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par

Grégory Faye

Some propagation phenomena in local and nonlocal reaction-diffusion equations

A dynamical systems approach

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Thierry Gallay (Université Grenoble Alpes)	Rapporteur
François Hamel (Aix-Marseille Université)	Rapporteur
Mariana Haragus (Université de Franche Comté)	Examinatrice
Philippe Laurençot (CNRS & Université Toulouse 3)	Parrain
Eric Lombardi (Université Toulouse 3)	Examinateur
Benoit Perthame (Sorbonne Université)	Examinateur
Arnd Scheel (Université du Minnesota)	Examinateur

après les rapports de Thierry Gallay, François Hamel & Björn Sandstede (Brown University).

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List of publications

Warning: Quotes along the text refer to the general bibliography. Corresponding designations are given here in square brackets. Papers marked with $\bullet$ (respectively $\star$) were completed after my PhD and are (not) presented in this memoir. Those marked with $\diamond$ are part of my PhD thesis are also not presented.

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- $\bullet$ [80] **G. Faye and M. Holzer**. *Asymptotic stability of the critical Fisher-KPP front using pointwise estimates.* Zeitschrift für angewandte Mathematik und Physik, 70:13, (2019), pp. 1–25.
- $\bullet$ [79] **G. Faye and M. Holzer**. *Bifurcation to locked fronts in two component reaction-diffusion systems.* Annales de l'Institut Henri Poincaré C, vol 39 (2019), pp. 545–584.
- $\bullet$ [82] **G. Faye and Z.P. Kilpatrick**. *Threshold of front propagation in neural fields: An interface dynamics approach.* SIAM J. Appl. Math., 78-5 (2018), pp. 2575–2596.
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Mentored undergraduate research

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Chapter 1

Introduction

This memoir collects the results I obtained since the end of my PhD on propagation phenomena in local and nonlocal reaction-diffusion equations. These results can be classified in three main categories: (i) existence and stability of traveling waves of various forms (fronts, pulses, modulated fronts), (ii) establishment of spreading speeds (*i.e.* asymptotic speed of propagation of compactly supported perturbations of an unstable state for a reaction-diffusion equation) and (iii) abstract results for nonlocal equations motived by questions raised in the study of propagation phenomena in nonlocal problems. My motivations for studying propagation phenomena are driven by some applications in neuroscience and ecology which I will explain later on in this memoir. However the aim of this memoir is to focus on the mathematical results I obtained, and I will be mainly giving references to the corresponding biological literature.

One of the specificity of my work is perhaps the diversity of models that I have investigated and, at this stage of the introduction, I will very briefly present the main equations that will be encountered in the forthcoming chapters to set a better stage and help the reader that might not be familiar with such models. The modeling assumptions of each equation together with their biological interpretations will be explained at the beginning of each chapter (or section if needed).

A large portion of my work focuses on the study of neural field equations which were heuristically derived in the early 70's by Wilson and Cowan [197, 198] and typically read as

$$\partial_t u(t,x) = -u(t,x) + \int_{\mathbb{R}} \mathcal{K}(x-y)S(u(t,y))\mathrm{d}y, \quad t>0, \quad x\in\mathbb{R}. \tag{1.0.1}$$

Here the scalar unknown $u(t,x)$ represents an average neural activity. In such equations, the connectivity kernel $\mathcal{K}$, which models nonlocal interactions, *i.e.* how populations of neurons are spatially inter-connected, and the firing rate function $S(u)$, which represents the intrinsic nonlinear behavior of neurons, entirely shape the dynamics. Neural field equations have been successfully used to explain many experimental observations including in particular visual hallucinations, binocular rivalry or working memory. I refer to the recent reviews [31, 33] for more details.

Another important class of equations encountered in this memoir are classical reaction-diffusion equations (or systems) which appeared in the celebrated works of Fisher [92] and Kolmogorov, Petrovsky and Piskunov [143] in the 30's, and that can be written in simplest form as

$$\partial_t u(t,x) = \underbrace{\partial_x^2 u(t,x)}_{\text{local diffusion}} + \underbrace{f(u(t,x))}_{\text{local reaction}}, \quad t>0, \quad x \in \mathbb{R}, \tag{1.0.2}$$

where the scalar unknown $u(t,x)$ represents for example a population density. Both the diffusion process and reaction terms in (1.0.2) are assumed to be local. I will also consider two types of nonlocal version of the reaction-diffusion equation (1.0.2) by either assuming that the diffusion process is nonlocal or that the reaction terms incorporate nonlocal interactions. In the former case, typical equations read as

$$\partial_t u(t,x) = \underbrace{-u(t,x) + \int_{\mathbb{R}} \mathcal{K}(x-y)u(t,y)\mathrm{d}y}_{\text{nonlocal diffusion}} + f(u(t,x)), \quad t>0, \quad x \in \mathbb{R}. \tag{1.0.3}$$

It is crucial to note that neural field equations (1.0.1) belong to the class of reaction-diffusion equations with the specificity that both reaction and diffusion processes are nonlocal. Indeed, if one denotes $f(u) := -u + S(u)$, the right-hand side of the above equation (1.0.1) can be written[1]

$$\partial_t u(t,x) = -u(t,x) + \int_{\mathbb{R}} \mathcal{K}(x-y)u(t,y)\mathrm{d}y + \underbrace{\int_{\mathbb{R}} \mathcal{K}(x-y)f(u(t,y))\mathrm{d}y}_{\text{nonlocal reaction}},$$

which should be compared to (1.0.3).

The typical questions that I have been mathematically investigating and of crucial importance for theoretical biologists can be formulated as follows. Suppose that at time $t=0$ one is given an initial data $u_0(x)$, what can be said about the asymptotic behavior of the corresponding solution $u(t,x)$ of (1.0.1)-(1.0.2)-(1.0.3) as $t \to +\infty$? Does each solution persist and invade the whole environment, and if yes, can one quantify this spreading? With these three model equations in mind and the typical questions that I would like to address, it is now time to explain what is presented in this memoir. I refer to Chapter 5 for a quick presentation of what is not presented in this memoir.

1.1 What is presented in this memoir

I have decided to divide my work on propagation phenomena in local and nonlocal reaction-diffusion equations into three different chapters. Nevertheless, it is important to emphasize that each chapter are interconnected and that some editorial choices had to be made.

[1] If one further assumes that $\int_{\mathbb{R}} \mathcal{K}(x)\mathrm{d}x = 1$, then (1.0.1) can also be written in the form

$$\partial_t u(t,x) = \underbrace{-S(u(t,x)) + \int_{\mathbb{R}} \mathcal{K}(x-y)S(u(t,y))\mathrm{d}y}_{\text{nonlinear nonlocal diffusion}} + f(u(t,x)), \quad t>0, \quad x \in \mathbb{R},$$

which can be seen as the combined effects of nonlinear nonlocal diffusion and local reaction.

Chapter 2 collects the results I obtained in [67, 73, 82] regarding propagation phenomena in scalar neural field equations:

- The first work in collaboration with J. Fang [67] deals with the existence of traveling wave solutions and spreading properties for scalar delayed neural field equations where the kinetic dynamics are of monostable type. We have characterized the invasion speed as a function of the asymptotic decay of the connectivity kernel. More precisely, we showed that for exponentially bounded kernels the minimal speed of traveling waves exists and coincides with the spreading speed, which further can be explicitly characterized under a KPP type condition. The proofs of these results rely on the application of the abstract monotone dynamical system theory recently developed in [69]. We also investigated the case of algebraically decaying kernels where we proved the non-existence of traveling wave solutions and show the level sets of the solutions eventually locate in between two exponential functions of time. The uniqueness of traveling waves modulo translation is also obtained.

- In [73], I have started a new study of traveling waves for scalar neural field equations set on a lattice, with infinite range interactions and in the regime where the kinetics of each individual neuron is of bistable type. I have shown existence of traveling fronts using a regularizing technique and proved that traveling front solutions which have nonzero wave speed are unique (up to translation) by constructing appropriate sub and super solutions. Finally, I have investigated spectral properties of the linearization around such traveling fronts in co-moving frame.

- The third work [82] has been obtained in collaboration with Z. Kilpatrick and pertains at deriving conditions for threshold of front propagations in neural field equations. The crucial modeling assumption is to work in the high gain limit for the firing rate function which allows to describe the dynamics of the interfaces, *i.e.* where the neural activity is at the firing threshold. We notably prove that there exists a critical "*size*" of the initial condition below which there is extinction of the activity (uniform convergence to the down state) and above which propagation does occur (local convergence to the up state).

Chapter 3 is about general nonlocal propagation problems and gathers the following works [5, 71, 78, 86–88]. I have decided to include [5, 71, 78] in this chapter as motivating examples for the analysis that I developed in [86] in collaboration with A. Scheel on Fredholm properties of nonlocal differential operators. This has then set the basis for our two following works [87, 88]. Let me first introduce the results obtained in [5, 71, 78]:

- In [71], I have proved the existence and stability of traveling pulses of neural field equations with synaptic depression (see (3.1.1)) with a specific connectivity kernel allowing to write the corresponding neural field equations as a higher-order PDE system. Here synaptic depression is a physiological form of negative feedback which evolves on a smaller time scale leading to a slow-fast dynamical system for the equations satisfied by the profile of the traveling pulse. The existence proof relies on geometric singular perturbation theory and blow-up techniques as one needs to track the solution near a point on the slow manifold that is not normally hyperbolic. The stability of the pulse is then investigated by computing the zeros of the corresponding Evans function. This study predicts that synaptic depression leads to the formation of stable traveling pulses with algebraic decay along the back of the profile.

- With M. Holzer, we have considered a nonlocal generalization of the Fisher-KPP equation (namely $f(u)$ in (1.0.2) is replaced by $\mu u(1-\mathcal{K}*u)$ for some parameter $\mu>0$ and $*$ stands for the convolution on the real line). As the parameter μ is varied and under some assumptions on the Fourier transform of the kernel $\mathcal{K}$, the system undergoes a Turing bifurcation. Near this bifurcation we have established two main results. First, we proved the existence of a two-parameter family of bifurcating stationary periodic solutions and derived a rigorous asymptotic approximation of these solutions. We also studied the spectral stability of the bifurcating stationary periodic solutions with respect to almost co-periodic perturbations. Secondly, restricting to a specific class of exponential kernels for which the nonlocal problem is transformed into a higher order partial differential equation, we proved the existence of modulated traveling fronts near the Turing bifurcation that describes the invasion of the Turing unstable homogeneous state by the periodic pattern established in the first part. Both results rely on a center manifold reduction to a finite dimensional ordinary differential equation.

- We investigated in [5] with A. Scheel and two undergraduate students T. Anderson and D. Stauffer pinning regions and unpinning asymptotics in nonlocal reaction-diffusion equations of the form (1.0.3). This work was also part of the REU program mentioned previously. Here *pinning* refers to propagation failure as it has been reported in discrete and inhomogeneous media. In our nonlocal setting, we find new unpinning asymptotics suggesting that the phenomena are different from the discrete and inhomogeneous cases. Using special kernels, we interpreted the unpinning transition as a slow passage through a fold in a singularly perturbed system. Indeed, for small speeds, the corresponding ODE system possesses a fast-slow structure, which we elucidated using geometric blowup methods.

A common idea used in [5, 71, 78] is to work with specific kernels which allows to transform the nonlocal problem into a higher order partial differential equation. Then, one can rely on spatial dynamics and classical dynamical systems tools (*e.g.* geometric singular perturbation theory for slow-fast system, center manifold theory near bifurcation) to study coherent structures, in the form of traveling pulses or fronts. This method is by nature restrictive as it only applies to the class of kernels whose Fourier transform is a rational fraction, that is $\widehat{\mathcal{K}}(\ell)=\int_{\mathbb{R}}\mathcal{K}(x)e^{-\mathbf{i}\ell x}\mathrm{d}x=\mathrm{Q}(\mathbf{i}\ell)/\mathrm{P}(\mathbf{i}\ell)$ with some polynomials Q and P with $\deg\mathrm{Q}<\deg\mathrm{P}$ such that $\mathcal{K}*u=v$ formally transforms into $\mathrm{Q}(\partial_x)u=\mathrm{P}(\partial_x)v$. This motivated the series of works [86–88] where the aim was to treat nonlocal problems in full generality without assuming any specific form on the kernels.

- In [86], we established Fredholm properties for a class of nonlocal differential operators that naturally arise when linearizing at coherent structures such as traveling fronts or pulses in nonlinear nonlocal differential equations of the form (1.0.1) or (1.0.3). Using mild convergence and exponential localization conditions on the nonlocal terms, we also showed how to compute Fredholm indices via a generalized spectral flow, using crossing numbers of generalized spatial eigenvalues.

- In [87], we proved the existence of fast traveling pulse solutions in excitable media with nonlocal coupling (*e.g.* equations (1.0.1) or (1.0.3) supplemented by a linear adaptation mechanism acting on a slower time scale). Our approach replaces methods from geometric singular perturbation theory, that had been crucial in previous existence proofs (such as in [71]), by a PDE oriented approach, relying on exponential weights, Fredholm theory (derived for that purpose in [86]), and commutator estimates. Our proof can be roughly sketched as follows. We construct a good enough singular solution by gluing front and back solutions to pieces of the slow manifolds (which we construct by hand). We then linearize around this approximate solution and count the number of parameters needed to compensate negative index Fredholm operators so that parameter derivatives span cokernels. Then the last step is to use a fixed-point argument to find a traveling pulse solution near the singular one. For this, one needs to control error terms and commutators estimates. This is achieved thanks to our careful choice for the singular approximate solution.

- Our last work [88] pertains at proving center manifold results for a large class of nonlocal differential equations which was in particular motivated by our proof of the existence of slow manifolds in [87]. The key remark (and difficulty) is that these nonlocal systems possess a natural spatial translation symmetry but local existence or uniqueness theorems for a spatial evolution associated with this spatial shift or even a well motivated choice of phase space for the induced dynamics do not seem to be available, due to the infinite range forward- and backward-coupling through nonlocal convolution operators. As a consequence, we performed a reduction relying entirely on functional analytic methods and leading to center manifold theorems. Despite the nonlocal nature of the problem, we did recover a local differential equation describing the dynamics on the set of small bounded solutions, exploiting that the translation invariance of the original problem induces a flow action on the center manifold. We showed how to apply our reduction to various problems (in particular to stationary and traveling wave solutions of neural field equations) to illustrate the new type of algebra necessary for the computation of Taylor jets of reduced vector fields.

In Chapter 4, I have gathered results on propagation phenomena for general reaction-diffusion equations:

- In [81, 83], I have investigated spreading speeds in systems due to nonlinear interactions. More precisely, in collaboration with M. Holzer and A. Scheel we have identified a new mechanism for propagation into unstable states in spatially extended systems, that is based on resonant interaction in the leading edge of invasion fronts [81]. Such resonant invasion speeds can be determined solely based on the complex linear dispersion relation at the unstable equilibrium, but rely on the presence of a nonlinear term that facilitates the resonant coupling. We proved that these resonant speeds give the correct invasion speed in a simple example, we showed that fronts with speeds slower than the resonant speed are unstable, and corroborated our speed criterion numerically in a variety of model equations (including a nonlocal scalar neural field equation). This study served as the foundations to the follow up paper [83], where with G. Peltier (a master student), we provided a complete description of the selected spreading speed of systems of reaction-diffusion equations with unilateral coupling and proved the existence of *anomalous* spreading speeds for systems with monostable nonlinearities. By anomalous spreading speed, we typically refer to the case of systems where the coupling between several populations enhances the propagation of one population in the sense that its spreading speed (in the coupled system) is strictly larger than the spreading speed of the population alone (uncoupled system).

- With M. Holzer, we studied in [79] transition from stage-invasion fronts to coherent traveling fronts (also referred to as *locked* fronts) in general two component reaction-diffusion systems which do not satisfy any comparison principle. Using a variation of Lin's method, we constructed traveling front solutions and showed the existence of a bifurcation to locked fronts where both components invade at the same speed. We also obtained expansions of the wave speed as a function of the diffusion constant of one species. The bifurcation can be sub or super-critical depending on whether the locked fronts exist for parameter values above or below the bifurcation value. A very interesting feature of this study is that in the sub-critical case numerical simulations reveal that the spreading speed of the PDE system does not depend continuously on the coefficient of diffusion. The mechanism at place in our setting appears to be different from the discontinuity of spreading speeds usually observed in systems as a parameter is altered from zero to some non-zero value (representing the onset of coupling of some previously uncoupled modes).

- The two works [72, 80] concentrate on the asymptotic stability of traveling fronts in local and nonlocal reaction-diffusion equations. I have proved in [72] the multidimensional stability of planar traveling fronts for the nonlocal reaction-diffusion equation (1.0.3) with bistable dynamics using semigroup estimates. More precisely, I showed that if the traveling front is spectrally stable in one space dimension, then it is stable in n-space dimension, $n \geq 2$, with perturbations of the traveling front decaying like $t^{-(n-1)/4}$ as $t \to +\infty$ in $H^k(\mathbb{R}^n)$ for $k \geq \left[\frac{n+1}{2}\right]$. In a second recent work in collaboration with M. Holzer, we proposed a simple alternative proof of a famous result of Gallay [97] regarding the nonlinear asymptotic stability of the critical front of the Fisher-KPP equation (*e.g.* equation (1.0.2) with $f(u) = u(1-u)$) which shows that perturbations of the critical front decay algebraically with rate $t^{-3/2}$ in a weighted L^∞ space. Our proof is based on pointwise semigroup methods and the key remark that the faster algebraic decay rate $t^{-3/2}$ is a consequence of the lack of an embedded zero of the Evans function at the origin for the linearized problem around the critical front.

1.2 Some terminology and known results

I will complete this introduction by presenting some terminology that I will be using throughout this memoir together with some known results.

1.2.1 Spreading speeds and pushed/pulled fronts

Let us consider the scalar local reaction-diffusion equation

$$\partial_t u = \partial_x^2 u + f(u), \quad t > 0, \quad x \in \mathbb{R}, \tag{1.2.1}$$

with **monostable** growth rate $f(u)$, *i.e.* f satisfies[2]

$$f \in \mathscr{C}^1([0,1]), \quad f(0) = f(1) = 0, \quad f'(0) > 0, f'(1) < 0, \quad f > 0 \text{ in } (0,1), \tag{1.2.2}$$

and associated traveling wave problem

$$\begin{cases} U'' + cU' + f(U) = 0, \text{ on } \mathbb{R}, \\ U(-\infty) = 1, \quad U(+\infty) = 0, \text{ and } 0 < U < 1 \text{ on } \mathbb{R}, \end{cases} \tag{1.2.3}$$

where $u(t,x) = U(x - ct)$ is a solution of (1.2.1). Traveling wave solutions of (1.2.1) are found as heteroclinic orbit of (1.2.3) connecting $U = 1$ to $U = 0$. In the following, I shall always refer to as **traveling front** any traveling wave solution obtained as a heteroclinic orbit of reaction-diffusion equation or system.

Theorem 1.2.1 ([9]). *Assume that f is monostable. Let $u_0 = u(0,\cdot) \neq 0$ be an initial condition compactly supported in $\mathbb{R}$ and satisfying $0 \leq u_0 \leq 1$. Let u be the solution of the Cauchy problem associated with (1.2.1) and u_0. Then, there exists $c_* \in (0, +\infty)$ such that*

(i) if $c > c_*$, then $u(t,x) \to 0$ uniformly in $\{[x| \geq ct\}$ as $t \to +\infty$;

(ii) if $0 \leq c < c_*$, then $u(t,x) \to 1$ uniformly in $\{[x| \leq ct\}$ as $t \to +\infty$.

Such a wave speed c_* is called a **spreading speed** as it characterizes the asymptotic speed of propagation of compactly supported perturbations of the unstable $u = 0$. It turns out that the spreading speed c_* also characterizes the minimal speed of traveling wave solutions of (1.2.3).

Theorem 1.2.2 ([9]). *Assume that f is monostable, then there is a traveling front of speed c solution of (1.2.3) if and only if $c \geq c_*$.*

I will be referring to the traveling front with minimal wave speed c_* as the **critical front**, while those with $c > c_*$ will be called **super-critical fronts**. A phase portrait analysis of (1.2.3) shows that necessarily $c_* \geq 2\sqrt{f'(0)}$ where $2\sqrt{f'(0)}$ is the linear spreading speed of perturbations of the linearized equation around the unstable state $u = 0$. Finally, a **pulled front** will be a critical front with $c_* = 2\sqrt{f'(0)}$ and a **pushed front** will be a critical front such that $c_* > 2\sqrt{f'(0)}$.

