choose $A$ arbitrarily close to 0, we have $s_{\text{sel}} \geq s$. The same construction holds for any $s < 2$, thus $s_{\text{sel}} \geq 2$. This ends the proof of the lemma. ∎

### 3.2.3 Lower bound $s_{\mathrm{sel}} \geq s_{\mathrm{anom}}$

**Lemma 3.9.** *For any $(d,\alpha) \in \widetilde{III}$, we have $s_{\mathrm{sel}} \geq s_{\mathrm{anom}}$.*

A similar proof has been done in [8] with a coupling term $\beta v$ instead of $\beta v^p(1-u)$. The adaptation to $p > 0$ is somehow straightforward, but the positive contribution $\beta v^p u$ in the expression of $N(u)$ requires some specific attention. This is why we will sketch the main lines of the proof. We first quote a result from [8].

**Lemma 3.10.** *Fix $\sigma > 2$ and $0 \leq v_0 \leq 1$ a compactly supported perturbation of the Heaviside step function $\mathbf{1}_{x\leq 0}$. Let $\delta > 0$. There exist $\tau_\pm(t;\delta,\sigma,v_0)$ and a $T^*(\delta,\sigma,v_0) > 0$ such that the function*

$$\underline{v}(t,x) = e^{\nu_v^-(\sigma)(x-\sigma t)} e^{-\delta t},$$

*is a sub-solution of the $v$ component (1.3) for $y \in [\tau_-(t), \tau_+(t)]$ and $t > T^*$.*

**Proof.** [of Lemma 3.9] We now turn our attention to the $u$ component. Consider $(d,\alpha) \in \widetilde{\mathrm{III}}$ and $\max(2, 2\sqrt{d\alpha}) < s < \sigma < s_{\mathrm{anom}}$. We will prove that there exist $T_\delta > 0$ such that

$$\underline{u}(t,x) = \begin{cases} U_r(x-st) & , x < \sigma t, \\ U_r((\sigma - s)t)\psi(x-\sigma t, t) & , \sigma t \leq x < \sigma t + \Theta_+(t) , \\ 0 & , x \geq \sigma t + \Theta_+(t), \end{cases}$$

is a sub-solution of the $u$ equation for any $t \geq T_\delta$. The different terms in $\underline{u}$ are as follows.

- The family of fronts $U_r(\cdot)$ indexed by $r \in \mathbb{R}$ represents the solutions of the $u$ equation in isolation. Using the invariance by translation, we can parameterize this family with the identity

  $$U_r(r) = h.$$

  with $h \in (0,1)$ fixed. As we choose $s > 2\sqrt{d\alpha}$, those are indeed fronts that tend to one as $x - st \to -\infty$, and decay to zero as $x - st \to +\infty$. In particular, a phase-plane analysis shows that

  $$U_r'(y) = \nu_u^+(s) U_r \left[1 + R(U_r)\right],$$

  with $|R(U_r)| < C U_r$ for $U_r$ small enough.

- The function $\psi$ is given by

  $$\psi(y,t) = c_1(t) e^{\nu_u^+(\sigma) y} - \frac{\beta}{D_u(p\nu_v^-(\sigma)) + \delta} e^{p\nu_v^-(\sigma) y} e^{-p\delta t}, \text{ with } c_1(t) = \left(1 + \frac{\beta}{D_u(p\nu_v^-(\sigma)) + \delta} e^{-\delta t}\right),$$

  where $D_u(p\nu_v^-(\sigma)) + \delta > 0$ for the values of parameters under consideration. Note that $\psi(y,t) \to 0^-$ when $y \to \infty$, and $\psi(0,t) = 1$. Finally, there exists $\Theta_+(t) \in (0,\infty)$ such that $\psi(\Theta_+(t), t) = 0$ and $\psi(y,t) > 0$ on $[0, \Theta_+(t))$.