[2] I have arbitrarily set the unstable state to be $u = 0$ and the stable one to be $u = 1$.

Theorem 1.2.1 says that the transition between the unstable state $u = 0$ and the stable state $u = 1$ asymptotically occurs near $|x| \sim c_*t$, while Theorem 1.2.2 asserts that the spreading speed is exactly the minimal possible wave speed of traveling wave solution. It is then natural to precise what is the asymptotic behavior of the solutions around this transition and ask if there is a link between spreading speed and traveling wave solutions. Under an extra condition on the nonlinearity f, namely $f(u) \leq f'(0)u$ for all $u \in [0, 1]$, it is well known that the solution of the Cauchy problem associated with a Heaviside or a compactly supported initial condition converges to the critical pulled front along the level lines. More precisely, there exists some $s(t)$ such that $u(t, x + s(t)) \to U_*(x)$, uniformly in x as $t \to +\infty$, where U_* denotes the critical front of (1.2.3) with $c = c_*$ [143]. The translation $s(t)$ behaves as c_*t at first order and is followed by logarithmic corrections [29, 108]

$$s(t) = c_*t - \frac{3}{2\lambda_*}\ln(t) + \mathcal{O}(1), \text{ as } t \to +\infty,$$

where $\lambda_* := c_*/2 = \sqrt{f'(0)}$.

1.2.2 Bistable dynamics

There will be another important class of growth rates that will be considered: those that are called **bistable**. That is f in (1.2.1) satisfies

$$\begin{cases} f \in \mathscr{C}^1([0,1]), \quad f(0) = f(1) = f(\theta) = 0 \text{ for some } \theta \in (0,1), \\ f'(0) < 0, f'(1) < 0, \text{ and } f'(\theta) > 0, \\ f < 0 \text{ in } (0,\theta), \quad f > 0 \text{ in } (\theta, 1). \end{cases} \tag{1.2.4}$$

In that case, Theorems 1.2.1 and 1.2.2 typically read as follows.

Theorem 1.2.3 ([91]). *Assume that f is bistable.*

(i) There exists a unique (up to translation) traveling front solution (U, c) of (1.2.3) which is monotone. The sign of the corresponding wave speed c is given by the sign of $\int_0^1 f(u)\mathrm{d}u$.

(ii) If $u_0 = u(0, \cdot) \neq 0$ is an initial condition satisfying $0 \leq u_0 \leq 1$ and

$$\limsup_{x\to-\infty} u_0(x) > \theta, \quad \liminf_{x\to+\infty} u_0(x) < \theta,$$

then there exists $x_0 \in \mathbb{R}$ so that the corresponding solution of the Cauchy problem associated with (1.2.1) and u_0 satisfies $u(t, x) \to U(x - ct + x_0)$ uniformly in x as $t \to +\infty$, with an exponential convergence rate.

Theorem 1.2.3 was later on extended to the case of neural field equations (1.0.1) in [66] for the first statement and [44] for the second one under the following conditions for the connectivity kernel $\mathcal{K}$:

$$\mathcal{K} \in \mathscr{C}^1(\mathbb{R}), \mathcal{K}(x) = \mathcal{K}(-x) \geq 0 \ \forall x \in \mathbb{R}, \quad \int_{\mathbb{R}} \mathcal{K}(x)\mathrm{d}x = 1, \int_{\mathbb{R}} \left(|x|\mathcal{K}(x) + |\mathcal{K}'(x)|\right)\mathrm{d}x < \infty. \tag{1.2.5}$$

Regarding the nonlinearity S, we assume that $S \in \mathscr{C}^1([0,1])$, $S' > 0$ and $f(u) = -u + S(u)$ is bistable. The associated traveling wave problem reads

$$\begin{cases} -cU' = -U + \mathcal{K} * S(U), \text{ on } \mathbb{R}, \\ U(-\infty) = 1, \quad U(+\infty) = 0, \text{ and } 0 < U < 1 \text{ on } \mathbb{R}. \end{cases} \tag{1.2.6}$$

Theorem 1.2.4 ([44, 66]). *Assume that the connectivity kernel $\mathcal{K}$ satisfies (1.2.5) and $S \in \mathscr{C}^1([0,1])$, $S' > 0$ with $f(u) = -u + S(u)$ bistable in the sens of (1.2.4).*

(i) There exists a unique (up to translation) traveling front solution (U, c) of (1.2.6) which is monotone. The sign of the corresponding wave speed c is given by the sign of $\int_0^1 (-u + S(u))\,\mathrm{d}u$.

(ii) If $u_0 = u(0,\cdot) \neq 0$ is an initial condition satisfying $0 \leq u_0 \leq 1$ and

$$\limsup_{x \to -\infty} u_0(x) > \theta, \quad \liminf_{x \to +\infty} u_0(x) < \theta,$$

then there exists $x_0 \in \mathbb{R}$ so that the corresponding solution of the Cauchy problem associated with (1.0.1) and u_0 satisfies $u(t,x) \to U(x - ct + x_0)$ uniformly in x as $t \to +\infty$, with an exponential convergence rate.

I will conclude this section by presenting some known results about the nonlocal reaction-diffusion equation (1.0.3) with bistable dynamics. In that case, the traveling wave problem is

$$\begin{cases} -cU' = -U + \mathcal{K} * U + f(U), \text{ on } \mathbb{R}, \\ U(-\infty) = 1, \quad U(+\infty) = 0, \text{ and } 0 < U < 1 \text{ on } \mathbb{R}. \end{cases} \tag{1.2.7}$$

One of the main difficulty in that case is that the profile U of the corresponding traveling front solution of (1.2.7) may not be smooth when the wave speed is zero. Indeed, rearranging the terms, we see that U satisfies when $c = 0$

$$\mathcal{K} * U = U - f(U) := g(U),$$

and depending on the properties of the function g, the profile U may have jump discontinuities [16, 44]. Nevertheless, the condition $g' > 0$ prevents this scenario to happen.

Theorem 1.2.5 ([16, 44]). *Assume that f is bistable and that $g' > 0$ on $[0,1]$ with $g(u) = u - f(u)$.*

(i) There exists a unique (up to translation) traveling front solution (U, c) of (1.2.7) which is monotone. The sign of the corresponding wave speed c is given by the sign of $\int_0^1 f(u)\mathrm{d}u$.

(ii) If $u_0 = u(0,\cdot) \neq 0$ is an initial condition satisfying $0 \leq u_0 \leq 1$ and

$$\limsup_{x \to -\infty} u_0(x) > \theta, \quad \liminf_{x \to +\infty} u_0(x) < \theta,$$

then there exists $x_0 \in \mathbb{R}$ so that the corresponding solution of the Cauchy problem associated with (1.0.3) and u_0 satisfies $u(t,x) \to U(x - ct + x_0)$ uniformly in x as $t \to +\infty$, with an exponential convergence rate.

Chapter 2

Traveling fronts in neural field equations

Biological motivation: traveling waves of neural activity. In the past few years, electrode recordings and imaging studies have revealed in multiple sensory, motor and cognitive systems a large variety of traveling waves of neural activity [123, 158, 180, 199] which are not only elicited by localized external stimuli but they can be spontaneously generated by recurrent circuits [99, 180, 199]. It has also been shown that they travel along brain networks at multiple scales, transiently modulating spiking activity (*e.g.* emission of action potentials) and excitability of the neural tissue as they pass [158]. However, despite the increasing number of experimental recordings, our theoretical understanding of wave generation and propagation, which is of crucial importance since they are at the basis of sensory stimuli processing and are often associated to pathological forms of cortical behaviors such as epileptic seizures and migraines, is far from being complete and serves as a motivation for the following studies.

Neural field equations: macroscopic models for traveling waves of neural activity. So far, most mathematical efforts have focused on a macroscopic description of cortical waves through traveling waves in continuous neural field models of the form (1.0.1) assuming homogeneity and isotropy of cortical tissue. As already explained in the introduction, an important property of homogenous neural fields is that they support traveling waves in the form of propagating fronts separating regions of high and low activity (see Theorem 1.2.4) which relate well to some of the experimental data. From a modeling point of view, the assumption that cortical tissue can be approximated by $\mathbb{R}$ is totally unrealistic. At least, one should look at two-dimensional neural field equations, that is (1.0.1) set on $\Omega \subset \mathbb{R}^2$, and even consider some curvature effects. Surprisingly enough, since the pioneer work of Ermentrout and McLeod [66], the mathematical study of traveling waves and more generally propagation phenomena in neural field equations has mainly relied on either formal computations and asymptotic expansions [30, 32, 52, 139, 166] or restricted to the special case where the nonlinearity S is assumed to be a Heaviside step function

[4, 53, 202] where explicit formula for traveling wave profile and wave speed can be derived, thus simplifying the analysis. As a consequence, even for the one-dimensional case, that is equation (1.0.1) set on $\mathbb{R}$, there remain a lot of open or partially answered questions.

Outline. In Section 2.1, I will present a fairly exhaustive study of propagation phenomena in neural field equations with monostable dynamics for general smooth nonlinearity S. The key result of the analysis is about spreading speeds where it is shown that for sufficiently localized connectivity kernels minimal spreading speeds exist (and thus finite) while for exponentially unbounded kernels the level sets of the solutions propagate with an infinite asymptotic speed. Then, in Section 2.2, I present some results regarding the existence, uniqueness and stability of monotone traveling fronts for a discrete version of the neural field equation (1.0.1). The motivation for such a study relies on my aim to develop a general program that would systematically analyze propagation phenomena in various types of neural networks. Here, I have started with the case of an infinite network with all to all connections. Finally, in the last section, I will present some partial results on threshold of front propagations in neural field equations when the nonlinearity S is assumed to be a Heaviside step function. It will illustrate the great efficiency of working in this regime.

2.1 Monotone traveling waves for delayed neural field equations

We consider the following single-layer delayed neural field equation

$$\partial_t u(t,x) = -u(t,x) + \int_{\mathbb{R}} \mathcal{K}(x-y) S(u(t-\tau,y)) \mathrm{d}y, \quad (t,x) \in (0,\infty) \times \mathbb{R}, \tag{2.1.1}$$

where the positive constant $0 \leq \tau < \infty$ models a synaptic delay. Here, we will consider the case where the dynamics of the system is of monostable type, that is $f(u) = -u + S(u)$ satisfies hypotheses similar to (1.2.2). The aim is to find conditions, especially on the connectivity kernel $\mathcal{K}$, such that similar results to Theorems 1.2.1 and 1.2.2 hold or do not hold for the delayed neural field equation (2.1.1) with monostable dynamics. Throughout this section, we will assume that the kernel $\mathcal{K}$ and the nonlinearity S satisfy the following hypotheses.

Hypothesis 2.1.1. *The function $\mathcal{K}$ defined on $\mathbb{R}$ is such that:*

(i) $\mathcal{K}$ *is uniformly bounded on* $\mathbb{R}$*;*

(ii) $\mathcal{K} \geq 0$*;*

(iii) $\int_{\mathbb{R}} \mathcal{K}(x) \mathrm{d}x = 1$.

Let us note that we assume weaker conditions on $\mathcal{K}$ than the ones given in (1.2.5). We will come back to these differences later on.

Hypothesis 2.1.2. *The nonlinearity S is such that:*

(i) S *is continuously differentiable with* $0 < S' \leq s_m$*;*

(ii) $f(u) := -u + S(u)$ *has precisely two zeros at* $u = 0$ *and* $u = 1$*, with* $f(u) > 0$ *for all* $u \in (0, 1)$*;*

(iii) $S'(0) > 1$.

The first result is about the large time behavior of solutions to the Cauchy problem

$$\begin{cases} \partial_t u(t,x) &= -u(t,x) + \int_{\mathbb{R}} \mathcal{K}(x-y) S(u(t-\tau,y))\mathrm{d}y & \text{if } (t,x) \in (0,\infty)\times\mathbb{R}, \\ u(t,x) &= \phi(t,x) & \text{if } (t,x) \in [-\tau,0]\times\mathbb{R}. \end{cases} \tag{2.1.2}$$

Theorem 2.1.1 (Spreading speeds)**.** *The Cauchy problem* (2.1.2) *admits leftward and rightward spreading speeds, denoted by* c_-^* *and* c_+^* *respectively, satisfying* $c_-^* + c_+^* > 0$ *and* $c_\pm^* \in (-\infty, +\infty]$*, in the following sense:*

(i) if $c > c_+^*$ *and* $c' > c_-^*$ *and the initial condition* ϕ *has compact support with* $0 \le \phi \le 1$ *and* $\phi \not\equiv 1$*, then the solution* u *of* (2.1.2) *has the property*

$$\lim_{t\to\infty,\ x\ge ct \text{ or } x\le -c't} u(t,x) = 0;$$

(ii) if $c < c_+^*$ *and* $c' < c_-^*$ *with* $c + c' > 0$ *and the initial condition* ϕ *has compact support with* $0 \le \phi \le 1$ *and* $\phi \not\equiv 0$*, then the solution* u *of* (2.1.2) *has the property*

$$\lim_{t\to\infty,\ c't\le x\le ct} u(t,x) = 1.$$

We readily remark that $c_-^* = c_+^* := c^*$ when the connectivity kernel $\mathcal{K}$ is symmetric and both of them could be infinite in specific evolutionary models, which will be discussed later. The second result is about the existence and uniqueness of monotone traveling wave solutions of (2.1.1). Hence, we are interested in solutions $u(t,x) = U(x-ct) = U(\xi)$, for $c \in \mathbb{R}$, where U is monotone and $0 \le U \le 1$, which satisfy

$$-cU'(\xi) = -U(\xi) + \int_{\mathbb{R}} \mathcal{K}(\xi-\xi') S(U(\xi'+c\tau))\mathrm{d}\xi', \tag{2.1.3}$$

together with the limits

$$\lim_{\xi\to-\infty} U(\xi) = 1 \text{ and } \lim_{\xi\to+\infty} U(\xi) = 0. \tag{2.1.4}$$

We call a leftward traveling wave with speed c of (2.1.1) a special solution having the form $u(t,x) = U(x+ct)$ and a rightward traveling wave with speed c is defined using $u(t,x) = U(x-ct)$.

Theorem 2.1.2 (Traveling waves)**.** *Let* $c_\pm^*$ *be the spreading speeds of* (2.1.2) *defined in Theorem 2.1.1. Then* c_-^* *is the minimal wave speed of the leftward nondecreasing traveling wave connecting* $u = 0$ *to* $u = 1$*, and* c_+^* *is the minimal wave speed of the rightward nonincreasing traveling wave connecting* $u = 1$ *to* $u = 0$*.*

The proofs of Theorems 2.1.1 and 2.1.2 rely on the application of the abstract monotone dynamical systems theory developed recently in [69, 150, 151]. It is important to note that such results on spreading properties and traveling waves solutions are the first rigorous results presented for neural field equations with synaptic delays and monostable kinetics. The only known results were the formal computations presented in [32]. The leftward or rightward spreading speeds obtained in Theorem 2.1.1 might be infinite. This actually depends on the asymptotic decaying properties of the connectivity kernel $\mathcal{K}$ (see Theorem 2.1.5). We summarize in the following theorem the results on the characterization of the minimal wave speed for exponentially bounded kernels. But first, we introduce the notion of exponentially bounded kernels.

We say that a kernel $\mathcal{K}$ is exponentially bounded if there exists $\mu_0 \in (0,\infty]$ such that

$$\int_{\mathbb{R}} \mathcal{K}(x)e^{\mu|x|}\mathrm{d}x < \infty, \quad \forall\ 0 \leq |\mu| < \mu_0 \text{ with } \lim_{\mu\uparrow\mu_0}\int_{\mathbb{R}} \mathcal{K}(x)e^{\mu|x|}\mathrm{d}x = +\infty. \tag{2.1.5}$$

Theorem 2.1.3 (Characterization of the minimal wave speed)**.** *Let suppose that the initial condition ϕ has compact support with $0 \leq \phi \leq 1$ and $\phi \not\equiv 0$ and that $\mathcal{K}$ is symmetric. If the kernel $\mathcal{K}$ is exponentially bounded and the nonlinearity S satisfies a KPP condition, then the minimal wave speed is bounded and can be explicitly characterized through the principal eigenvalue of the linearized equation around the unstable steady state $u = 0$. Furthermore, as a function of the delay, the minimal wave speed $\tau \to c^*(\tau)$ is a monotone continuously decreasing function which converges to zero as $\tau \to +\infty$.*

In the above Theorem 2.1.3, what is referred to as a KPP condition is the following set of assumptions. We say that S satisfies a KPP condition if

$$S \in \mathscr{C}^2([0,1]) \text{ is concave }, \quad S(0) = 0, \quad S(1) = 1, \quad \text{ and } 1 < S'(0). \tag{2.1.6}$$

As stated in Theorem 2.1.3, the minimal wave speed $c^*(\tau)$ can be explicitly characterized. Indeed, $c^*(\tau)$ satisfies

$$0 < c^*(\tau) := \min_{0<\mu<\mu_0} \frac{\lambda^\tau(\mu)}{\mu} < \infty \tag{2.1.7}$$

where $\lambda^\tau(\mu) > 0$ is the principal eigenvalue of

$$\lambda = -1 + S'(0)e^{-\lambda\tau}\widetilde{\mathcal{K}}(\mu),$$

with $\widetilde{\mathcal{K}}(\mu) := \int_{\mathbb{R}} \mathcal{K}(x)e^{\mu x}\mathrm{d}x$. In fact, we verify that

$$\lambda^\tau(\mu) = \frac{1}{\tau}W_0\left(\tau e^\tau S'(0)\widetilde{\mathcal{K}}(\mu)\right) - 1, \tag{2.1.8}$$

where W_0 is the principal branch of the Lambert function which satisfies $W_0(x)e^{W_0(x)} = x$ for any $x \geq 0$. As a consequence, we have

$$c^*(\tau) = \min_{0<\mu<\mu_0}\left\{\frac{1}{\tau\mu}\left(W_0\left(\tau e^\tau S'(0)\widetilde{\mathcal{K}}(\mu)\right) - \tau\right)\right\}.$$

We now provide another result regarding the uniqueness of monotone traveling wave solutions given in Theorem 2.1.2 in the special case of symmetric exponentially bounded kernels $\mathcal{K}$ and nonlinearity S which satisfies the KPP condition (2.1.6).

Theorem 2.1.4 (Uniqueness of traveling waves)**.** *Let suppose that $\mathcal{K}$ is a symmetric exponentially bounded kernel and that $S \in \mathscr{C}^2$ in a neighborhood of $u = 0$. Then the monotone traveling wave solutions of (2.1.3) and (2.1.4) are unique up to translation. Furthermore if S satisfies the KPP condition (2.1.6), each monotone traveling wave solution u of (2.1.3) and (2.1.4) satisfies*

$$u(\xi) \underset{\xi\to+\infty}{=} \mathcal{O}\left(\xi e^{-\mu^*\xi}\right) \quad \text{for } c = c^*, \quad \text{and} \quad u(\xi) \underset{\xi\to+\infty}{=} \mathcal{O}\left(e^{-\mu_1(c)\xi}\right) \quad \text{for } c > c^*,$$

for some $0 < \mu_1(c)$ and $0 < \mu^*$.

Finally, our last results deal with kernels that are symmetric but not exponentially bounded. We say that a kernel $\mathcal{K}$ is exponentially unbounded if $\mathcal{K}$ is a $\mathscr{C}^1$ function for large x, and

$$\mathcal{K}'(x) = o\left(\mathcal{K}(x)\right) \quad \text{as} \quad |x| \to +\infty. \tag{2.1.9}$$

This condition implies that $\mathcal{K}$ decays more slowly than any exponentially decaying functions, in the sense that

$$\mathcal{K}(x)e^{\mu|x|} \to +\infty \quad \text{as} \quad |x| \to +\infty,$$

for all $\mu > 0$. Our main theorem is the following one.

Theorem 2.1.5 (Exponentially unbounded kernels)**.** *Let suppose that the initial condition ϕ has compact support with $0 \leq \phi \leq 1$ and $\phi \not\equiv 0$ and that $\mathcal{K}$ is symmetric.*

- *If the kernel $\mathcal{K}$ is exponentially unbounded, then the level sets of the solution u to (2.1.2) propagate with an infinite asymptotic speed.*
- *In the specific case of algebraically decaying kernels, the position of any level sets moves exponentially fast as time goes to infinity.*

Roughly speaking, the second point of Theorem 2.1.5 can be stated as follows. Suppose that $\mathcal{K}(x)$ is defined as

$$\mathcal{K}(x) = \frac{C_\alpha}{1 + |x|^{2\alpha}}, \quad \alpha > 1/2,$$

and $C_\alpha > 0$ is a normalizing constant. Then, for any $\tau \geq 0$ and any $\kappa \in (0,1)$ there exists positive constants $0 < \rho^*(\tau) \leq \bar{\rho}(\tau)$ such that

$$\frac{\rho^*(\tau)}{2\alpha} \leq \liminf_{t\to\infty} \frac{\log\left(\min\{x : u(t,x) = \kappa\}\right)}{t} \leq \limsup_{t\to\infty} \frac{\log\left(\max\{x : u(t,x) = \kappa\}\right)}{t} \leq \frac{\bar{\rho}(\tau)}{2\alpha}. \tag{2.1.10}$$

While our approach is built on the one developed in the work of Garnier [101] for nonlocal reaction-diffusion equations of the form (1.0.3) with KPP nonlinearity, it differs greatly on a technical level. Indeed, our equation presents two main difficulties: the nonlinear part of the equation is nonlocal and we have to deal with time delays. Nonetheless, we are able to take the advantage of the strategy developed by Cabré and Roquejoffre [40] in the context of Fisher-KPP equations with fractional diffusion to establish the lower bound in (2.1.10). A key ingredient is a subtle estimate for the solutions of some linear nonlocal Cauchy problems with delays. Let us also comment on the sharpness of the bounds found in (2.1.10). The upper bound is definitely not sharp as it is obtained by constructing a super solution with the same asymptotic behavior as the kernel $\mathcal{K}$ where some rough estimates on convolutions of the kernel are used. The lower bound is expected to be sharp as $\rho^*(\tau) > 0$, which is the unique positive solution of $\rho + 1 - S'(0)e^{-\rho\tau} = 0$, is derived from the linearization of the equation around the unstable state $u = 0$. Actually, the recent study [28] has demonstrated the sharpness of the lower bound found by Garnier in [101] in the context of nonlocal reaction-diffusion equations. The idea of [28] is to rescale the equation appropriately and analyze the thin-front limit in the spirit of geometric optics approach to reaction-diffusion equations [10, 95] via Hamilton-Jacobi dynamics of sharp interfaces. Similar techniques could be applied to our neural field setting to obtain sharp upper bound in (2.1.10).

Theorems 2.1.3 and 2.1.5 have several important implications from a modeling point of view. First of all, we clearly see that the asymptotic decaying properties of the connectivity kernel $\mathcal{K}$ are crucial and that one needs strong enough decay (exponential) in order to have finite spreading speeds. Of course, from a neurobiological point of view [123, 180], infinite asymptotic speed does not seem plausible and have, so far, never been recorded in the literature. This result should then be interpreted as follows. In order to have a representative model of neuronal excitatory connections within cortical areas, one should use exponentially bounded kernels. Finally, we recover the fact that constant synaptic delays slow down the spreading speed of the system for exponentially bounded kernels in the case of monostable type of nonlinearity. This was already established in previous studies when the nonlinearity S was idealized with a Heaviside step function, see [32, 33] and references therein.

2.2 Traveling fronts for lattice neural field equations

For $n \in \mathbb{Z}$, we consider the following lattice differential equation

$$\dot{u}_n(t) = -u_n(t) + \sum_{j \in \mathbb{Z}} K_j S(u_{n-j}(t)), \quad t > 0, \tag{2.2.1}$$

where $\dot{u}_n$ stands for $\dfrac{\mathrm{d}u_n}{\mathrm{d}t}$ and $u_n(t)$ represents the membrane potential of neuron labelled n at time t. Here K_j represents the strength of interactions associated to the neural network at position j on the lattice and the firing rate of neurons $S(u)$ is a nonlinear function. Such an equation can be seen as a Hopfield neural network model with infinite range interactions [121] or more simply as a discrete neural field equation [65] where each neuron is set on the lattice $\mathbb{Z}$ with all to all couplings. In that later respect, we will call equations such as (2.2.1) Lattice Neural Field Equations (LNFEs).

Our aim is to initiate a series of work on neural field dynamics set on various types of networks and equation (2.2.1) is one of the very first model to study as it consists of a network composed of infinite neurons indexed on $\mathbb{Z}$ with all to all couplings represented by the interaction communication rates K_j for $j \in \mathbb{Z}$, see Figure 2.1 for an illustration. There is a second natural motivation for studying LNFEs which stems from the numerical study of the continuous neural field equation (1.0.1). Indeed, if one is looking for a numerical approximation of the solutions

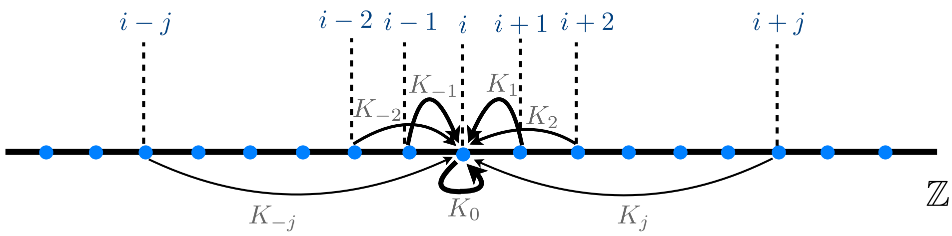


Figure 2.1: Topology of the network associated with (2.2.1).

of (1.0.1), one may discretize space and recover an equation similar to (2.2.1) depending on the quadrature rule used to approximate the integral in (1.0.1).

We would like to study special entire solutions of (2.2.1). Let first suppose that there exists two homogeneous stationary states $(u_n(t))_{n\in\mathbb{Z}} = (u)_{n\in\mathbb{Z}}$ with $u = 0$ and $u = 1$ for the dynamics of (2.2.1). Hence, we are interested in particular solutions of (2.2.1) of the form $u_n(t) = \mathbf{u}(n-ct)$ for some $c \in \mathbb{R}$ where the profile $\mathbf{u} : \mathbb{R} \to \mathbb{R}$ satisfies

$$\begin{cases} -c\mathbf{u}'(x) = -\mathbf{u}(x) + \displaystyle\sum_{j\in\mathbb{Z}} K_j S(\mathbf{u}(x-j)), \quad x \in \mathbb{R}, \\ \displaystyle\lim_{x\to-\infty} \mathbf{u}(x) = 1 \text{ and } \lim_{x\to+\infty} \mathbf{u}(x) = 0, \end{cases} \tag{2.2.2}$$

where we set $x := n - ct$ and $\mathbf{u}'$ stands for $\dfrac{\mathrm{d}\mathbf{u}}{\mathrm{d}x}$. It is understood that when $c = 0$, a stationary wave solution of (2.2.1) is a sequence $(u_n(t))_{n\in\mathbb{Z}} = (\tilde{\mathbf{u}}_n)_{n\in\mathbb{Z}}$, independent of time, which verifies

$$\begin{cases} \tilde{\mathbf{u}}_n = \displaystyle\sum_{j\in\mathbb{Z}} K_j S(\tilde{\mathbf{u}}_{n-j}), \quad n \in \mathbb{Z}, \\ \displaystyle\lim_{n\to-\infty} \tilde{\mathbf{u}}_n = 1 \text{ and } \lim_{n\to+\infty} \tilde{\mathbf{u}}_n = 0. \end{cases} \tag{2.2.3}$$

Throughout this section, we will suppose that the following condition on the weights K_n is satisfied

$$\sum_{n\in\mathbb{Z}} K_n = 1. \tag{2.2.4}$$

Then, steady homogeneous states of the form $(u_n(t))_{n\in\mathbb{Z}} = (u)_{n\in\mathbb{Z}}$ for some $u \in \mathbb{R}$ satisfy the equation $0 = -u + S(u) = f(u)$. We will assume the following hypotheses for the nonlinear function S.

Hypothesis 2.2.1. *We suppose that:*

(i) $S \in \mathscr{C}_b^r(\mathbb{R})$ for $r \geq 2$ with $S(0) = 0$ and $S(1) = 1$ together with $S'(0) < 1$ and $S'(1) < 1$;

(ii) there exists a unique $\theta \in (0,1)$ such that $S(\theta) = \theta$ with $S'(\theta) > 1$;

(iii) $u \mapsto S(u)$ is strictly nondecreasing on $[0,1]$ and there exists $s_m > 1 > s_0 > 0$ such that $s_0 < S'(u) \leq s_m$ for all $u \in [0,1]$.

The assumption (i)-(ii) ensures that $(u_n(t))_{n\in\mathbb{Z}} = (u)_{n\in\mathbb{Z}}$ with $u \in \{0, \theta, 1\}$ are stationary homogeneous solutions of the LNFE (2.2.1) and that the function f is of bistable type. The third condition ensures that S is an increasing function, which is natural for a firing rate function. We also ask for some regularity for S, at least $\mathscr{C}_b^2(\mathbb{R})$. This will be necessary in order to prove our uniqueness result. Regarding the interaction weights $(K_n)_{n\in\mathbb{Z}}$, we will work with the following conditions.

Hypothesis 2.2.2. *We suppose that:*

(i) the normalization condition (2.2.4) *is satisfied;*

(ii) for all $n\in\mathbb{Z}$, *we have* $K_n = K_{-n} \geq 0$ *and* $K_{\pm 1} > 0$;

(iii) $\sum_{n\in\mathbb{Z}} |n| K_n < \infty$.

The second condition is a natural biological assumption and expresses the symmetric and excitatory nature of the considered neural network. The third condition is a technical assumption that is necessary in the process of proving the existence and uniqueness of traveling front solutions. Let us remark that our results cover both the case of finite and infinite range interactions, although we are primarily interested in the later one where one may further assume that $K_n > 0$ for all $n \in \mathbb{Z}$. Let us also precise that if the support of the interactions were to be finite, then one could rely of the theory developed by Mallet-Paret [155] for the study of traveling front solutions in general lattice differential equations. Let us also mention the work of Bates and coauthors [13, 14] who studied traveling waves in infinite range lattice differential equations with bistable dynamics.

Our first result is about the existence of monotone traveling front solutions of (2.2.1).

Theorem 2.2.1 (Existence of monotone traveling waves)**.** *Suppose that the Hypotheses 2.2.1 and 2.2.2 are satisfied then there exists a traveling wave solution* $u_n(t) = \mathbf{u}_*(n - c_* t)$ *of* (2.2.1) *such that the profile* $\mathbf{u}_*$ *satisfies* (2.2.2) *when* $c_* \neq 0$ *or* (2.2.3) *if* $c_* = 0$. *In the later case, we denote* $(\tilde{\mathbf{u}}_n^*)_{n\in\mathbb{Z}}$ *the stationary wave solution. Moreover,*

(i) $\mathrm{sgn}(c_*) = \mathrm{sgn} \int_0^1 f(u)\mathrm{d}u$ *if* $c_* \neq 0$;

(ii) if $\int_0^1 f(u)\mathrm{d}u = 0$ *then* $c_* = 0$;

(iii) if $c_* \neq 0$ *then* $\mathbf{u}_* \in \mathscr{C}^{r+1}(\mathbb{R})$ *and* $\mathbf{u}_*' < 0$ *on* $\mathbb{R}$;

(iv) if $c_* = 0$ *then* $(\tilde{\mathbf{u}}_n^*)_{n\in\mathbb{Z}}$ *is a strictly decreasing sequence.*

The proof of Theorem 2.2.1 relies on a strategy developed by Bates and Chmaj [14] for a discrete convolution model for phase transitions where the idea is to regularize the traveling wave problem (2.2.2). This amounts to considering a sequence of traveling waves problems for continuous neural field equations of the form of (1.0.1) and applying Theorem 1.2.4(i) of Ermentrout & McLeod [66]. The final step is to pass to the limit and verify that the limiting front profiles satisfy all the properties stated in Theorem 2.2.1. One of the main differences between Theorem 2.2.1 and its continuous counterpart (Theorem 1.2.4(i)) comes from the fact that $c_* = 0$ does not necessarily imply that $\int_0^1 f(u)\mathrm{d}u = 0$. Actually, depending on the specific form of the firing rate function S, it is possible that $c_* = 0$ and at the same time $\int_0^1 f(u)\mathrm{d}u \neq 0$, which is often referred to as propagation failure or pinning [125, 135, 155]. Roughly speaking, propagation failure means that there exists fronts with nonzero wave speed for the continuous system while for the discrete one fronts have zero wave speed ($c_* = 0$), see also Section 3.3. We do not pursue in that direction, and we will consider in the next two results only fronts having nonzero wave speed. But before presenting these results, we briefly sketch the first steps of the proof of Theorem 2.2.1.

Proof. Following the original idea of Bates & Chmaj in [14], we define

$$\mathcal{K}_\delta(x) := \sum_{j\in\mathbb{Z}} K_j \delta(x-j) \tag{2.2.5}$$

where $\delta(x-j)$ stands for the delta Dirac mass at $x = j$. Using this notation, we can write

$$\sum_{j\in\mathbb{Z}} K_j S(\mathbf{u}(x-j)) = \mathcal{K}_\delta * S(\mathbf{u})[x],$$

where $*$ denotes the convolution on the real line. As a consequence, the traveling wave problem (2.2.2) can be written as

$$\begin{cases} -c\mathbf{u}' = -\mathbf{u} + \mathcal{K}_\delta * S(\mathbf{u}), \text{ on } \mathbb{R}, \\ \lim\limits_{x\to-\infty} \mathbf{u}(x) = 1 \text{ and } \lim\limits_{x\to+\infty} \mathbf{u}(x) = 0. \end{cases} \tag{2.2.6}$$

Now, in order to use Theorem 1.2.4, we need to regularize the kernel $\mathcal{K}_\delta$ in the following way. Let $\Psi \in \mathscr{C}^\infty(\mathbb{R})$, $\Psi \geq 0$, $\int_\mathbb{R} \Psi(x)\mathrm{d}x = 1$, even and with compact support. Finally, define $\rho_m(x) := m\Psi(mx)$ for all $x \in \mathbb{R}$ and

$$\mathcal{K}_m(x) := \sum_{j=-m}^{m} \frac{1}{\omega_m} K_j \rho_m(x-j), \tag{2.2.7}$$

where $\omega_m := \sum_{j=-m}^m K_j$. It then is easy to check (see [14]) that for all $\phi \in \mathscr{C}_c^\infty(\mathbb{R})$ we have

$$\mathcal{K}_m * \phi \underset{m\to\infty}{\longrightarrow} \mathcal{K}_\delta * \phi$$

uniformly on compact sets. As a consequence, we can consider the sequence of traveling waves problems

$$\begin{cases} -c_m\mathbf{u}_m' = -\mathbf{u}_m + \mathcal{K}_m * S(\mathbf{u}_m), \text{ on } \mathbb{R}, \\ \lim\limits_{x\to-\infty} \mathbf{u}_m(x) = 1 \text{ and } \lim\limits_{x\to+\infty} \mathbf{u}_m(x) = 0. \end{cases} \tag{2.2.8}$$

With the definition of $\mathcal{K}_m$ in (2.2.7) we can also easily check that all the conditions listed in (1.2.5) are satisfied. Then, there exists a unique solution (modulo translation) $(\mathbf{u}_m, c_m)$ of (2.2.8) which further satisfies $\mathbf{u}_m' < 0$ on $\mathbb{R}$. Moreover, we have that $c_m = 0$ if and only if $\int_0^1 f(u)\mathrm{d}u = 0$ and otherwise $\mathrm{sgn}(c_m) = \mathrm{sgn} \int_0^1 f(u)\mathrm{d}u$. The solutions $(\mathbf{u}_m, c_m)$ are of course weak solutions of (2.2.8), i.e. for any $\phi \in \mathscr{C}_c^\infty(\mathbb{R})$ they satisfy

$$-c_m \int_\mathbb{R} \mathbf{u}_m \phi' \mathrm{d}x + \int_\mathbb{R} \left(-\mathbf{u}_m + \mathcal{K}_m * S(\mathbf{u}_m)\right) \phi \mathrm{d}x = 0. \tag{2.2.9}$$

Let suppose that $c_m \geq 0$ and take $\alpha \in (0, \theta)$ and translate each $\mathbf{u}_m$ so that $\mathbf{u}_m(0) = \alpha$. As $(\mathbf{u}_m)_{m\geq 0}$ is a sequence of strictly monotone functions, by Helly's theorem, we can extract a subsequence of $\mathbf{u}_m$, which we still denote by $\mathbf{u}_m$, converging pointwise to a monotone function $\mathbf{u}_*$ as $m \to \infty$. Note that by construction, we have $0 \leq \mathbf{u}_m \leq 1$ and thus $0 \leq \mathbf{u}_* \leq 1$. Let us show that the sequence $(c_m)_{m\geq 0}$ is also uniformly bounded. Assume the contrary, that there is a sequence $c_m \to +\infty$ as $m \to \infty$. From (2.2.8) we have

$$|-c_m \mathbf{u}_m'(x)| = |-\mathbf{u}_m(x) + \mathcal{K}_m * S(\mathbf{u}_m)(x)| \leq 2, \qquad \text{for all } x \in \mathbb{R} \text{ and } m \geq 0,$$

and thus $\|\mathbf{u}_m'\|_\infty \to 0$ as $m \to \infty$. This implies that $\mathbf{u}_*$ is constant and thus $\mathbf{u}_* = \alpha$. This is a contradiction. Indeed, as $\alpha \in (0, \theta)$, we have $f(\alpha) = -\alpha + S(\alpha) < 0$ but

$$-c_m \mathbf{u}_m' = -\mathbf{u}_m + \mathcal{K}_m * S(\mathbf{u}_m) = -\mathbf{u}_m + \mathcal{K}_m * \mathbf{u}_m + \mathcal{K}_m * f(\mathbf{u}_m) \geq 0 \text{ on } \mathbb{R},$$

and we deduce

$$-\mathcal{K}_m * f(\mathbf{u}_m) \leq -\mathbf{u}_m + \mathcal{K}_m * \mathbf{u}_m,$$

that is

$$0 < -f(\alpha) = \lim_{m\to\infty} (-\mathcal{K}_m * f(\mathbf{u}_m)) \leq \lim_{m\to\infty} (-\mathbf{u}_m + \mathcal{K}_m * \mathbf{u}_m) = 0.$$

Finally, by passing to another subsequence, we also have that $c_m \to c_*$, for some $c_* \geq 0$, as $m \to \infty$. We can now pass to the limit in (2.2.9), and we obtain that $\mathbf{u}_*$ is a weak solution of (2.2.6) as it satisfies

$$-c_* \int_{\mathbb{R}} \mathbf{u}_* \phi' \mathrm{d}x + \int_{\mathbb{R}} (-\mathbf{u}_* + \mathcal{K}_\delta * S(\mathbf{u}_*))\, \phi \mathrm{d}x = 0, \tag{2.2.10}$$

for all $\phi \in \mathscr{C}_c^\infty(\mathbb{R})$. This follows from Lebesgue's dominated convergence theorem, the continuity of S and the limit

$$\int_{\mathbb{R}} (\mathcal{K}_m * S(\mathbf{u}_m))\phi \mathrm{d}x = \int_{\mathbb{R}} (\mathcal{K}_m * \phi) S(\mathbf{u}_m) \mathrm{d}x \underset{m\to\infty}{\longrightarrow} \int_{\mathbb{R}} (\mathcal{K}_\delta * \phi) S(\mathbf{u}_*) \mathrm{d}x = \int_{\mathbb{R}} (\mathcal{K}_\delta * S(\mathbf{u}_*))\phi \mathrm{d}x.$$

As a consequence, when $c_* \neq 0$, the equality (2.2.10) implies that $\mathbf{u}_* \in W^{1,\infty}(\mathbb{R})$. A bootstrap argument then show that $\mathbf{u}_* \in \mathscr{C}^{r+1}(\mathbb{R})$ and thus a traveling wave solution of (2.2.1). If $c_* = 0$, then

$$\int_{\mathbb{R}} (-\mathbf{u}_* + \mathcal{K}_\delta * S(\mathbf{u}_*))\, \phi \mathrm{d}x = 0, \text{ for all } \phi \in \mathscr{C}_c^\infty(\mathbb{R}),$$

so that

$$\mathbf{u}_* = \mathcal{K}_\delta * S(\mathbf{u}_*) \text{ a.e. on } \mathbb{R}.$$

Note that $\mathcal{K}_\delta$ is not a regularization kernel and thus $\mathbf{u}_*$ need not be continuous. However, $\mathbf{u}_*$ is monotone with $0 \leq \mathbf{u}_* \leq 1$, therefore it has only jump discontinuities and the set of these jump discontinuities is at most countable. Thus we can find a sequence $(\iota_k)_{k\geq 0}$ with $\iota_k \to 0$ as $k \to \infty$ such that $\mathbf{u}_*(n + \iota_k)$ is continuous at $n + \iota_k$ for all $n \in \mathbb{Z}$ and $k > 0$. We get that

$$\mathbf{u}_*(n + \iota_k) = \mathcal{K}_\delta * S(\mathbf{u}_*)(n + \iota_k) = \sum_{j\in\mathbb{Z}} K_j S(\mathbf{u}_*(n + \iota_k - j))$$

for all $n \in \mathbb{Z}$ and $k > 0$. It follows that the sequence

$$\tilde{\mathbf{u}}_n^* := \lim_{k\to\infty} \mathbf{u}_*(n + \iota_k), \quad n \in \mathbb{Z},$$

satisfies

$$\tilde{\mathbf{u}}_n^* = \sum_{j\in\mathbb{Z}} K_j S\left(\tilde{\mathbf{u}}_{n-j}^*\right),$$

so is a stationary solution of (2.2.1). We refer to [73] for the end of the proof which consists in checking the monotony property of the traveling front profiles together with their asymptotic limits. ■

The second result is about the uniqueness of traveling front solutions having nonzero wave speed.