As a consequence, if one chooses,

$$\delta = \delta_c = \frac{1}{p}\sqrt{\sigma^2 - 4}(p\nu_v^-(\sigma) - \nu_u^+(\sigma)) > 0,$$

then one can ensure that

$$\tau_-(t;\delta,\sigma,v_0) < \Theta_+(t) < \tau_+(t;\delta,\sigma,v_0),$$

holds for all $t$ greater than some $T_\delta(\sigma, v_0) \geq T^*(\delta,\sigma,v_0)$. Besides, if $\sigma$ is close enough to $s_{\text{anom}}$, then $\tau_-(t) > 0$ for $t > T_\delta$. Then, for all $t > T_\delta$, in a moving frame $y = x - \sigma t$, the real line can be decomposed into

$$I_a = (-\infty, 0], \quad I_b = (0, \tau_-(t)], \quad I_c = (\tau_-(t), \Theta_+(t)], \quad I_d = (\Theta_+(t), \infty).$$

We now prove that $\underline{u}$ is a sub-solution on each interval. Regions $I_a$ and $I_d$ are trivial. Furthermore, an easy computation shows that $\underline{u}$ is a sub-solution at the matching point $= x - \sigma t = 0$. In both regions $I_b$ and $I_c$ we have

$$\begin{aligned} N(\underline{u}) &= (\sigma - s)U_r'(\cdot)\psi + U_r(\cdot)c_1'(t)e^{\nu_u^+(\sigma)(x-\sigma t)} \\ &+ U_r(\cdot)\beta\underline{v}^p - \beta v^p(t,x)(1 - U_r\psi) + \mathcal{F}(U_r\psi), \end{aligned}$$

where $\mathcal{F}(u) := f(u) - \alpha u = \mathcal{O}(u^2)$ as $u \to 0$. Since $\sigma > s$, by taking $T_\delta$ perhaps even larger, we have that $U_r(x - st)$ is very close to 0 when $x \in I_b \cup I_c$. Thus we can simplify the expression and we obtain

$$\begin{aligned} N(\underline{u}) &= \left[(\sigma - s)\nu_u^+(s) + R(U_r) + \beta v^p(t,x)\right] U_r\psi + \mathcal{F}(U_r\psi) \\ &+ \left[U_r(\cdot)c_1'(t)e^{\nu_u^+(\sigma)(x-\sigma t)}\right] + \left[\beta U_r(\cdot)\underline{v}^p(t,x) - \beta v^p(t,x)\right]. \end{aligned}$$

**Sub-solution on $I_c$.** Note first that we have $c_1'(t) < 0$. Since $U_r \to 0$ as $r \to -\infty$, there exists $r_0(s,\sigma,v_0)$ such that for all $r < r_0(s,\sigma,v_0)$ and $t > T_\delta(\sigma, v_0)$ we have for all $x \in I_c$ that

$$\left[(\sigma - s)\nu_u^+(s) + R(U_r) + \beta v^p(t,x) + CU_r\psi\right] < 0$$

as $(\sigma - s)\nu_u^+(s) < 0$ is a fixed negative number and $\beta v^p(t,x)$ converges to zero uniformly for $x \in I_c$ and $\mathcal{F}(u) \leq Cu^2$ as $u \to 0$ for some positive constant $C > 0$. We have thus proved that $N(\underline{u}) < 0$ on $I_c$.

**Sub-solution on $I_b$.** The same reasoning does not hold on region $I_b$, since $\underline{v}$ is not a sub-solution here. However, it remains true that the first and second bracket are negative if $T_\delta$ is large enough. Indeed, it is enough to control $\beta U_r(\cdot)\underline{v}^p$. If we divide and multiply this term by $\psi$, we can enter it into the first bracket. Then similarly to the control we applied in region $I_c$, it suffices to show that $\beta U_r \underline{v}^p/\psi$ can be made arbitrarily small, which is satisfied by taking $T_\delta$ possibly even larger. Thus $N(\underline{u}) < 0$ on $I_b$.