Theorem 2.2.2 (Uniqueness of traveling waves with nonzero speed)**.** *Let $(\mathbf{u}_*, c_*)$ be a solution to (2.2.2) as given in Theorem 2.2.1, such that $c_* \neq 0$. Let $(\hat{\mathbf{u}}, \hat{c})$ be another solution to (2.2.2). Then $c = \hat{c}$ and, up to a translation, $\mathbf{u}_* = \hat{\mathbf{u}}$.*

The strategy of proof of Theorem 2.2.2 is to use a "*squeezing*" technique developed by Chen in [44] by constructing appropriate sub and super solutions for (2.2.1). The principal difficulty is that the nonlinearity enters in a non trivial way in the infinite sum, and thus we need to adapt all the arguments in our specific context. We say that a sequence $(u_n(t))_{n\in\mathbb{Z}}$ is a sub-solution of (2.2.1) if it satisfies for all $n \in \mathbb{Z}$ and all $t > 0$

$$\dot{u}_n(t) \leq -u_n(t) + \sum_{j\in\mathbb{Z}} K_j S(u_{n-j}(t)). \tag{2.2.11}$$

A super-solution is defined by reversing the inequality in (2.2.11). The sub and super-solutions are built on the following sequences

$$w_n^{\pm}(t) := \mathbf{u}_*\left(n - c_* t + \xi_0 \mp \sigma\gamma(1 - e^{-\beta t})\right) \pm \gamma e^{-\beta t}, \quad \forall n \in \mathbb{Z} \tag{2.2.12}$$

for some well chosen parameters ξ_0, σ, γ and β. More precisely, the key result in the proof of Theorem 2.2.2 is the following lemma.

Lemma 2.2.1. *Assume that Hypotheses 2.2.1 and 2.2.2 hold and let $(\mathbf{u}_*, c_*)$ with $c_* \neq 0$ be as in Theorem (2.2.1). Then, there exists a small positive constant γ_0 and a large positive constant σ such that for any $\gamma \in (0, \gamma_0]$ and every $\xi_0 \in \mathbb{R}$, the sequences $w_n^{\pm}(t)$ defined by (2.2.12) are a respectively sub ($w_n^-(t)$) and super ($w_n^+(t)$) solutions with $\beta := \frac{1}{2}\min\{1 - S'(0); 1 - S'(1)\} > 0$.*

We now turn our attention to the linear stability of the traveling fronts with nonzero wave speed. First, we require an extra assumption on the sequence of weights $(K_j)_{j\in\mathbb{Z}}$.

Hypothesis 2.2.3. *We suppose that:*

(i) $(K_j)_{j\in\mathbb{Z}}$ *satisfies (H2);*

(ii) *there exists* $\eta > 0$*, such that* $\sum_{j\in\mathbb{Z}} K_j e^{\eta|j|} < \infty$.

Our stability result will be obtained for the continuous version of (2.2.1). That is we interpret solutions of (2.2.1) as $u_n(t) = \mathbf{u}(t, n - c_* t)$ for some function $\mathbf{u} \in \mathscr{C}^1([0,\infty)\times\mathbb{R},\mathbb{R})$, intuitively filling the gap between each lattice site, which satisfies a nonlocal partial differential equation of the form

$$\partial_t \mathbf{u}(t,x) = c_* \partial_x \mathbf{u}(t,x) - \mathbf{u}(t,x) + \sum_{j\in\mathbb{Z}} K_j S(\mathbf{u}(t,x-j)), \quad (t,x)\in(0,\infty)\times\mathbb{R}.$$

By definition, $\mathbf{u}_*$ from Theorems 2.2.1 & 2.2.2 is a stationary solution of the above equation and we will be interested in the spectral properties of its associated linear operator

$$\mathcal{L}\mathbf{v} := c_* \mathbf{v}' - \mathbf{v} + \mathcal{K}_\delta * [S'(\mathbf{u}_*)\mathbf{v}], \tag{2.2.13}$$

where $\mathcal{K}_\delta * \mathbf{v} := \sum_{j\in\mathbb{Z}} K_j \mathbf{v}(\cdot - j)$. From its definition, the operator $\mathcal{L}$ is a closed unbounded operator on $L^2(\mathbb{R})$ with dense domain $H^1(\mathbb{R})$ in $L^2(\mathbb{R})$. Furthermore, it is not difficult to check that $\mathcal{L}$ is the infinitesimal generator of a strongly continuous semigroup on $L^2(\mathbb{R})$. Our main result regarding $\mathcal{L}$ reads as follows.

Theorem 2.2.3 (Spectral properties of $\mathcal{L}$)**.** *Suppose that the Hypotheses 2.2.1 and 2.2.3 are satisfied and let $(\mathbf{u}_*, c_*)$, with $c_* \neq 0$, be the unique (up to translation) strictly monotone traveling wave solution to (2.2.1) as given in Theorem 2.2.1. Let $\mathcal{L} : H^1(\mathbb{R}) \to L^2(\mathbb{R})$ be the operator defined in (4.3.5). We have:*

(i) 0 is an algebraically simple eigenvalue of $\mathcal{L}$ with a negative eigenfunction $\mathbf{u}_*'$;

(ii) the adjoint operator $\mathcal{L}^*$ has a negative eigenfunction, $\mathbf{q} \in \mathscr{C}^1(\mathbb{R})$, corresponding to the simple eigenvalue 0;

(iii) for all $0 < \kappa < \min\{1 - S'(0), 1 - S'(1)\}$ the operator $\mathcal{L} - \lambda$ is invertible as an operator from $H^1(\mathbb{R})$ to $L^2(\mathbb{R})$ for all $\lambda \in \mathbb{C}\backslash 2\pi \mathbf{i} c_* \mathbb{Z}$ such that $\Re(\lambda) \geq -\kappa$;

(iv) there exist $\eta_*, \eta_{**} \in (0,\eta)$ and some constants $C_* > 0$, $C_{**} > 0$ such that

$$|\mathbf{u}_*'(x)| \leq C_* e^{-\eta_*|x|} \|\mathbf{u}\|_{L^\infty(\mathbb{R})}, \text{ and } |\mathbf{q}(x)| \leq C_{**} e^{-\eta_{**}|x|} \|\mathbf{q}\|_{L^\infty(\mathbb{R})},$$

for all $x \in \mathbb{R}$.

The main ingredient of the proof is to show that the operator $\mathcal{L} - \lambda$ is Fredholm on the some region in the complex plane. This analysis relies on some recent work [86] on Fredholm properties of nonlocal differential operators with infinite range interactions which will be presented later on in Section 3.4. Theorem 2.2.3 can be seen as preliminary result towards the nonlinear stability of traveling fronts of equation (2.2.1). Let us explain how such a spectral analysis could be used to get insight on the asymptotic behavior of solutions of (2.2.1) starting from an initial condition close to a traveling front solution. Let us introduce the nonlinear operator

$$\begin{array}{rccl} \mathcal{F}: & \ell^\infty(\mathbb{R}) & \longrightarrow & \ell^\infty(\mathbb{R}) \\ & \mathfrak{u} & \longmapsto & \mathcal{F}(\mathfrak{u}) = -\mathfrak{u} + \mathcal{K} *_d S(\mathfrak{u}), \end{array} \tag{2.2.14}$$

where for all $n \in \mathbb{Z}$ we have set

$$(\mathcal{K} *_d S(\mathfrak{u}))_n := \sum_{j\in\mathbb{Z}} K_j S(\mathfrak{u}_{n-j}),$$

and

$$\ell^\infty(\mathbb{R}) := \left\{ \mathfrak{u} = (\mathfrak{u}_n)_{n\in\mathbb{Z}} \in \mathbb{R}^{\mathbb{Z}} \mid \|\mathfrak{u}\|_{\ell^\infty(\mathbb{R})} := \sup_{n\in\mathbb{Z}} |\mathfrak{u}_n| < \infty \right\}.$$

Using this notation, we can then write (2.2.1) as

$$\dot{\mathfrak{u}}(t) = \mathcal{F}(\mathfrak{u}(t)), \quad t > 0. \tag{2.2.15}$$

for which $\mathfrak{u}^*(t) = (\mathbf{u}_*(n - c_* t))_{n\in\mathbb{Z}}$ with $c_* \neq 0$ is a solution, where the existence of the profile $\mathbf{u}_*$ is given by Theorems 2.2.1 and 2.2.2. To study the stability of $\mathfrak{u}^*(t)$, we look for solutions of (2.2.15) that can be written as $\mathfrak{u}(t) = \mathfrak{u}^*(t) + \mathfrak{v}(t)$ where $\mathfrak{v}(t)$ is a perturbation of the traveling wave solution $\mathfrak{u}^*(t)$. We then find that $\mathfrak{v}(t)$ must satisfy the time-dependent lattice neural field equation

$$\dot{\mathfrak{v}}(t) = D\mathcal{F}(\mathfrak{u}^*(t))\mathfrak{v}(t) + \mathcal{N}(t, \mathfrak{v}(t)), \tag{2.2.16}$$

in which

$$\mathcal{N}(t, \mathfrak{v}(t)) = \mathcal{F}(\mathfrak{u}^*(t) + \mathfrak{v}(t)) - \mathcal{F}(\mathfrak{u}^*(t)) - D\mathcal{F}(\mathfrak{u}^*(t))\mathfrak{v}(t). \tag{2.2.17}$$

Then the strategy would be to obtain spectral properties for $\dot{\mathfrak{v}}(t) = D\mathcal{F}(\mathfrak{u}^*(t))\mathfrak{v}(t)$ from those of the operator $\mathcal{L}$ in order to be able to close a nonlinear stability argument. This method was introduced and successfully implemented by Benzoni-Gavage and coauthors in [20] by analyzing associated Green's functions for the nonlinear stability analysis of semidiscrete shock waves and more recently reused in the context of nonlinear stability analysis of traveling pulses in the discrete FitzHugh-Nagumo equations with finite and infinite range interactions [126, 183]. We conjecture the following result with perturbations measured in the Banach spaces $\ell^p(\mathbb{R})$, which are defined by

$$\ell^p(\mathbb{R}) := \left\{ \mathfrak{u} = (\mathfrak{u}_n)_{n\in\mathbb{Z}} \in \mathbb{R}^{\mathbb{Z}} \mid \|\mathfrak{u}\|_{\ell^p(\mathbb{R})} := \left(\sum_{n\in\mathbb{Z}} |\mathfrak{u}_n|^p \right)^{\frac{1}{p}} < \infty \right\}$$

for $1 \leq p < \infty$.

Conjecture 2.2.1 (Nonlinear stability)**.** *Suppose that the Hypotheses 2.2.1 and 2.2.3 are satisfied and let $(\mathbf{u}_*, c_*)$, with $c_* \neq 0$, be the unique (up to translation) strictly monotone traveling wave solution to (2.2.1) as given in Theorem 2.2.1. We denote by $\mathfrak{u}^*(t) := (\mathbf{u}_*(n - c_* t))_{n\in\mathbb{Z}}$. Then for all $1 \leq p \leq \infty$, there exist constants $\delta > 0$, $C > 0$, $\omega > 0$ such that for all sequences $\mathfrak{u}^0 = (u_n^0)_{n\in\mathbb{Z}}$ which satisfy $\|\mathfrak{u}^0 - \mathfrak{u}^*(0)\|_{\ell^p(\mathbb{R})} \leq \delta$, there exists an asymptotic phase shift $\xi_0 \in \mathbb{R}$ such that the unique solution $t \to \mathfrak{u}(t) = (u_n(t))_{n\in\mathbb{Z}}$ of (2.2.1), with initial condition $\mathfrak{u}(0) = \mathfrak{u}^0$, verifies*

$$\|\mathfrak{u}(t) - \mathfrak{u}^*(t + \xi_0)\|_{\ell^p(\mathbb{R})} \leq C e^{-\omega t} \|\mathfrak{u}^0 - \bar{\mathfrak{u}}(0)\|_{\ell^p(\mathbb{R})},$$

for all $t \geq 0$.

2.3 Threshold of front propagations in neural field equations

In this section, we aim at characterizing the asymptotic behavior of the solutions of the Cauchy problem

$$\begin{cases} \partial_t u(t,x) = -u(t,x) + \int_{\mathbb{R}} \mathcal{K}(x-y) S(u(t,y)) \mathrm{d}y, & t > 0, \quad x \in \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{2.3.1}$$

for various non wave-like initial conditions u_0 for which the stability result of Theorem 1.2.4 does not apply. More specifically, for a given initial condition, we would like to characterize when the corresponding solution u of the Cauchy problem (2.3.1) will be attracted by the steady state $u = 0$ and goes extinct or by the steady state $u = 1$ and propagates. This question of characterizing the transition from extinction to propagation has been addressed for the local reaction-diffusion equation (1.0.2) where the first optimal result was obtained by Zlatoš [203] and later on refined in [59, 159] for larger class of initial conditions.

Theorem 2.3.1 ([203]). *Let f be a bistable nonlinearity satisfying (1.2.4) and $\int_0^1 f(u)\mathrm{d}u > 0$. Let u be the solution of the Cauchy problem associated to (1.0.2) with initial condition $u_0 = \chi_{[-\ell,\ell]}$, the indicator of the interval $[-\ell,\ell] \subset \mathbb{R}$, for $\ell > 0$. Then, there is $\ell_c > 0$ such that*

(i) if $\ell < \ell_c$, then $u(t,x) \to 0$ uniformly on $\mathbb{R}$ as $t \to +\infty$;

(ii) if $\ell = \ell_c$, then $u(t,x) \to U_b$ uniformly on $\mathbb{R}$ as $t \to +\infty$;

(iii) if $\ell > \ell_c$, then $u(t,x) \to 1$ uniformly on compacts as $t \to +\infty$;

where U_b is the unique even positive solution of $0 = U'' + f(U)$ satisfying $U(\pm\infty) = 0$[1].

There is thus a sharp threshold between extinction and propagation for the local reaction-diffusion equation (1.0.2) starting from initial conditions $u_0 = \chi_{[-\ell,\ell]}$. By propagation, we refer to the fact that when $\ell > \ell_c$ the solution u will asymptotically converge as $t \to +\infty$ toward two counter-propagating fronts

$$U(x - ct + x_0) + U(-x - ct - x_0) - 1.$$

Our aim is to prove that a similar result holds for the neural field equation (2.3.1), and more generally to the nonlocal reaction-diffusion equation (1.0.3). As equation (2.3.1) has no regularizing effect, we will need to consider smoother initial conditions. We define the one-parameter family $(\varphi_\ell)_{\ell>0}$ of functions that satisfy for each $\ell > 0$:

- $\varphi_\ell : \mathbb{R} \mapsto [0,1]$ in $H^k(\mathbb{R})$ for $k \geq 2$;
- $\varphi_\ell(x) = \varphi_\ell(-x) \geq 0$ and $\varphi_\ell'(x) < 0$ for all $x > 0$;
- $\varphi_\ell(\pm\ell) = \theta$.

[1] Note that the existence of the ground state U_b can be found in [21].

We can then state the following conjecture.

Conjecture 2.3.1. *Let $\mathcal{K}$ be a connectivity kernel that satisfies (1.2.5) and $S \in \mathscr{C}^1([0,1])$, $S' > 0$ with $f(u) = -u + S(u)$ bistable in the sens of (1.2.4) with $\int_0^1 [-u + S(u)]\mathrm{d}u > 0$. Let $u(t,x)$ be the solution of the Cauchy problem associated to (2.3.1) with initial condition $u_0 = \varphi_\ell$ for $\ell > 0$. Then, there is $\ell_c > 0$ such that*

(i) if $\ell < \ell_c$, then $u(t,x) \to 0$ uniformly on $\mathbb{R}$ as $t \to +\infty$;

(ii) if $\ell = \ell_c$, then $u(t,x) \to U_b$ uniformly on $\mathbb{R}$ as $t \to +\infty$;

(iii) if $\ell > \ell_c$, then $u(t,x) \to 1$ uniformly on compacts as $t \to +\infty$;

where U_b is the unique even positive solution of $0 = -U + \mathcal{K} * S(U)$ satisfying $U(\pm\infty) = 0$.

The existence of a ground state U_b follows from [45] whereas the uniqueness is still an open problem. In the specific case where $\mathcal{K}(x) = e^{-|x|}/2$, the equation of the ground state is equivalent to $0 = U'' - U + S(U)$ for which the uniqueness result of [21] applies. It is reasonable to expect that uniqueness should hold within the class of kernels which are exponentially bounded. Regarding the sharp threshold result, all the proofs available in the literature [59, 159, 203] rely heavily on the local nature of the reaction-diffusion equation (1.0.2), and at the moment only perturbative arguments could yield to weaker statements of Theorem 2.3.1 in the limits $\ell \ll 1$ and $\ell \gg 1$.

In order to make some progress, we use a special form for the nonlinearity: $S(u) = H(u - \kappa)$ for $\kappa \in (0,1)$ where $H(u)$ stands for the Heaviside function defined as

$$H(u) := \begin{cases} 0 & \text{if } u < 0, \\ 1 & \text{if } u \geq 0. \end{cases} \tag{2.3.2}$$

Although such a nonlinearity simplifies some of the computations and allows one to derive closed form formula, it introduces a discontinuity in the equation and thus one needs to be careful when talking about solutions of (2.3.1) in that case. We refer to the recent work of Krüger and Stannat [145] which provides a rigorous setting regarding the existence and uniqueness of mild solutions of (2.3.1) in the form of

$$u(t,x) = e^{-t}u_0(x) + \int_0^t e^{-(t-s)} \int_{\mathbb{R}} \mathcal{K}(x-y)H(u(s,y) - \kappa)\mathrm{d}y\mathrm{d}s, \quad t > 0, \quad x \in \mathbb{R}. \tag{2.3.3}$$

Let us note that it is the uniqueness part that is difficult to establish. Our hypotheses on $\mathcal{K}$ (see (1.2.5)) and the class of initial conditions that we will consider all fit within the framework developed in [145] such that the Cauchy problem (2.3.1) will always have well defined solutions that are global and $\mathscr{C}^1(\mathbb{R})$ in space. Our first result reads as follows.

Theorem 2.3.2. *Let $\mathcal{K}$ be a connectivity kernel that satisfies (1.2.5) and $S = H(\cdot - \kappa)$ with $\kappa \in (0, 1/2)$. Let $u(t,x)$ be the mild solution of the Cauchy problem associated to (2.3.3) with initial condition $u_0 = \varphi_\ell$ for $\ell > 0$. Then, there is $\ell_c > 0$ such that*

(i) if $\ell < \ell_c$, then $u(t,x) \to 0$ uniformly on $\mathbb{R}$ as $t \to +\infty$;

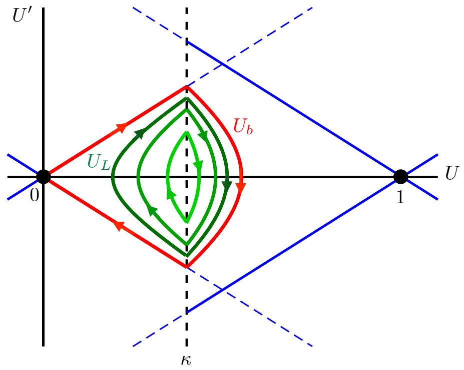


Figure 2.2: Phase portrait of $0 = U'' - U + H(U-\kappa)$, describing stationary solutions of Eq. (2.3.1) with an exponential kernel $\mathcal{K}(x) = e^{-|x|}/2$, with $\kappa \in (0, 1/2)$. Solid black horizontal and blue diagonal lines are nullclines of U and U', respectively. Homogeneous states $\bar{U} = 0, 1$ occur at their intersection. Homoclinic orbits arise about the point $(U, U') = (\kappa, 0)$, crossing the threshold κ twice. The single bump U_b (red outer trajectory) forms a separatrix, bounding all other nontrivial stationary solutions. There exists an infinite number of periodic solutions U_L inside (green inner trajectories), whose orbits shrink as L is decreased from infinity.

(ii) if $\ell = \ell_c$, then $u(t,x) \to U_b$ uniformly on $\mathbb{R}$ as $t \to +\infty$;

(iii) if $\ell > \ell_c$, then $u(t,x) \to 1$ uniformly on compacts as $t \to +\infty$;

where U_b is the unique unimodal even positive solution of $0 = -U + \mathcal{K} * H(U-\kappa)$ satisfying $U(\pm\infty) = 0$. The critical value $\ell_c > 0$ is uniquely defined via

$$\kappa = \int_0^{2\ell_c} \mathcal{K}(y)\mathrm{d}y. \tag{2.3.4}$$

Few remarks are in order. First, the uniqueness of the ground state is only among unimodal even positive solutions. Second, it is straightforward to check that if such a solution exists then it should satisfy

$$U_a(x) = \int_{-a}^{a} \mathcal{K}(x-y)\mathrm{d}y, \text{ with } \quad U_0(\pm a) = \kappa,$$

for a unique value of a that solves $\kappa = \int_0^{2a} \mathcal{K}(y)\mathrm{d}y$. Thus, the half-width of the ground state precisely corresponds to the critical value ℓ_c of equation (2.3.4) of Theorem 2.3.2. Finally, for the sake of illustration, in Figure 2.2, it is shown the phase portrait of the stationary solutions of $0 = -U + \mathcal{K} * H(U-\kappa)$ in the very specific case where $\mathcal{K}(x) = e^{-|x|}/2$ for which the equation can be recast as $0 = U'' - U + H(U-\kappa)$.

We now present the proof of Theorem 2.3.2 which relies on describing the time evolution of the active region $\{x \in \mathbb{R} \mid u(t,x) \geq \kappa\}$. Roughly speaking, we show that for the one-parameter family of initial conditions $(\varphi_\ell)_{\ell>0}$ the active region $\mathcal{A}(t)$ can either (i) shrink to reduce to a single point leading to extinction, (ii) stagnate for all time or (iii) expand leading to propagation.

Proof. Symmetry of equation (2.3.1) ensures solutions with even initial conditions are always even, so the active region $\mathcal{A}(t) = \{x \in \mathbb{R} \mid u(t,x) \geq \kappa\}$ is symmetric for all $t > 0$. The dynamics of the symmetric active region $\mathcal{A}(t) = [-a(t), a(t)]$ can be described with interface equations for the two points $x = \pm a(t)$ (see [4, 54]). We start by rewriting equation (2.3.1) as

$$\partial_t u(t,x) = -u(t,x) + \int_{\mathcal{A}(t)} \mathcal{K}(x-y)\mathrm{d}y, \tag{2.3.5}$$

which can be further simplified:

$$\partial_t u(t,x) = -u(t,x) + \mathbf{K}(x+a(t)) - \mathbf{K}(x-a(t)), \text{ with } \mathbf{K}(x) := \int_0^x \mathcal{K}(y)\mathrm{d}y.$$

Let us note that equation (2.3.5) remains well defined even in the case where $a(t)$ vanishes. We can describe the dynamics of the two interfaces by the implicit equations

$$u(t, \pm a(t)) = \kappa. \tag{2.3.6}$$

Differentiating equation (2.3.6) with respect to t, we find the total derivative is:

$$\pm\alpha(t)a'(t) + \partial_t u(t, \pm a(t)) = 0, \tag{2.3.7}$$

where we define $a'(t) = \frac{\mathrm{d}a(t)}{\mathrm{d}t}$ and $\pm\alpha(t) = \partial_x u(t, \pm a(t))$. The symmetry of equation (2.3.7) allows us to reduce to a single differential equation for the dynamics of $a(t)$:

$$a'(t) = -\frac{1}{\alpha(t)}\left[\mathbf{K}(2a(t)) - \kappa\right], \tag{2.3.8}$$

where we have substituted equation (2.3.5) at $a(t)$ for $\partial_t u(t, a(t))$. Equation (2.3.8) is not well-defined for $\alpha(t) = 0$, but we will show how to circumvent this difficulty. Furthermore, we can obtain a formula for $\alpha(t)$ by defining $z(t,x) := \partial_x u(t,x)$ and differentiating equation (2.3.5) with respect to x to find [54]

$$\partial_t z(t,x) = -z(t,x) + w(x+a(t)) - w(x-a(t)),$$

which we can integrate and evaluate at $a(t)$ to find

$$\alpha(t) = u_0'(a(t))e^{-t} + \int_0^t e^{-(t-s)}\left[\mathcal{K}(a(t)+a(s)) - \mathcal{K}(a(t)-a(s))\right]\mathrm{d}s. \tag{2.3.9}$$

Thus, we have a closed system describing the evolution of the right interface $a(t)$ of the active region $\mathcal{A}(t)$, given by equations (2.3.8) and (2.3.9), along with the initial conditions $a(0) = \ell$ and $\alpha(0) = u_0'(\ell) < 0$, as long as $\alpha(t) < 0$. Criticality occurs for initial conditions such that $a'(t) = 0$, which means $\mathbf{K}(2\ell) = \kappa$, *i.e.* for $\ell = b = \mathbf{K}^{-1}(\kappa)/2$, so the critical ℓ is precisely the half-width of the stationary bump solution U_b.

Propagation. If $\ell > \mathbf{K}^{-1}(\kappa)/2$ then $a'(t) > 0$ and, due to the monotonicity of $\mathcal{K}$ and equation (2.3.9), $\alpha(t) < 0$ for all $t > 0$ so $\lim_{t\to\infty} a(t) = \infty$, and the active region $A(t)$ expands indefinitely. As a consequence, for any compact set $K = [-k, k]$ with $k > 0$ given and any $\epsilon > 0$, we can find $t_* > 0$ large enough such that $K \subset A(t_*)$ and

$$|\mathbf{K}(x+a(t_*)) - \mathbf{K}(x-a(t_*)) - 1| \leq \epsilon, \quad \forall x \in K,$$

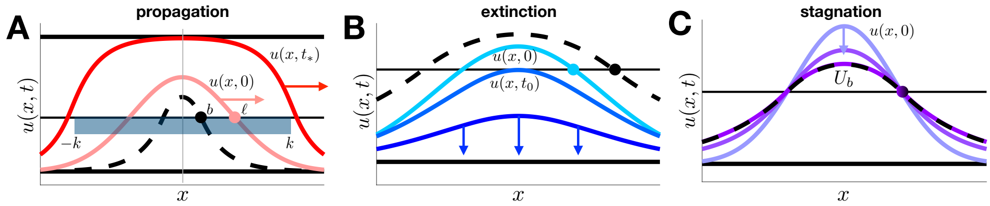


Figure 2.3: Long term behavior of $u(t,x)$ depends only on how the initial interface location $a(0) = \ell$ compares to the bump half-width, $b = \mathbf{K}^{-1}(\kappa)/2$. (A) If $\ell > b$, propagation occurs and $\lim_{t\to\infty} u(t,x) \equiv 1$, $\forall x \in K = [-k,k]$ for $k < \infty$. This follows from the fact that for any K, we can find a time t_* for which $u(t_*,x) > \kappa$, $\forall x \in K$. (B) If $\ell < b$, eventually $u(t,x) < \kappa$, right after the time t_0 when $u(t_0,0) = \kappa$, and so $\lim_{t\to\infty} u(t,x) \equiv 0$. (C) If $\ell = b$, stagnation occurs and $\lim_{t\to\infty} u(t,x) = U_b(x)$.

so that for any equal or later time $s \geq t_*$ we have

$$|\mathbf{K}(x + a(s)) - \mathbf{K}(x - a(s)) - 1| \leq \epsilon, \quad \forall x \in K.$$

We can solve for $u(t,x)$ starting for time t_* to obtain

$$u(t,x) = u(t_*,x)e^{t_*-t} + e^{-t} \int_{t_*}^{t} e^s \left(\mathbf{K}(x + a(s)) - \mathbf{K}(x - a(s))\right) \mathrm{d}s.$$

Using the fact that any solution is continuous, we have that $|u(t_*,x)| \leq M$ for all $x \in K$. As a consequence, we get that $\forall x \in K$,

$$\begin{aligned}|u(t,x) - 1| &= \left|(u(t_*,x) - 1)e^{t_*-t} + e^{-t} \int_{t_*}^{t} e^s \left(W(x + a(s)) - W(x - a(s)) - 1\right) \mathrm{d}s\right| \\ &\leq (1 + M)e^{t_*-t} + \epsilon.\end{aligned}$$

This implies that $\lim_{t\to\infty} |u(t,x) - 1| = 0$, $\forall x \in K$. As a consequence, the solutions of equation (2.3.1) locally uniformly converge to the homogeneous state $u \equiv 1$ as $t \to \infty$ (Figure 2.3A). Thus, we have propagation of $u \equiv 1$ into $u \equiv 0$ as time evolves.

Extinction. If $\ell < \mathbf{K}^{-1}(\kappa)/2$, then $a'(t) < 0$ and $0 < a(t) < \ell$ on $t \in (0, t_0)$. By continuity, there exists a finite $t_0 > 0$ such that $a(t_0) = 0$, at which point the interface dynamics, equations (2.3.8) and (2.3.9), breaks down. We know this because $\mathbf{K}(2a(t)) - \kappa < 0$ and decreases as $a(t)$ decreases. Note also that for $t \in (0, t_0)$ we consistently have $\alpha(t) < 0$. Inspecting equation (2.3.9) shows that $\lim_{t\to t_0^-} \alpha(t) = 0$ since $u_0'(0) = 0$. Thus, at time $t = t_0$, we have $0 \leq u(t_0,x) \leq \kappa$, and for $t \geq t_0$, $\partial_t u(t,x) = -u(t,x)$, so $u(t,x) = e^{t_0-t}u(t_0,x)$ for $t \geq t_0$, and $\lim_{t\to\infty} u(t,x) \equiv 0$, uniformly on $x \in \mathbb{R}$ (Figure 2.3B).

Stagnation. If $\ell = \mathbf{K}^{-1}(\kappa)/2$, then $a'(t) = 0$ for all times assuming $\alpha(t) < 0$, implying $a(t) \equiv \ell$. Plugging into equation (2.3.9) yields $\alpha(t) = (\mathcal{K}(2b) - \mathcal{K}(0))(1 - e^{-t}) + u_0'(\ell)e^{-t} < 0$ for $t > 0$. As a consequence, $a(t) = \ell$ for all times and $\lim_{t\to\infty} \alpha(t) = \mathcal{K}(2b) - \mathcal{K}(0)$. Furthermore, we can explicitly solve for

$$u(t,x) = \mathbf{K}(x + b) - \mathbf{K}(x - b) + e^{-t}\left[u_0(x) - \mathbf{K}(x + b) + \mathbf{K}(x - b)\right],$$

so $\lim_{t\to\infty} u(t,x) \equiv U_b(x)$, uniformly on $\mathbb{R}$. We call this case stagnation as the active region remains fixed for $t > 0$ (Figure 2.3C). ∎

It is possible to push further the interface dynamics approach to tackle a variety of other scenarios and treat the case of multiple active regions. In [82], we investigated the cases of periodic initial conditions and initial conditions with two symmetric active regions, but also the case where equation (2.3.1) is forced by an external input $I(t, x)$. In all cases, we managed to derive conditions relating the initial conditions (or external input), the connectivity kernel $\mathcal{K}$ and the threshold κ leading to either extinction, propagation or stagnation.

2.4 Perspectives

Propagation phenomena in heterogeneous neural fields. Although the assumption of cortical homogeneity is reasonable at a first place, there are lots of evidence that cortical areas should be thought as a heterogeneous medium. It is for example well known that the visual cortex have functional maps such as preferred orientation or ocular dominance maps which are superimposed on the underlying retinotopic representation of the visual field. This functional architecture is spatially organized (often assumed to be periodic) and lead to heterogeneities in connectivity kernels. In the framework of neural fields, such heterogeneities have been incorporated by taking connectivity kernels of the form $\mathcal{K}(x,y) = \widehat{\mathcal{K}}(x-y, y/\epsilon)$ for some parameter $\epsilon > 0$ that can be taken very small [30, 52] or very large [32]. In the one-dimensional case ($d = 1$), it has been reported that propagation failure can occur when the medium is sufficiently heterogeneous [30] and traveling pulsating waves have been shown to exist numerically [32, 52]. A first perspective could be to set a rigorous theoretical framework for the existence of traveling pulsating waves of (1.0.1) with connectivity kernels of the form $\widehat{\mathcal{K}}(x-y, y/\epsilon)$ in both limit $\epsilon \to 0$ and $\epsilon \to +\infty$ in the case where $\widehat{\mathcal{K}}$ is assumed to be periodic in its second argument. The question of the stability of such pulsating waves is also of interest and should be investigated. An important extension will be to that of temporal heterogeneities where the connectivity kernel is allowed to vary with time and a first natural step will be to consider periodic modulations. Finally, in a more exploratory direction, one could study the propagation of waves in disordered medium in which random spatial fluctuations are taken into account by for example assuming that the connectivity function is some stationary random field over a probability space indexed by a random variable.

Spreading properties in discrete neural networks. A first natural perspective is to complete the stability analysis of the traveling fronts constructed in Theorem 2.2.1 and obtain a nonlinear stability result along the lines of Conjecture 2.2.1. In parallel, I would like to investigate other network topologies by studying neural networks on various types of graphs (trees, Erdös-Rényi graphs, power law graphs, W-random graphs, etc...) which could be dense or sparse.

Threshold of propagation in neural fields and waves interactions. One perspective is to present a proof of Conjecture 2.3.1. The very first step will to be characterize the set of even stationary solutions of (1.0.1) which are monotone on the half line, and prove a uniqueness result. Indeed, in the local case [59, 203], this stationary solution (which is unique up to translation) serves as a separatrix for the dynamics starting from smooth unimodal even initial conditions and is of crucial importance. Similar techniques as the ones in [203] based on comparison principles could be extended to (1.0.1) and the key challenge will be to handle the nonlinear nonlocal term. It is interesting to investigate similar questions for the classical nonlocal reaction-diffusion equation (1.0.3). Although in this context the reaction term is local, it is expected the analysis in this setting to be more intricate (see Section 3.3). Indeed, for such equations, Chamj and Ren [45] have shown the existence of families of discontinuous even solutions and monotone on the half line which can be stable in $L^\infty(\mathbb{R})$ for the dynamics. It is of sharp contrast with the local case where these stationary solutions are always smooth and unstable for the dynamics. New threshold phenomena are expected to take place in that nonlocal case.

Recent experiments in rat visual cortex [99] have shown that when two visual stimuli are successively presented in the visual field of a rat three patterns of neural activity can emerge: fusion, suppression or independence. More precisely, these cortical states depend on the time delay of presentation in between two stimuli. If the two stimuli are well separated in time, then the cortical responses induced by each stimulus are independent and do not see each other. If this delay is reduced, the cortical response of the second stimulus can be totally suppressed. When this delay is even further reduced, the two cortical responses fuse. I would like to model and analyze this paradigm using neural field formalism. The suppression regime suggests an adaptation mechanism that needs to be incorporated into the neural field equation (1.0.1) in the form of linear adaptation [87] or synaptic depression [71]. In fact, it is my aim to show that this paradigm of fusion, suppression or independence is the counterpart for excitable systems of threshold of propagation found in scalar reaction-diffusion equations in the context of traveling pulse initiation.

Chapter 3

Nonlocal propagation problems: a dynamical systems approach

We start this chapter by presenting three examples (see Sections 3.1, 3.2 and 3.3) where the use of kernels with explicit form allows a fairly detailed analysis of the problems under consideration. Note that each example (and thus corresponding section) is independent from one another. The first example shows the existence and stability of fast traveling pulses for neural field equations supplemented with synaptic depression. The second one deals with modulated fronts for the Fisher-KPP equation with nonlocal interactions. Finally, the last example explores pinning regions and unpinning asymptotics for the transition from stationary to traveling fronts in scalar reaction-diffusion equations with nonlocal diffusion and bistable dynamics. Then, in Section 3.4 we present a key result about Fredholm properties of nonlocal differential operators which is used in the subsequent two sections to prove the existence of traveling pulses for the nonlocal FitzHugh-Nagumo equations and establish center manifold theorems for nonlocal equations.

3.1 Existence and stability of traveling pulses in neural field equations with synaptic depression

We consider a neural network which includes synaptic depression [137, 201] such that the neural field equation (1.0.1) is modified according to the following system of equations:

$$\left\{ \begin{array}{l} \partial_t u(t,x) = -u(t,x) + \displaystyle\int_{\mathbb{R}} \mathcal{K}(x-y)q(t,y)S(u(t,y))\mathrm{d}y, \\ \partial_t q(t,x) = \epsilon\left(1-q(t,x)-\beta q(t,x)S(u(t,x))\right), \end{array} \right. \tag{3.1.1}$$

for $t>0$ and $x\in\mathbb{R}$. The first equation describes the evolution of the synaptic current $u(t,x)$ in the presence of synaptic depression which takes the form of a synaptic scaling factor $q(t,x)$ evolving according to the second equation. This factor can be interpreted as a measure of available presynaptic resources, which are depleted at a rate $\epsilon\beta S$, and recovered on a time scale specified by the constant ϵ. We assume units of time t to be 10ms each. Experimental recordings [188] suggest that synaptic depression recovers on a timescale of $200-800$ms, so that $1/\epsilon$ typically ranges from 20 to 80 and thus $\epsilon \sim 0.01-0.05$ can be consider as a small parameter in equations (3.1.1). The range of allowable values for β, as used in [137] with $\epsilon \sim 0.01-0.05$, is $\beta \sim 1-20$. The nonlinear firing-rate function is taken to be the following smooth function

$$S(u) = \frac{1}{1+e^{-\lambda(u-\kappa)}} \tag{3.1.2}$$

with threshold κ and gain λ. We take the excitatory weight function $\mathcal{K}$ to be a normalized exponential [137],

$$\mathcal{K}(x) = \frac{b}{2}e^{-b|x|} \tag{3.1.3}$$

where $b > 0$ is the effective range of excitatory distribution.