**Conclusion of the proof.** The last step is to prove there exists $T_u(s,\sigma,q_0) \geq T_\delta$ such that $\underline{u}(T_u, x) \leq u(T_u, x)$. As the proof is identical to the one in [8] we do not detail it here. Thus we obtain $\underline{u} \leq u$ for all $t \geq T_u$ and $x \in \mathbb{R}$. For any threshold $h \in (0,1)$, we have chosen $U_r$ so that $U_r(r) = h$. For large values of $t$ we have $\sigma t > r$ so that the invasion point of $\underline{u}$ associated to the threshold $h$ satisfies $\kappa_h(t) = st + r$. Then the selected speed of $\underline{u}$ is equal to $s$ for any $h \in (0,1)$. This implies $s_{\text{sel}} \geq s$. As $s < s_{\text{anom}}$ was taken arbitrarily, we deduce $s_{\text{sel}} \geq s_{\text{anom}}$. This ends the proof. ∎

# 4 Existence of anomalous spreading speed in monostable systems – A case study

In this section, we investigate the existence of anomalous spreading speed in system (1.1) when we relax the Fisher-KPP condition (1.6) and only suppose that $f$ is monostable *i.e.* only satisfies (1.2). As already explained in the introduction, in general, for the $u$ component in isolation there is no explicit expression for the selected spreading speed $s_0$. Also, recall that in that case $s_0$ verifies $s_0 \geq 2\sqrt{df'(0)}$. In order to slightly simplify our presentation and better illustrate our result, we consider (1.1) with a specific $f$. Namely, we will suppose throughout this sequel that

$$f(u) = \alpha u(1-u)(1+au), \tag{4.1}$$

where $a > 0$ is a varying parameter. The motivation for such a choice comes from the fact that one can exactly compute the spreading speed $s_0$ as shown in the following section. Of course, all results of this section can be generalized to any monostable nonlinearities but one cannot get as fine statements as the ones we present here (see Conjecture 1 & 2).

## 4.1 Equation $u$ in isolation

For the moment, we consider the $u$ component in isolation

$$u_t = du_{xx} + \alpha u(1-u)(1+au).$$

We look for solutions of the form $u(t,x) = U_s(x-st)$ which connects monotonically the homogeneous states $u = 0$ and $u = 1$. Such solutions satisfy

$$dU_s'' + sU_s' + \alpha U_s(1-U_s)(1+aU_s) = 0, \quad U_s' < 0 \quad \text{with } U_s(-\infty) = 1 \text{ and } U_s(+\infty) = 0.$$

A phase-plane analysis [1, 6] shows that there exists such $U_s$, unique up to a translation, for any $s \geq s_0$ where

$$s_0 := \begin{cases} 2\sqrt{d\alpha} & \text{, if } a \leq 2, \\ \left(\frac{2+a}{\sqrt{2a}}\right)\sqrt{d\alpha} & \text{, if } a \geq 2. \end{cases}$$

Note that $s_0 > 2\sqrt{d\alpha}$ whenever $a > 2$. As shown in [1], compactly supported, positive initial condition, will spread at speed $s_0$, and thus for the full system (1.1) we necessarily have that $s_{\text{sel}}^{\text{m}}(p) \geq s_0$ where we denote by $s_{\text{sel}}^{\text{m}}(p)$ the selected speed for system (1.1) the subscript $m$ referring to the monostable nature of the nonlinearity $f$.

**Remark 4.1.** *Notice that $f$ can be written*

$$f(u) = \alpha u - \alpha a u^3 + \alpha u^2(a-1),$$

*such that for any $a \in (0,1]$, $f$ naturally satisfies the Fisher-KPP condition (1.6). And one can also note that even for $a \in (1,2]$, the spreading speed $s_0$ is linearly determined.*

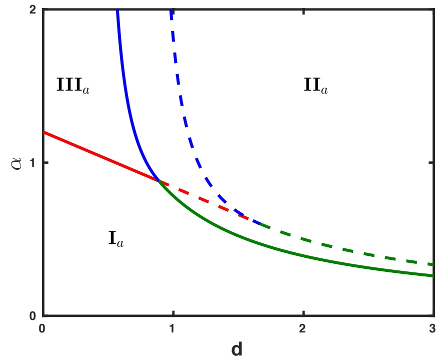



Figure 4.1: *Selected speed of* (1.1) *with* $f(u) = \alpha u(1-u)(1+au)$, $p = 0.6$, $a = 5.5$ *as stated in Conjecture 2. The domain* $\widetilde{\mathrm{I}}_a$ *corresponds to* $s_{\mathrm{sel}}^{\mathrm{m}}(p) = 2$, *the domain* $\widetilde{\mathrm{II}}_a$ *corresponds to* $s_{\mathrm{sel}}^{\mathrm{m}}(p) = s_0$, *and the domain* $\widetilde{\mathrm{III}}_a$ *corresponds to the anomalous speed* $s_{\mathrm{sel}}^{\mathrm{m}}(p) = s_{\mathrm{anom}} > \max(2, c_0)$. *The dotted lines represent the shifted boundaries of the domains when* $a \leq 2$, *that is when* $s_{\mathrm{sel}}^{\mathrm{m}}(p)$ *is linearly determined (see Conjecture 1). In that case, the respective domains do not depend on* $a \in (0, 2]$.