We are interested to prove the existence and stability of traveling wave solutions for equation (3.1.1) in the regime $0 < \epsilon \ll 1$. To do so, we introduce a new coordinate $\xi = x + ct$ for some unknown $c \in \mathbb{R}$ and then express the neural field system (3.1.1) in these new coordinates as

$$\left\{\begin{array}{l} \partial_t u(t,\xi) = -c\partial_\xi u(t,\xi) - u(t,\xi) + \displaystyle\int_{\mathbb{R}} \mathcal{K}(\xi-y)q(t,y)S(u(t,y))\mathrm{d}y, \\ \partial_t q(t,\xi) = -c\partial_\xi q(t,\xi) + \epsilon\left(1-q(t,\xi)-\beta q(t,\xi)S(u(t,\xi))\right). \end{array}\right. \tag{3.1.4}$$

Traveling wave solutions are thus time independent solutions of these equations that satisfy the functional differential equations of mixed type

$$\left\{\begin{array}{l} cu'(\xi) = -u(\xi) + \displaystyle\int_{\mathbb{R}} \mathcal{K}(\xi-\xi')q(\xi')S(u(\xi'))\mathrm{d}\xi', \\ cq'(\xi) = \epsilon\left(1-q(\xi)-\beta q(\xi)S(u(\xi))\right), \end{array}\right. \tag{3.1.5}$$

where u' stands for $\dfrac{\mathrm{d}u}{\mathrm{d}\xi}$. We now use the specific form of the kernel $\mathcal{K}$ to write (3.1.5) as the system of ordinary differential equations

$$\left\{\begin{array}{rl} b^2 v(\xi) - v''(\xi) & = b^2 q(\xi)S(u(\xi)), \\ cu'(\xi) & = -u(\xi) + v(\xi), \\ cq'(\xi) & = \epsilon\left(1-q(\xi)-\beta q(\xi)S(u(\xi))\right), \end{array}\right.$$

which can be converted into a system of first-order equations

$$\left\{\begin{array}{l} u' = \frac{1}{c}\left(-u+v\right), \\ v' = w, \\ w' = b^2\left(v - qS(u)\right), \\ q' = \dfrac{\epsilon}{c}\left(1-q-\beta qS(u)\right). \end{array}\right. \tag{3.1.6}$$

In order to prove the existence of a traveling pulse solution of equation (3.1.1), we will need to have some hypotheses on the different parameters of our system $(\lambda, \kappa, \beta, b)$. The following hypothesis ensures that there exists a unique stationary homogeneous solution of system (3.1.1).

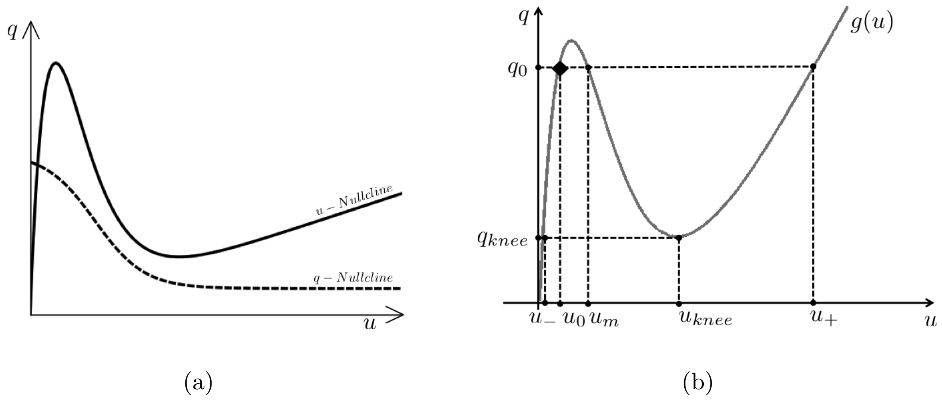


Figure 3.1: (a) A typical graph of the nullclines of system (3.1.6) in the (u, q)-plane when the conditions of Hypothesis 3.1.1 are satisfied.(b) Illustration of the assumptions on the function g of Hypothesis 3.1.2. The *large diamond* is the fixed point of the system (u_0, q_0).

Hypothesis 3.1.1. *Suppose that* $(\lambda, \kappa) \in (0, \infty) \times (0, 1)$ *satisfy the relation*

$$2 - 2\ln(2) \leq \lambda\kappa - \ln(\lambda)$$

and $\beta > \beta_c(\lambda, \kappa)$ *for some explicit* $\beta_c(\lambda, \kappa) > 0$[1]*, then there exists a unique stationary homogeneous solution of system* (3.1.1) *that we denote* (u_0, q_0)*.*

The second hypothesis that we formulate is on the shape of our nonlinear function S.

Hypothesis 3.1.2. *Let* g *be the* $\mathscr{C}^\infty$*-smooth function defined through*

$$g(u) = \frac{u}{S(u)}. \tag{3.1.7}$$

We suppose that (λ, κ) *are such that there exist* $u_+ > u_m > 0$ *with* $g(u_0) = g(u_m) = g(u_+) = q_0$ *together with* $g'(u_0) > 0$*,* $g'(u_m) < 0$ *and* $g'(u_+) > 0$*. We further suppose that* (λ, κ) *are such that*

$$\int_{u_0}^{u_+} -u + q_0 S(u) du > 0. \tag{3.1.8}$$

On account of Hypothesis 3.1.2, we may choose closed intervals I_L and I_R with $u_0 \in I_L$ and $u_+ \in I_R$, that have nonempty interiors and in addition have $g'(u) > 0$ for all $u \in I_L \cup I_R$. There exist constants $q_{knee} < q_0 < q_{max}$ in such way that we can define two $\mathscr{C}^\infty$-smooth function $s_L : (q_{knee}, q_{max}) \to I_L$ and $s_R : (q_{knee}, q_{max}) \to I_R$ with

$$g(s_L(q)) = g(s_R(q)) = q$$

for all $q \in (q_{knee}, q_{max})$. Notice that $s_L(q_0) = u_0$ and $s_R(q_0) = u_+$. We define by continuity $u_{knee} = s_R(q_{knee})$ and $u_- = s_L(q_{knee})$. We thus have the ordering:

$$u_- < u_0 < u_m < u_{knee} < u_+.$$

[1] See [71, Lemma 2.1]

We refere to Figure 3.1(b) for an illustration. Let us finally remark that under the assumptions of Hypothesis 3.1.1 the couple (u_{knee}, q_{knee}) is the unique solution of

$$\begin{cases} 0 = -u + qS(u), \\ 0 = -1 + qS'(u). \end{cases} \tag{3.1.9}$$

The key point for proving the existence of traveling pulse solutions of (3.1.6), homoclinic to (u_0, q_0), is to first construct a singular traveling wave solution for $\epsilon = 0$, and then show that this singular solution persists for small positive values of $0 < \epsilon \ll 1$. The singular solution is constructed by looking at the slow and fast subsystems associated to (3.1.6). The slow subsystem can be found by rescaling the independent variable $z = \epsilon\xi$, and then formally setting $\epsilon = 0$ in the resulting equation to obtain

$$\begin{cases} 0 = \frac{1}{c}\left(-u + v\right), \\ 0 = w, \\ 0 = b^2\left(v - qS(u)\right), \\ q_z = \dfrac{1}{c}\left(1 - q - \beta qS(u)\right). \end{cases} \tag{3.1.10}$$

For the above reduced slow system (3.5.8), there exists associated leading order slow manifolds given by two pieces:

$$\mathcal{M}_L := \{(s_L(q), q)\} \text{ and } \mathcal{M}_R := \{(s_R(q), q)\}.$$

The slow dynamics on these manifolds is given by

$$q_z = \frac{1}{c}\left(1 - q - \beta qS(u)\right) \text{ for } (u,q) \in \mathcal{M}_j, \quad j = L, R.$$

The fast system is obtained by simply setting $\epsilon = 0$ in (3.1.6)

$$\begin{cases} u' = \frac{1}{c}\left(-u + v\right), \\ v' = w, \\ w' = b^2\left(v - qS(u)\right), \\ q' = 0. \end{cases} \tag{3.1.11}$$

For the fast system, we seek a leading order solution connecting the reduced fixed point on $\mathcal{M}_L$ at $(u, v, w, q) = (u_0, u_0, 0, q_0)$ to the fixed point on $\mathcal{M}_R$ at $(u, v, w, q) = (u_+, u_+, 0, q_0)$ for some value of the wave speed c. In the neural field formalism, this is equivalent to find a traveling wave solution $u(t, x) = u_f(x + c_* t)$ of

$$\partial_t u(t,x) = -u(t,x) + q_0 \int_{\mathbb{R}} \mathcal{K}(x-y) S(u(t,y)) \mathrm{d}y \tag{3.1.12}$$

for some wave speed $c_* \in \mathbb{R}$ and profile $u_f \in \mathscr{C}^1(\mathbb{R}, \mathbb{R})$ that satisfies the limits

$$\lim_{\xi \to -\infty} u_f(\xi) = u_0 \text{ and } \lim_{\xi \to +\infty} u_f(\xi) = u_+. \tag{3.1.13}$$

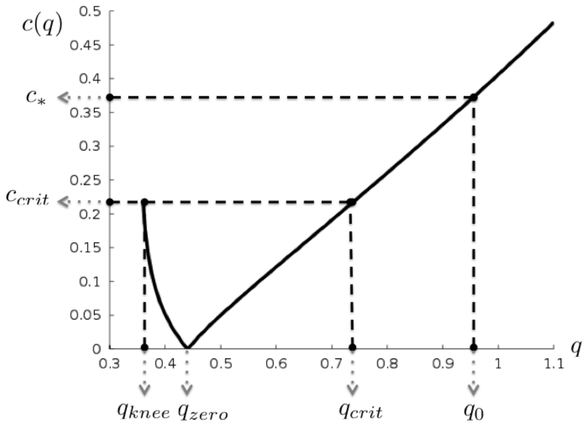


Figure 3.2: A typical graph of the wave speed for the front and back of the wave, with $\lambda = 20$, $\kappa = 0.22$, $b = 4.5$ and $\beta = 5$. Note that $q_{knee} = 0.3605$, $q_{zero} = 0.4405$ and $q_{crit} = 0.7352$ with $c_{crit} = 0.2197$. The reduced model admits traveling front solutions for $q \in [q_{zero}, q_{max}]$ and traveling back solutions for $q \in [q_{knee, q_{zero}}]$. Note that this picture is typical of the homoclinic orbits that we study: no value of $q \in [q_{crit}, q_{max}]$ leads to a singular connection between $\mathcal{M}_L$ and $\mathcal{M}_R$. Therefore, the jump back must occur at the knee.

If Hypothesis 3.1.2 holds, then we know that such a front exists from Theorem 1.2.4. Adapting the proof of [66], we obtain the following formula for the wave speed c_* as a function of q_0:

$$c_* = c(q_0) = \frac{1}{q_0} \frac{\displaystyle\int_{u_0}^{u_+} [-u + q_0 S(u)]\, \mathrm{d}u}{\displaystyle\int_{-\infty}^{+\infty} \left(u_f'(\xi)\right)^2 S'(u_f(\xi))\mathrm{d}\xi} > 0. \tag{3.1.14}$$

A quick look at formula (3.1.14) shows that there should be a switch from positive to negative $c(q)$ at some value q_{zero} where $c(q_{zero}) = 0$. Then for all $q > q_{zero}$, $c(q) > 0$. The front selects the wave speed of the pulse. In turn, the wave speed of the pulse selects the particular value of q for which a jump back exists connecting the right slow manifold to the left slow manifold. It turns out that for typical values of the parameters, as illustrated in Figure 3.2, we observe that above a particular value of q, there exists no choice of q for which such a connection between the right slow manifold and left slow manifold can be found. We will label this value of q as $q_{crit} \in [q_{zero}, q_{max}]$. It is defined by the condition that

$$c(q_{crit}) = c(q_{knee}) := c_{crit}.$$

Therefore, in that case, the only possibility is that the jump back from the right slow manifold to the left slow manifold occurs at the knee of the right slow manifold. Here, the knee is precisely the point (u_{knee}, q_{knee}). Note that the knee is not a hyperbolic fixed point, as a consequence the results of Theorem 1.2.4 are no longer valid. However, the existence of such a connection is well understood in the case of the generalized Fisher-KPP equation of order 2 [25, 26] and can be extended to the neural field formalism. We have the following result which relies on the specific form of the kernel $\mathcal{K}$.

Proposition 3.1.1. *For each $c \geq c_{crit}$ there exists a traveling back solution $u_b(x+ct)$ to*

$$\partial_t u(t,x) = -u(t,x) + q_{knee} \int_{\mathbb{R}} \mathcal{K}(x-y) S(u(t,y)) \mathrm{d}y \tag{3.1.15}$$

with profile $u_b \in \mathscr{C}^1(\mathbb{R},\mathbb{R})$ and that satisfies the limits

$$\lim_{\xi \to -\infty} u_b(\xi) = u_{knee} \text{ and } \lim_{\xi \to +\infty} u_b(\xi) = u_-. \tag{3.1.16}$$

Moreover, u_b satisfies the asymptotic expansions at $\xi = -\infty$

$$u_b(\xi) \sim \begin{cases} u_{knee} - \alpha \exp\left(\frac{-1+\sqrt{1+4b^2c^2}}{2c}\xi\right) & c = c_{crit} \\ u_{knee} - \left(\dfrac{q_{knee} S''(u_{knee})}{2c}\right)^{-1} \dfrac{1}{\xi} & c > c_{crit} \end{cases} \quad \text{as } \xi \to -\infty \tag{3.1.17}$$

with α some positive constant.

Thus, this result ensures that for any $c \geq c_{crit}$ there exists a connection between the knee at $(u_{knee}, u_{knee}, 0, q_{knee})$ and the left branch of the slow manifold at $(u_-, u_-, 0, q_{knee})$. Putting the information from the reduced slow and fast dynamics together, the singular solution consists of four pieces as follows:

(1) a fast jump from $(u_0, u_0, 0, q_0)$ to $(u_+, u_+, 0, q_0)$, which is given by the profile u_f solution of (3.1.12) with speed $c_* > 0$;

(2) slow decay along $\mathcal{M}_R$ from $(u_+, u_+, 0, q_0)$ to $(s_R(q_{knee}), s_R(q_{knee}), 0, q_{knee})$;

(3) a fast jump from $(s_R(q_{knee}), s_R(q_{knee}), 0, q_{knee})$ to $(s_L(q_{knee}), s_L(q_{knee}), 0, q_{knee})$ that departs along the center-unstable manifold;

(4) slow growth along $\mathcal{M}_L$ back to $(u_0, u_0, 0, q_0)$.

Definition 3.1.1. *The set, labelled Π, of allowable parameters $(\lambda, \kappa, b, \beta)$ for the model in (3.1.1) consists of those parameters such that Hypotheses 3.1.1 and 3.1.2 are satisfied together with $b > 0$, and such that the wave speed selected by the front is strictly greater than the wave speed selected by the back.*

We can now state the first main result.

Theorem 3.1.1. *Suppose that $(\lambda, \kappa, b, \beta) \in \Pi$. Then there exists $\epsilon_1 > 0$ such that for all $0 < \epsilon < \epsilon_1$, there exists $c(\epsilon) = c_* + \mathcal{O}(\epsilon)$ for which problem (3.1.1) has a traveling pulse solution of the form $(u(x+ct), q(x+ct))$ with $\lim_{\xi \pm \infty}(u,q) = (u_0, q_0)$.*

The idea of the proof is to demonstrate that the singular pulse described above persists for small positive $0 < \epsilon \ll 1$. The main difficulty relies on the fact that the solution passes close to the knee of the slow manifold that is no longer normally hyperbolic. One needs to rely on blow-up techniques to overcome this issue [147]. Let us note that this type of problem has already been encountered in biological model of electrical cardiac wave [18] and in the propagation of wave in deformable media [118].

We can then turn to the stability of traveling pulse given by Theorem 3.1.1.

Theorem 3.1.2. *Suppose that $(\lambda, \kappa, b, \beta) \in \Pi$. Then there exists $\epsilon_2 > 0$ such that for all $0 < \epsilon < \epsilon_2$, the traveling pulse solution from Theorem 3.1.1 is spectrally stable with a simple zero eigenvalue at $\lambda = 0$ due to translational invariance of the pulse.*

The proof of the above theorem follows several steps. First, we linearize the system (3.1.4) at the traveling pulse and show that the resulting essential spectrum is bounded to the left of the imaginary axis, where the bound depends upon ϵ. Then, we construct Evans functions associated (i) with the full problem and (ii) with the reduced fast pieces along the front and the back of the pulse. In a third step, we show that the eigenvalues of the full Evans function are solely determined by those of the reduced problems which allows us to determine the spectral stability of the pulse. More precisely, we show that the only zero of the Evans function in the right-half plane is zero, and its geometric and algebraic multiplicity is one. The crucial step in this analysis is to show that the knee does not produce any additional eigenvalue which can be shown through winding number computations. Our definition of the Evans function is somewhat different from the one previously used for neural field models with Heaviside firing rate function [53, 176]. In our case, the Evans function is not known through an explicit formula due to our choice of the nonlinearity. However, we can collect enough information to determine the location of its zeros. Finally, we further note that the linear stability of the traveling pulse solution from Theorem 3.1.1 follows from a spectral mapping theorem [164] for the strongly continuous semigroup generated by the associated linear operator around the traveling pulse. In addition, we can use standard invariant manifold theory and [17, Theorem 4.3] to show that the traveling pulse is nonlinearly stable as well. Indeed, the zero eigenvalue of Theorem 3.1.2 is isolated.

Recall that in [137], Kilpatrick & Bressloff have used a Heaviside firing rate function instead of the smooth firing rate function (3.1.2). It is important to note that the singular perturbation analysis presented here breaks down if one uses a Heaviside function as Hypothesis 3.1.2 is no longer satisfied. However, Kilpatrick & Bressloff [137] have explicitly derived closed formulas for the traveling pulse solution. Their constructive approach predicts the existence of two types of traveling pulse solutions: a fast pulse (with a wide profile) and a slow pulse (with a narrow profile). The fast traveling pulse solution, obtained with a Heaviside function, is qualitatively similar to the pulse found in our study. Indeed, in both models, the fast pulse is found to be spectrally stable. Both fast pulses have a wide profile. Note however that the Heaviside model predicts an exponential decay along the back of the pulse while our model predicts an algebraic decay, so that there still exists a slight difference in the profile of the solution. As we have used singular perturbation theory, it is not possible to directly predict the existence of the slow pulse without doing some modifications that we now outline. In order to prove the existence of a slow pulse, one needs to rescale the wave speed c appropriately. We anticipate that the wave speed will scale as $c = \widetilde{c}\sqrt{\epsilon}$ with $\widetilde{c} = \mathcal{O}(1)$. With this new scaling, one can use singular perturbation theory to prove the existence of a traveling pulse solution along similar lines of the proof of Theorem 3.1.1. The existence of slow traveling pulse was recently achieved in [113] using ODE techniques only. As for the Heaviside model, we expect this slow pulse solution to be unstable.

3.2 Modulated traveling fronts for a nonlocal Fisher-KPP equation

We consider the following nonlocal version of the Fisher-KPP equation

$$\partial_t u(t,x) = \partial_x^2 u(t,x) + \mu u(t,x)\left(1 - \int_{\mathbb{R}} \mathcal{K}(x-y)u(t,y)\mathrm{d}y\right), \quad t>0 \quad x \in \mathbb{R}, \tag{3.2.1}$$

where $\mu > 0$ represents the strength of the nonlocal competition. Throughout this section, we will assume that the kernel $\mathcal{K}$ satisfies the following hypotheses.

Hypothesis 3.2.1. *The kernel $\mathcal{K}$ satisfies:*

$$\mathcal{K} \geq 0, \quad \mathcal{K}(0) > 0, \quad \mathcal{K}(-x) = \mathcal{K}(x), \quad \int_{\mathbb{R}} \mathcal{K}(x)dx = 1, \text{ and } \int_{\mathbb{R}} x^2 \mathcal{K}(x)dx < \infty.$$

In the limiting case where $\mathcal{K}$ is replaced by the Dirac δ-function, the nonlocal partial differential equation (3.2.1) reduces to the classical Fisher-KPP equation [92, 143]

$$\partial_t u = \partial_x^2 u + \mu u(1-u), \quad t>0, \quad x \in \mathbb{R}. \tag{3.2.2}$$

Such an equation (3.2.2) arises naturally in many mathematical models in biology, ecology or genetics, and u typically stands the density of some population. The nonlocal equation (3.2.1) can then be interpreted as a generalization of the local Fisher-KPP equation (3.2.2) in which interactions among individuals are nonlocal. For more details on such nonlocal models, we refer to [96, 102, 106] among others.