### 4.2 Anomalous spreading speed

Let us first remark that for any $a > 0$, one can find $\tilde{f} \in C^2$ such that $\tilde{f}(u) \leq \alpha u(1-u)(1+au)$ for all $u \in [0,1]$ and $\tilde{f}$ satisfies (1.2) and (1.6) with $\tilde{f}'(0) = f'(0) = \alpha$. In fact, this a general statement for monostable nonlinearities which verify conditions (1.2). We will denote by $\tilde{s}_{\mathrm{sel}}^{\mathrm{KPP}}(p)$ the selected speed for system (1.1) when the nonlinearity is given by $\tilde{f}$ satisfying the above conditions. As a consequence, we can apply our main Theorem 1 and we readily obtain that

$$s_{\mathrm{sel}}^{\mathrm{m}}(p) \geq \tilde{s}_{\mathrm{sel}}^{\mathrm{KPP}}(p) = \begin{cases} 2 & , (d,\alpha) \in \mathrm{I}, \\ 2\sqrt{d\alpha} & , (d,\alpha) \in \mathrm{II}, \\ s_{\mathrm{anom}}(d,\alpha,p) & , (d,\alpha) \in \mathrm{III}, \end{cases}$$

with $s_{\mathrm{anom}}(d,\alpha,p) > \max(2, 2\sqrt{d\alpha})$ for $(d,\alpha) \in \mathrm{III}$. From there, if $a \leq 2$, we can conclude the existence of an anomalous speed $s_{\mathrm{sel}}^{\mathrm{m}}(p) > \max(2, s_0)$ for $(d,\alpha) \in \mathrm{III}$. Besides, that anomalous speed is at least greater or equal than $s_{\mathrm{anom}}(d,\alpha,p)$. On the other hand, when $a > 2$, we have $s_0 > 2\sqrt{d\alpha}$ and a speed is considered anomalous if strictly greater than $\max(2, s_0)$. By solving $s_{\mathrm{anom}}(d,\alpha,p) > \max(2, s_0)$ for $(d,\alpha) \in \mathrm{III}$, one is able to conclude the existence of an anomalous speed on the domain

$$\mathrm{III}_a = \left\{ 2p - dp^2 < \alpha \,,\, dp \leq \frac{2}{a+2} \right\} \cup \left\{ 2p - dp^2 < \alpha < \frac{adp^2}{(2+a)dp - 2} \,,\, \frac{2}{a+2} < dp < \frac{4}{a+2} \right\}.$$

This is illustrated in Figure 4.1 and for $(d,\alpha) \in \mathrm{III}_a$, we have an anomalous speed $s_{\mathrm{sel}}^{\mathrm{m}}(p) \geq s_{\mathrm{anom}}(d,\alpha,p) > \max(2, s_0)$.

### 4.3 Conjectures & numerical illustrations

We conclude our study by stating two conjectures that we illustrate with numerical simulations.

**Conjecture 1** (Case $1 < a \leq 2$)**.** *Consider (1.1) with $f$ defined in* (4.1) *with* $a \in (1,2]$ *and* $d, \beta, p, \alpha > 0$. *Fix initial data* $0 \leq u(0,x) \leq 1$ *and* $0 \leq v(0,x) \leq 1$, *each consisting of a compactly supported perturbation* *of the Heaviside step function* $\mathbf{1}_{x\leq 0}$*. Then, the selected speed* $s_{\text{sel}}^{\text{m}}(p)$ *of* (1.1) *is given by*

$$s_{\text{sel}}^{\text{m}}(p)=\begin{cases}2 & ,\ (d,\alpha)\in\text{I},\\ 2\sqrt{d\alpha} & ,\ (d,\alpha)\in\text{II},\\ s_{\text{anom}}(d,\alpha,p) & ,\ (d,\alpha)\in\text{III},\end{cases}$$