As emphasized in the introduction, the behavior of the solutions of the local equation (3.2.2) has been studied for decades and is now well understood while much less is known about solutions to the nonlocal equation (3.2.1). Indeed, from a mathematical point of view, the analysis of (3.2.1) is quite involved since this class of equations with a nonlocal competition term generally does not satisfy the comparison principle. Recently, theoretical and numerical studies [22, 68, 161] have shown that for sufficiently small μ, the solutions share many of the same properties of the local Fisher-KPP equation in that there exists a family of traveling wave solutions of the form

$$u(t,x) = U(x-ct), \quad \lim_{\xi \to -\infty} U(\xi) = 1, \quad \lim_{\xi \to +\infty} U(\xi) = 0, \quad U \text{ decreasing}. \tag{3.2.3}$$

It is known that, again for μ sufficiently small, these traveling waves and the homogeneous stationary solutions $u(t,x)=1$ and $u(t,x)=0$ are the only entire bounded solutions to (3.2.1), see [1, 3, 22, 68]. On the other hand, when μ is large, some other bounded solutions may exist as suggested by the numerical exploration of [161]. More precisely, if the Fourier transform of the kernel $\mathcal{K}$ takes some negative values, then for sufficiently large μ, the trivial state $u(t,x)=1$ is Turing unstable for (3.2.1). This suggests the emergence of non-monotonic bounded solutions [7, 102]. Indeed, recent work by Hamel and Ryzhik [109] has shown the existence of stationary periodic solutions u satisfying,

$$0 = \partial_x^2 u + \mu u(1 - \mathcal{K} * u), \quad x \in \mathbb{R}, \tag{3.2.4}$$

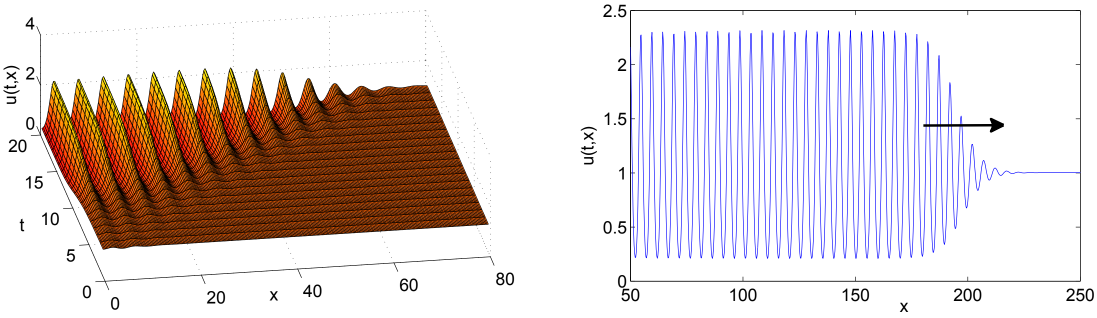


Figure 3.3: A modulated traveling front obtained from direct numerical simulation of (3.2.1) with kernel (3.2.8) for $a = 0.7$ and $\mu = 32$. The front invades the Turing unstable state $u = 1$ and leaves a stationary periodic pattern in its wake.

for large μ when the Fourier transform of the kernel attains negative values.

Recent numerical studies of (3.2.1) also suggest the existence of modulated traveling fronts where these stationary periodic solutions invade the Turing unstable state $u = 1$, see [161] and Figure 3.3. These modulated traveling fronts are the focus of our study. For a certain class of kernels, we will prove the existence of modulated traveling fronts of the form

$$u(t,x) = U(x - ct, x), \quad \lim_{\xi \to -\infty} U(\xi, x) = 1 + P(x), \quad \lim_{\xi \to +\infty} U(\xi, x) = 1, \tag{3.2.5}$$

where $P(x)$ is a stationary periodic solution of the shifted problem

$$0 = \partial_x^2 v - \mu v - \mu v \mathcal{K} * v, \quad x \in \mathbb{R}. \tag{3.2.6}$$

The first step of our analysis will be to refine the existence result of [109] for parameter values near the onset of Turing instability and then use center manifold techniques to construct modulated traveling fronts of the form (3.2.5). In that direction, we also point out that explicit examples of wave-train solutions have been recently constructed in [61, 162] for a different nonlocal problem.

Before stating our main results, we first make some further assumptions on the kernel $\mathcal{K}$. Linearizing equation (3.2.1) around the stationary homogeneous state $u = 1$, we find the following dispersion relation,

$$d(\lambda, k, \mu) := -k^2 - \mu \widehat{\mathcal{K}}(k) - \lambda. \tag{3.2.7}$$

Hypothesis 3.2.2. *For $\mathcal{K}$ satisfying Hypothesis 3.2.1, we further assume that there exists unique $k_c > 0$ and $\mu_c > 0$, such that the following conditions are satisfied,*

(i) $d(0, k_c, \mu_c) = 0$.

(ii) $\partial_k d(0, k_c, \mu_c) = 0$.

(iii) $\partial_{kk} d(0, k_c, \mu_c) < 0$.

The first condition imposes that $\widehat{\mathcal{K}}(k_c) < 0$ as from the dispersion relation $d(0, k_c, \mu_c) = 0$, we have that $\widehat{\mathcal{K}}(k_c) = -\frac{k_c^2}{\mu_c} < 0$. The second condition ensures that k_c is a double root of the dispersion relation and combined with the third condition, that μ_c represents the onset of instability. For $\mu < \mu_c$ all the spectrum is to the left of the imaginary axis while for $\mu > \mu_c$ there is a band of wave numbers surrounding $k = \pm k_c$ that are unstable.

For some part of our analysis, we will work with a specific kernel that satisfies Hypothesis 3.2.1, namely we will choose

$$\mathcal{K}(x) := Ae^{-a|x|} - e^{-|x|}, \tag{3.2.8}$$

for some values of $A > 0$ and $a > 0$. Recall that Hypothesis 3.2.1 requires that $\mathcal{K}(x) > 0$ and $\int_{\mathbb{R}} \mathcal{K}(x)\mathrm{d}x = 1$. The second condition implies that $A = 3a/2$, and the first condition in turn implies that $a \in (2/3, 1)$. The choice of such a specific kernel is motivated by the fact that equation (3.2.1) can be reduced to a system of partial differential equations. Indeed, define

$$v(t,x) := Ae^{-a|x|} * u(t,x), \quad w(t,x) := -e^{-|x|} * u(t,x),$$

we find that (3.2.1) reduces to the following system,

$$\begin{cases} \partial_t u = \partial_x^2 u + \mu u(1 - v - w), \\ 0 = \partial_x^2 v - a^2 v + 3a^2 u, \\ 0 = \partial_x^2 w - w - 2u. \end{cases} \tag{3.2.9}$$

The first result concerns the existence of stationary periodic solutions of the nonlocal equation (3.2.1) and can be stated as follows.

Theorem 3.2.1. *Assume that Hypothesis 3.2.1 and 3.2.2 are satisfied. Let $\mu := \mu_c + \epsilon^2$ and $k := k_c + \delta$. There exists $\epsilon_0 > 0$, such that for all $\epsilon \in (0, \epsilon_0]$ and all $\delta^2 < \dfrac{-\widehat{\mathcal{K}}(k_c)}{1 + \frac{\mu_c}{2}\widehat{\mathcal{K}}''(k_c)}\epsilon^2$ there is a stationary $\frac{2\pi}{k}$-periodic solution of (3.2.1) with leading expansion of the form*

$$\mathbf{u}_{\epsilon,\delta}(x) = 1 + \sqrt{\frac{\widehat{\mathcal{K}}(k_c)\epsilon^2 + \left(1 + \frac{\mu_c}{2}\widehat{\mathcal{K}}''(k_c)\right)\delta^2}{\omega}} \cos\left((k_c + \delta)x\right) + \mathcal{O}\left(\left|\epsilon^2 - \delta^2\right|\right), \tag{3.2.10}$$

where $\omega < 0$[2]*. Moreover, for any $\tau \in [0, 2\pi/k]$, $(x) \mapsto \mathbf{u}_{\epsilon,\delta}(x + \tau)$ is also a solution of (3.2.1).*

First, note that the results of Theorem 3.2.1 do not rely on a specific form of the kernel. This theorem complements the study of Hamel & Ryzhik [109] where they also proved the existence of stationary periodic solutions of (3.2.1). While the analysis in [109] is global and relies on degree theory and in the regime μ large, our study is local and uses center manifold theory. To some extent, our approach gives sharper results close to the bifurcation point μ_c as we obtain a complete description of all bounded stationary solutions of (3.2.1) in some neighborhood of the solution $u = 1$. Furthermore, we show the existence of a family of periodic solutions indexed by their spatial frequency $k \approx k_c$.

The second main result is a spectral analysis of the stationary periodic solutions found in Theorem 3.2.1 and our results are summarized as follows.

[2] The explicit expression of ω can be found in [78, Lemma 2.1].

Theorem 3.2.2. *Assume that Hypothesis 3.2.1 and 3.2.2 are satisfied. Then, the following assertions are true.*

(i) The periodic solutions $\mathbf{u}_{\epsilon,\delta}$ *given in Theorem 3.2.1 are neutrally stable with respect to perturbations of the same period* $2\pi/k$*, where* $k = k_c + \delta$ *and* δ *satisfies the relation* $\delta^2 < \dfrac{-\widehat{\mathcal{K}}(k_c)}{1 + \frac{\mu_c}{2}\widehat{\mathcal{K}}''(k_c)}\epsilon^2$ *for all* $\epsilon \in (0, \epsilon_0]$.

(ii) The periodic solutions $\mathbf{u}_{\epsilon,\delta}$ *given in Theorem 3.2.1 are spectrally unstable with respect to perturbations of the form* $e^{i\sigma x}V(x)$*, where* $V \in H^2_{per}[0, 2\pi]$ *is a solution the spectral problem obtained by linearizing (3.2.1) around* $\mathbf{u}_{\epsilon,\delta}$*, in the limit* $\sigma \to 0$ *and whenever* δ *satisfies*

$$\frac{-\widehat{\mathcal{K}}(k_c)}{3\left(1 + \frac{\mu_c}{2}\widehat{\mathcal{K}}''(k_c)\right)}\epsilon^2 < \delta^2 < \frac{-\widehat{\mathcal{K}}(k_c)}{1 + \frac{\mu_c}{2}\widehat{\mathcal{K}}''(k_c)}\epsilon^2, \tag{3.2.11}$$

for $\epsilon \in (0, \epsilon_0]$.

The first part of this theorem is a direct consequence of the center manifold reduction used in the existence proof of Theorem 3.2.1. Indeed, the spectral analysis of the periodic solutions with respect to perturbations of the same period can be directly done on the reduced two dimensional equation on the center manifold where one finds two eigenvalues $\lambda_c = 0$ and $\lambda_s < 0$. The fact that there exists a critical eigenvalue is due to the translation of invariance of the problem, namely $\frac{\mathrm{d}}{\mathrm{d}x}\mathbf{u}_{\epsilon,\delta}$ is always in the kernel of the linearized operator. The second part of the theorem is a perturbation analysis, using Lyapunov Schmidt reduction, where we show that the critical eigenvalue $\lambda_c = 0$ is perturbed into $\lambda_c = \mathbf{g}(\epsilon, \delta)\sigma^2 + \mathcal{O}(\sigma^4)$, as $\sigma \to 0$ when we are looking for perturbations of the form $e^{i\sigma x}V(x)$, $V \in H^2_{per}[0, 2\pi]$. The region in parameter space (ϵ, δ) where $\mathbf{g}(\epsilon, \delta) > 0$ will then give spectral instability with respect to such perturbations, see Figure 3.4. Such a technique was introduced by Mielke for the study of sideband instabilities in the Swift-Hohenberg equation [157].

The central result of this study is the proof of the existence of modulated traveling front solutions that are asymptotic at infinity to the stationary periodic solutions found in Theorem 3.2.1 and the homogeneous state $u = 1$. We will realize these modulated traveling fronts as heteroclinic orbits of a reduced system of ODEs in normal form. Roughly speaking, and fixing the frequency to k_c, we look for solutions of (3.2.1) that can be written as

$$u(t, x) = U(x - \epsilon st, x) = \sum_{n\in\mathbb{Z}} U_n(x - \epsilon st)e^{-\mathbf{i}nk_c x},$$

where $\epsilon = \sqrt{\mu - \mu_c}$. Replacing this ansatz into the equivalent system (3.2.9) will lead to the study of an infinite dimensional dynamical system of the form

$$\partial_\xi U_n = \mathcal{L}_n^\epsilon U_n + \mathcal{R}_n(U, \epsilon). \tag{3.2.12}$$

The main difficulty in studying (3.2.12) comes from the presence at onset ($\epsilon = 0$) of an infinite dimensional central part. However, as $0 < \epsilon \ll 1$, these eigenvalues will leave the imaginary axis with different velocities. A finite number will stay close ($\mathcal{O}(\epsilon)$) to the imaginary axis while all

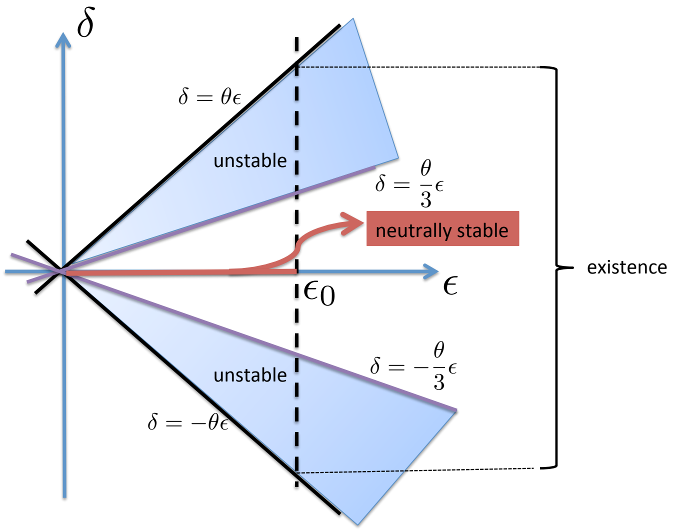


Figure 3.4: Sketch in the (ϵ, δ)-plane of the regions of existence and spectral stability given by Theorem 3.2.1 and 3.2.2. Here, we have denoted $\theta := \sqrt{\frac{-\widehat{\mathcal{K}}(k_c)}{1+\frac{\mu_c}{2}\widehat{\mathcal{K}}''(k_c)}}$.

other eigenvalues leave fast enough ($\mathcal{O}(\sqrt{\epsilon})$) so that a spectral gap exists. This gap will allow for small $\epsilon > 0$ the construction of a finite dimensional invariant manifold of size $\mathcal{O}(\epsilon^{2/3+\gamma})$, for $\gamma > 0$. This manifold will contain the modulated traveling fronts that we are looking for. The result that we obtain can be formulated as follows.

Theorem 3.2.3. *Assume that $\mathcal{K}$ is the kernel given in (3.2.8) and that Hypothesis 3.2.2 is satisfied. Provided that $s > \sqrt{-4\widehat{\mathcal{K}}(k_c)\zeta}$, where $\zeta := 1 + \mu_c\widehat{\mathcal{K}}''(k_c)/2$, there is an $\epsilon_0 > 0$ such that for all $\epsilon \in (0, \epsilon_0)$, and all $\delta^2 < -\frac{\widehat{\mathcal{K}}(k_c)}{1+\frac{\mu_c}{2}\widehat{\mathcal{K}}''(k_c)}\epsilon^2$, equation (3.2.1) has modulated traveling front solutions of frequency $k_c + \delta$ and of the form*

$$u(t, x) = U(x - \epsilon st, x) = \sum_{n\in\mathbb{Z}} U_n(x - \epsilon st)e^{-\mathbf{i}n(k_c+\delta)x},$$

with the boundary conditions at infinity

$$\lim_{\xi\to-\infty} U(\xi, x) = \mathbf{u}_{\epsilon,\delta}(x) \text{ and } \lim_{\xi\to+\infty} U(\xi, x) = 1.$$

The first known existence results of modulated traveling waves are due to Collet & Eckmann [50, 51] and Eckmann & Wayne [63], who proved the existence of such solutions in the Swift-Hohenberg equation with cubic nonlinearities. The techniques developed in [63] have then been generalized for the problem of bifurcating fronts for the Taylor-Couette problem in infinite cylinders by Haragus & Schneider in [110] with quadratic nonlinearities, and our proof of Theorem 3.2.3 will rely on a center manifold result presented in [110]. Finally, note that similar results have been obtained in the two-dimensional Swift-Hohenberg equation for more general modulated fronts, for example modulated fronts that connect stable hexagons with unstable roll solutions [58].

Perspective. Let us conclude by noticing that from the perspective of the original problem (3.2.1) and the related Fisher-KPP equation it is often the dynamics for initial data near the state $u = 0$ that is of interest. Here, one observes traveling fronts where the zero state is invaded by a periodic stationary state around $u = 1$. Sometimes an intermediate region where the solution is approximately in the state $u = 1$ is observed. Invasion fronts of this form were numerically computed in [161]. Since the stationary periodic solutions come in families, one expects that the invasion process is dynamically selecting a particular pattern amongst this family of solutions. When $\mu \approx \mu_c$, the primary front where $u = 1$ replaces $u = 0$ travels much faster than the secondary modulated front and the selected pattern is determined by the modulated traveling front propagating with the minimal speed. However, when μ is large the numerically observed speeds of the secondary modulated traveling front exceed or are of the same order as that of the primary front and the pattern selection mechanism is more difficult to characterize.

3.3 Pinning and upinning in nonlocal systems

Relaxation to the energy minimum in spatially extended systems is often mediated by the propagation of fronts, separating globally and locally minimizing states. The speed of propagation of such fronts gives crucial information on time scales for relaxation. In the simplest, typical scenario, the front motion is driven by the energy difference between local and global minimizers, yielding an effective force on the interface. The speed of the front is then proportional to this effective force. Stationary fronts correspond to the situation where the states on either side of the interface have equal energy. In formulas, the speed c depends in a smooth and monotone fashion on the energy difference $\mathcal{E}$, $c = c(\mathcal{E})$, $c'(\mathcal{E}) > 0$, so that for $\mathcal{E}$ small,

$$c \sim \mathcal{E}. \tag{3.3.1}$$

A prototypical example for this scenario is the reaction-diffusion equation (1.0.2) with cubic bistable nonlinearity

$$f_a(u) := u(1-u)(u-a), \quad a \in (0,1), \tag{3.3.2}$$

where in that case, it is well-known[3] that $\mathcal{E} = a - \frac{1}{2}$, and $c = \sqrt{2}\mathcal{E}$ is linear in $\mathcal{E}$.

It has been well known that this simple picture fails in many important contexts. In particular, the speed of fronts may vanish for sufficiently small yet non-zero energy differentials. Such propagation failure is usually referred to as *pinning*, alluding to a simple scenario where energy depends on space x. In the following, we briefly describe this scenario from several view points. Our contribution is to provide a new, different view point on pinning, with fundamentally different characteristic expansions and analytic tools.

[3] The corresponding front is exactly $U(\xi) = \frac{1}{1+e^{-\xi/\sqrt{2}}}$. Note that it connects 0 at $-\infty$ to 1 at $+\infty$.

The possibly simplest case where pinning is observed are spatially periodic media, such as

$$\partial_t u = \partial_x^2 u + u(1-u)(u-a+\varepsilon\sin(x)), \quad t>0, \quad x\in\mathbb{R}. \tag{3.3.3}$$

The parameter a still detunes the relative energy of the equilibria $u=0$ and $u=1$. Since, however, this relative energy difference varies in x, fronts may have to overcome a barrier, pointwise in x, in order to reduce energy in the system. For $\varepsilon>0$, one typically finds a *pinning region* (a_-,a_+) where two stationary fronts exist, one stable and one unstable. At the boundary of this interval, the two stationary fronts disappear in a saddle-node bifurcation. For values $a=a_++\mathcal{E}$, $\mathcal{E}>0$, the speed of the interface scales as

$$c\sim\mathcal{E}^{1/2}. \tag{3.3.4}$$

This can be formally (and more rigorously) understood as induced by the time that a moving interface spends near a saddle-node bifurcation, where time scales with $\mathcal{E}^{-1/2}$. Let us further remark that stationary profiles in spatially periodic media solve a non-autonomous differential equation,

$$u_x = v, \qquad v_x = -u(1-u)(u-a+\varepsilon\sin(x)),$$

whose time evolution $\Psi_{2\pi,0}$ defines a diffeomorphism of the plane with hyperbolic fixed points $(1,0)$ and $(0,0)$. Intersections of stable and unstable manifolds are typically transverse since time-translation symmetry is broken, hence robust with respect to changes in the parameter a.