*with* $s_{\text{anom}}(d,\alpha,p)$ *defined in* (1.7) *and domains* I, II *and* III *defined in Theorem 1.*

Let us define the following two domains

$$\begin{aligned}\text{I}_a&=\left\{\alpha\leq 2p-dp^2\ ,\ dp\leq\frac{4}{a+2}\right\}\cup\left\{\alpha\leq\frac{8a}{d(2+a)^2}\ ,\ dp\geq\frac{4}{a+2}\right\},\\ \text{II}_a&=\left\{\alpha\geq\frac{adp^2}{(2+a)dp-2}\ ,\ \frac{2}{a+2}<dp<\frac{4}{a+2}\right\}\cup\left\{\alpha\geq\frac{8a}{d(2+a)^2}\ ,\ dp\geq\frac{4}{a+2}\right\}.\end{aligned}$$

**Conjecture 2** (Case $a>2$)**.** *Consider (1.1) with* $f$ *defined in* (4.1) *with* $a>2$ *and* $d,\beta,p,\alpha>0$*. Fix initial data* $0\leq u(0,x)\leq 1$ *and* $0\leq v(0,x)\leq 1$*, each consisting of a compactly supported perturbation of the Heaviside step function* $\mathbf{1}_{x\leq 0}$*. Then, the selected speed* $s_{\text{sel}}^{\text{m}}(p)$ *of* (1.1) *is given by*

$$s_{\text{sel}}^{\text{m}}(p)=\begin{cases}2 & ,\ (d,\alpha)\in\text{I}_a,\\ \left(\frac{2+a}{\sqrt{2a}}\right)\sqrt{d\alpha} & ,\ (d,\alpha)\in\text{II}_a,\\ s_{\text{anom}}(d,\alpha,p) & ,\ (d,\alpha)\in\text{III}_a,\end{cases}$$

*with* $s_{\text{anom}}(d,\alpha,p)$ *defined in* (1.7)*.*

The two conjectures 1 & 2 assess that the lower bound $s_{\text{sel}}^{\text{m}}(p)\geq\max\left(\tilde{s}_{\text{sel}}^{\text{KPP}}(p),s_0\right)$ that we found is actually also an upper-bound. And as consequence, the *linear* anomalous spreading speed we derived in the previous sections is also the selected spreading speed for system (1.1) when $f$ is given by (4.1) in the monostable regime. In Figure 4.2, we confirm these conjectures numerically by direct simulations of (1.1) where we compare the numerical spreading speed with selected speed given in conjectures 1 & 2. It will be the subject of future work to prove these two conjectures. Actually, we suspect that such results should also apply to general monostable nonlinearities.

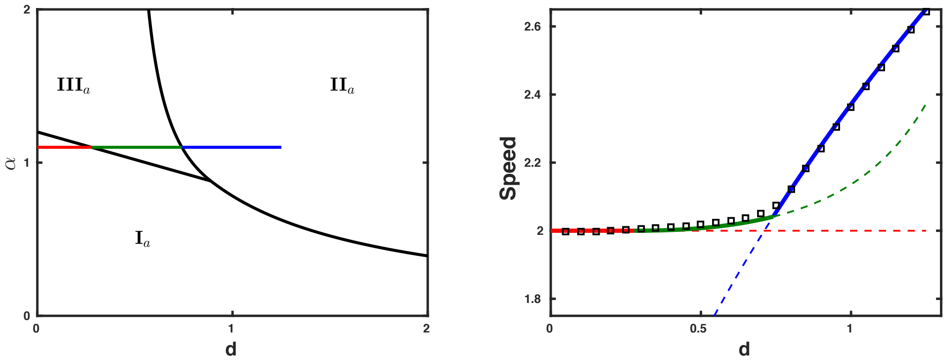



Figure 4.2: *Simulations of the selected speed with parameters* $a = 5.5$, $p = 0.6$, $\alpha = 1.1$, $\beta = 1$, *and for* $d \in [0.05, 1.25]$. *The maximum relative error is inferior to* $9.10^{-3}$.

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