Similar phenomena are found in spatially discrete media

$$u_i' = \frac{d}{2}\left(u_{i+1}-2u_i+u_{i-1}\right)+f_a(u_i), \qquad i\in\mathbb{Z}, \tag{3.3.5}$$

where stationary fronts solve the two-term recursion

$$0 = \frac{d}{2}\left(u_{i+1}-2u_i+u_{i-1}\right)+f_a(u_i).$$

Roughly speaking, the phenomena mirror the case of spatially periodic media: fronts are stationary in the pinning region $a\in(a_-,a_+)$ and propagate with speed $c\sim\mathcal{E}^{1/2}$ for $a-a_+=\mu>0$. In that case, pinned fronts can be viewed as heteroclinic orbits of the two-term recursion, which defines a (local) diffeomorphism of the plane,

$$u_{i+1}=u_i+w_i, \qquad w_{i+1}=w_i-\frac{1}{d}f_a(u_i+w_i).$$

Indeed, $u=0,1$ and $w=0$ define hyperbolic fixed points of this diffeomorphism. The corresponding stable and unstable manifolds intersect along orbits that yield stationary fronts. Such an intersection of stable and unstable manifolds in diffeomorphisms is typically (beyond this particular case) transverse.

In both cases, the boundary of the pinning region is given by a parameter value where stable and unstable manifolds intersect non-transversely, typically with a quadratic tangency that reflects the generic saddle-node bifurcation alluded to earlier. Summarizing, the traditional view of pinning associates open pinning regions with the absence of a continuous translational symmetry in the system.

We are interested here in, apparently quite different, nonlocal systems,

$$\partial_t u = d(-u + \mathcal{K} * u) + f_a(u), \quad t > 0, \quad x \in \mathbb{R}, \tag{3.3.6}$$

with $d > 0$ and a convolution kernel $\mathcal{K}$ that satisfies Hypothesis 1.2.5. Here, in order to simplify the presentation, we assume that f_a is given by (3.3.2). Existence and stability of fronts solutions of (1.2.7) are typically given by Theorem 1.2.5. As mentioned in the introduction, it was noted in [16, 44] that for weak coupling strength, $d \ll 1$, fronts are discontinuous and do not propagate, for values of a in a pinning region (a_-, a_+). Actually, reinterpreting the analysis of [16], the boundary of the pinning region in that case can be precisely characterized as explained in the following lemma.

Lemma 3.3.1. *The pinning region of the cubic function $f_a(u) = u(1-u)(u-a)$ in the (a,d)-plane is bounded by*

$$d(a) = \begin{cases} \frac{1}{3}(1 - a + a^2 - \sqrt{1-2a}) & a \leq \frac{1}{2}, \\ \frac{1}{3}(1 - a + a^2 - \sqrt{-1+2a}) & a \geq \frac{1}{2}. \end{cases}$$

In particular, robust pinning occurs for $d < 1/4$, in an interval $(a_-(d), a_+(d))$, with

$$a_\pm(d) = \frac{1}{2} \pm \frac{9}{2}\left(d - \frac{1}{4}\right)^2 + \mathcal{O}\left(\left(d - \frac{1}{4}\right)^4\right). \tag{3.3.7}$$

We denote $(a_*, d_*) := (1/2, 1/4)$ the tip of the pinning region.

For (3.3.6), the presence of an open pinning region is associated with a lack of regularity in the profile, rather than the absence of a translational symmetry. One can however emphasize similarities with lattice systems by embedding the lattice system (3.3.5) into a system on the real line,

$$\partial_t u(t,x) = d\left(-u(t,x) + \frac{1}{2}(u(t,x+1) + u(t,x-1)\right) + f(u(t,x)), \quad t > 0, \quad x \in \mathbb{R}. \tag{3.3.8}$$

Of course, this system decouples in an infinite family of lattice systems $x \in x_* + \mathbb{Z}$, $x_* \in [0,1)$, each of which is equivalent to (3.3.5). While quite artificial, (3.3.8) exhibits the similarities between the different nonlocal and discrete pinning when written in the form (3.3.6) with $\mathcal{K}(x) = \frac{1}{2}(\delta(x-1)+\delta(x+1))$ (although such kernels are not covered by assumptions in the references cited above). More explicitly, (3.3.8) possesses a continuous translational symmetry, but stationary fronts are discontinuous, given for instance as $u(x) = u_{[x]}$, where $[x]$ is the integer part of x and u_j is the stationary interface in (3.3.5)[4].

The point of view taken here is that (3.3.8) is a special element of the class of equations (3.3.6), in the sense that its kernel possesses very low regularity. One can then consider smoothed out versions of $\frac{1}{2}(\delta(x-1)+\delta(x+1))$ and ask about pinning regions and unpinning asymptotics. Our results indicate that both depend in a crucial fashion on the regularity of the approximation. While pinning is generic for the discrete kernel, pinning occurs only for sufficiently strong

[4] In this sense, the profiles have countably many discontinuities at locations $x \in \mathbb{Z}$.

coupling in smooth kernels. Unpinning asymptotics are changed from speeds scaling with an exponent 1/2 power law (see (3.3.4)) to

$$c \sim \mathcal{E}^{3/2},$$

in the pinning regime, or smooth speed asymptotics (3.3.1) in the unpinned regime. More precisely we have the following theorem.

Theorem 3.3.1. *Consider $\mathcal{K}(x) = e^{-|x|}/2$ and f_a given by (3.3.2). For fixed $d < d_* = 1/4$, as the parameter a approaches the left boundary of the pinning region at a_- the asymptotics of the wave speed of the unique front given in Theorem 1.2.5 are:*

$$c = k_1(a_- - a)^{\frac{3}{2}} + \mathcal{O}\left((a_- - a)^2 \ln(a_- - a)\right), \text{ as } a \nearrow a_-,$$

with some explicit formula for k_1. Note that a_- depends on d; see (3.3.7). The equivalent result (with same constant) holds for the right boundary of the pinning region.

Using the specific form of $\mathcal{K}$, the strategy of the proof is to realize the traveling front solutions of (1.2.7) as solution of

$$\begin{cases} w_x = v, \\ v_x = w - u, \\ -cu_x = d(-u+w) + f_a(u). \end{cases} \tag{3.3.9}$$

Traveling fronts are now heteroclinic solutions of (3.3.9) that connect the saddle equilibria $\underline{u}_+ = (0,0,0)$ and $\underline{u}_- = (1,0,1)$ in $\underline{u} = (w,v,u)$-space. With the parameter c, the system has a natural slow-fast structure close to the boundary of the pinning region. We find that, at the pinning boundary, there exists a *singular trajectory*, patched together from solutions of (3.3.9) at $c = 0$, and a singular fast jump through a fold point. This slow passage near a fold-point is precisely responsible for the peculiar scaling of wave speeds. The construction of heteroclinic solutions of (3.3.9) in the singular limit $c \ll 1$ is actually very close to the one of Theorem 3.1.1 from Section 3.1 and relies on blow-up techniques [147] near the passage through the fold.

Inspecting the shape of the pinning region from Lemma 3.3.1, we expect a transition at $d = 1/4$. The previous result, Theorem 3.3.1, describes speeds for a close to the left boundary, when $d < 1/4$. For $d > 1/4$, fronts are stationary at $a = 1/2$, only, and $c \sim \mathcal{E}$ is smooth. It is therefore interesting to examine speeds at criticality, fixing $d = 1/4$ and varying $\mathcal{E} = a - 1/2$ near the origin. Following the proof of Theorem 3.3.1, it is possible to formally see that a very similar strategy leads to expansions with a new exponent, 5/4. The heart of the analysis, however, relies on a singular perturbation problem that involves the slow passage through an inflection point, which has not been studied in a rigorous fashion, to our knowledge. For a rigorous geometric approach of the related slow passage through a cusp, we refer to the recent study in [37]; the results there cover a more general unfolding but do not give expansions for our case. Thus, we state this result as a conjecture.

Conjecture 3.3.1. *For fixed $d = d_* = 1/4$, as a approaches the boundary of the pinning region, $a \nearrow a_* = 1/2$, the wave speed of the front from Theorem 1.2.5 is*

$$c = k_c(a_* - a)^{\frac{5}{4}} + \mathcal{O}((a_* - a)^{\frac{5}{4}}),$$

with some explicit formula for k_c.

Let us finally conclude that unpinning asymptotics have recently been investigated in continuous and discrete ergodic media [6, 182], where now asymptotics are observed with power law exponent that depends on the dimension of the ergodic associated measure.

3.4 Fredholm properties of nonlocal differential operators

We have seen in the previous sections that the use of specific kernels could allow one to replace the initial nonlocal problem into high order PDE or ODE system where classical tools from dynamical systems theory (geometric singular perturbations, center manifolds, etc...) can be applied. Although this approach has the merit to provide interesting theoretical results, it only gives insight to nonlocal problems where the Fourier symbol of the kernel is a rational fraction. On the other hand, it is expected that all results stated above should still be valid for a large class of kernels, at least within the class of exponentially localized kernels. In that effort to go beyond specific kernels, we have developed in [86–88] a series of work that precisely provide analytic tools to study nonlocal problems with exponentially localized kernels. At the heart of our work is the development of Fredholm properties of nonlocal differential operators that we now present.

Setting. We consider linear nonlocal differential equations that can be written as:

$$\frac{\mathrm{d}}{\mathrm{d}\xi}U(\xi) = \int_{\mathbb{R}} \mathcal{K}(\xi-\xi';\xi)U(\xi')d\xi' + \sum_{j\in\mathcal{J}} A_j(\xi)U(\xi-\xi_j) + F(\xi). \tag{3.4.1}$$

Here $U(\xi), F(\xi) \in \mathbb{C}^n$, and $\mathcal{K}(\zeta;\xi), A_j(\xi) \in \mathscr{M}_n(\mathbb{C})$, $n \geq 1$, the space of $n\times n$ complex matrices. The set $\mathcal{J}$ is countable and the shifts ξ_j satisfy (without loss of generality)

$$\xi_1 = 0, \quad \xi_j \neq \xi_k, \quad j \neq k \in \mathcal{J}. \tag{3.4.2}$$

For each $\xi \in \mathbb{R}$, we define $\mathcal{A}(\xi)$ by

$$\mathcal{A}(\xi) := \Big(\mathcal{K}(\ \cdot\ ;\xi), (A_j(\xi))_{j\in\mathcal{J}}\Big), \tag{3.4.3}$$

so we may write (3.6.1) as

$$\frac{\mathrm{d}}{\mathrm{d}\xi}U(\xi) = \mathcal{N}[\mathcal{A}(\xi)]\cdot U(\xi) + F(\xi), \tag{3.4.4}$$

where $\mathcal{N}[\mathcal{A}(\xi)]$ denotes the linear nonlocal operator

$$\mathcal{N}[\mathcal{A}(\xi)]\cdot U(\xi) := \int_{\mathbb{R}} \mathcal{K}(\xi-\xi';\xi)U(\xi')d\xi' + \sum_{j\in\mathcal{J}} A_j(\xi)U(\xi-\xi_j). \tag{3.4.5}$$

We denote $\mathcal{K}_\xi := \mathcal{K}(\ \cdot\ ;\xi)$ and write (3.4.5) as a generalized convolution

$$\mathcal{N}[\mathcal{A}(\xi)]\cdot U = \left[\mathcal{K}_\xi + \sum_{j\in\mathcal{J}} A_j(\xi)\delta_{\xi_j}\right] * U. \tag{3.4.6}$$

Here $*$ refers to convolution on $\mathbb{R}$ and δ_{ξ_j} is the Dirac delta at $\xi_j \in \mathbb{R}$. Setting $F \equiv 0$, we obtain the homogeneous system

$$\frac{\mathrm{d}}{\mathrm{d}\xi}U(\xi) = \mathcal{N}[\mathcal{A}(\xi)] \cdot U(\xi). \tag{3.4.7}$$

A special case of (3.4.6) are constant coefficient operators $\mathcal{A}(\xi)$

$$\mathcal{A}(\xi) = \left(\mathcal{K}^0(\;\cdot\;), \left(A_j^0\right)_{j\in\mathcal{J}}\right) := \mathcal{A}^0, \quad \forall\, \xi \in \mathbb{R}.$$

We have

$$\mathcal{N}[\mathcal{A}^0] \cdot U = \left[\mathcal{K}^0 + \sum_{j\in\mathcal{J}} A_j^0 \delta_{\xi_j}\right] * U \tag{3.4.8}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}\xi}U(\xi) = \mathcal{N}[\mathcal{A}^0] \cdot U(\xi). \tag{3.4.9}$$

Finally, associated with (3.4.7), we have the linear operator

$$\mathcal{T}_{\mathcal{A}} := \frac{\mathrm{d}}{\mathrm{d}\xi} - \mathcal{N}[\mathcal{A}(\xi)]. \tag{3.4.10}$$

Example. Operators such as (3.4.10) appear naturally when linearizing at coherent structures such as traveling fronts or pulses in nonlinear nonlocal differential equations. Typical examples are linearization at a traveling pulse solution in neural field equations. Recall from Section 3.1 that, in traveling wave coordinates, neural field equations with synaptic depression read

$$\left\{\begin{array}{l} \partial_t u(t,\xi) = -c\partial_\xi u(t,\xi) - u(t,\xi) + \displaystyle\int_{\mathbb{R}} \mathcal{K}(\xi - y) q(t,y) S(u(t,y))\mathrm{d}y, \\ \partial_t q(t,\xi) = -c\partial_\xi q(t,\xi) + \epsilon\left(1 - q(t,\xi) - \beta q(t,\xi) S(u(t,\xi))\right). \end{array}\right. \tag{3.4.11}$$

The linearization of the above equation at a particular stationary solution $(u_0(\xi), q_0(\xi))$ takes the form

$$\left\{\begin{array}{l} \partial_t u(t,\xi) = -c\partial_\xi u(t,\xi) - u(t,\xi) + \displaystyle\int_{\mathbb{R}} \mathcal{K}(\xi - y)\left[q(t,y)S(u_0(y)) + q_0(y)S'(u_0(y))u(t,y)\right]\mathrm{d}y, \\ \partial_t q(t,\xi) = -c\partial_\xi q(t,\xi) - \epsilon\left[q(t,\xi) + \beta q(t,\xi)S(u_0(\xi)) + \beta q_0(\xi)S'(u_0(\xi))u(t,\xi)\right]. \end{array}\right. \tag{3.4.12}$$

Denoting $U := (u,q)$ and $\mathcal{L}_0$ the right-hand side of (3.4.12), the eigenvalue problem associated with the linearization of (3.4.11) at (u_0, q_0) reads

$$\lambda U = \mathcal{L}_0 U. \tag{3.4.13}$$

This eigenvalue problem can be cast as a first-order nonlocal differential equation

$$\frac{\mathrm{d}}{\mathrm{d}\xi}U(\xi) = \widetilde{\mathcal{K}}_\xi^\lambda * U(\xi) \tag{3.4.14}$$

where

$$\widetilde{\mathcal{K}}_\xi^\lambda(\zeta) = \frac{1}{c}\begin{pmatrix} -(1+\lambda)\delta_0 + \mathcal{K}(\zeta)q_0(\xi-\zeta)S'(u_0(\xi-\zeta)) & \mathcal{K}(\zeta)S(u_0(\xi-\zeta)) \\ -\epsilon\beta q_0(\xi)S'(u_0(\xi))\delta_0 & -(\epsilon+\lambda+\beta S(u_0(\xi)))\delta_0 \end{pmatrix}$$

and δ_0 denotes the Dirac delta at 0.

The differential system (3.4.14) can be viewed as systems of functional differential equations of mixed type since the convolutional term introduces both advanced and retarded terms. Such equations are notoriously difficult to analyze. Our goal here is threefold. First, we establish Fredholm properties of such operators. Second we give algorithms for computing Fredholm indices. Last, we show how such Fredholm properties can be used to analyze perturbation and stability problems.

Notations and hypotheses. We denote by $\mathcal{H}$ and $\mathcal{W}$ the Hilbert spaces $L^2(\mathbb{R},\mathbb{C}^n)$ and $H^1(\mathbb{R},\mathbb{C}^n)$ equipped with their usual norm

$$\|U\|_{\mathcal{H}} := \max_{k=1,\cdots,n} \|U_k\|_{L^2(\mathbb{R})},$$

and

$$\|U\|_{\mathcal{W}} := \left\| \frac{\mathrm{d}}{\mathrm{d}\xi} U \right\|_{\mathcal{H}} + \|U\|_{\mathcal{H}}.$$

For a function $\mathcal{K}_\xi = \mathcal{K}(\ \cdot\ ;\xi) : \mathbb{R} \to L^1_\eta(\mathbb{R},\mathscr{M}_n(\mathbb{C}))$, $\eta > 0$, we define its norm as

$$||\mathcal{K}_\xi||_\eta := \max_{(k,l)\in[\![1,n]\!]^2} \|\mathcal{K}_{k,l}(\ \cdot\ ;\xi)e^{\eta|\ \cdot\ |}\|_{L^1(\mathbb{R})}.$$

We also introduce the following norm for the kernel $\mathcal{K} \in \mathscr{C}^1\left(\mathbb{R}, L^1_\eta(\mathbb{R},\mathscr{M}_n(\mathbb{C}))\right)$,

$$|||\mathcal{K}|||_{\infty,\eta} := \sup_{\xi\in\mathbb{R}} \|\mathcal{K}_\xi\|_\eta + \sup_{\xi\in\mathbb{R}} \left\| \frac{\mathrm{d}}{\mathrm{d}\xi}\mathcal{K}_\xi \right\|_\eta.$$

For a function $A \in \mathscr{C}^1(\mathbb{R},\mathscr{M}_n(\mathbb{C}))$ we define its norm as

$$\|A\|_n := \sup_{\xi\in\mathbb{R}} \|A(\xi)\|_{\mathscr{M}_n(\mathbb{C})} + \sup_{\xi\in\mathbb{R}} \left\| \frac{\mathrm{d}}{\mathrm{d}\xi} A(\xi) \right\|_{\mathscr{M}_n(\mathbb{C})}.$$

Finally we denote by τ the linear transformation that acts on $\mathcal{K}_\xi$ as $\tau\cdot\mathcal{K}_\xi := \mathcal{K}(\ \cdot\ ;\xi+\tau)$ and we naturally define $\tau\cdot\mathcal{K} : \xi \longmapsto \tau\cdot\mathcal{K}_\xi$. We can now give further assumptions on the maps $\mathcal{K}$ and $(A_j)_{j\in\mathcal{J}}$.

Hypothesis 3.4.1. *There exists $\eta > 0$ such that the matrix kernel $\mathcal{K}$ satisfies the following properties:*

1. *$\mathcal{K}$ belongs to $\mathscr{C}^1\left(\mathbb{R}, L^1_\eta(\mathbb{R},\mathscr{M}_n(\mathbb{C}))\right)$;*
2. *$\mathcal{K}$ is localized, that is,*
$$\left\{ \begin{array}{l} |||\mathcal{K}|||_{\infty,\eta} < \infty\ , \\ |||\tau\cdot\mathcal{K}|||_{\infty,\eta} < \infty\ ; \end{array} \right. \tag{3.4.15}$$
3. *there exist two functions $\mathcal{K}^\pm \in L^1(\mathbb{R},\mathscr{M}_n(\mathbb{C}))$ such that*
$$\lim_{\xi\to\pm\infty} \mathcal{K}(\zeta;\xi) = \mathcal{K}^\pm(\zeta) \tag{3.4.16}$$

uniformly in $\zeta \in \mathbb{R}$ *and*

$$\begin{cases} \lim_{\xi\to\pm\infty} \|\mathcal{K}_\xi - \mathcal{K}^\pm\|_\eta = 0 \\ \lim_{\xi\to\pm\infty} \|\tau\cdot\mathcal{K}_\xi - \mathcal{K}^\pm\|_\eta = 0. \end{cases} \tag{3.4.17}$$

Hypothesis 3.4.2. *The matrices* A_j *satisfy the properties:*

1. $A_j \in \mathscr{C}^1(\mathbb{R}, \mathscr{M}_n(\mathbb{C}))$ *for all* $j \in \mathcal{J}$*;*
2. *with* η *defined in Hypothesis 1.2.5, we have,*

$$\sum_{j\in\mathcal{J}} \|A_j\|_n e^{\eta|\xi_j|} < \infty \ ; \tag{3.4.18}$$

3. *there exist* $A_j^\pm \in \mathscr{M}_n(\mathbb{C})$ *such that*

$$\lim_{\xi\to\pm\infty} A_j(\xi) = A_j^\pm, \quad \sum_{j\in\mathcal{J}} \|A_j^\pm\|_{\mathscr{M}_n(\mathbb{C})} e^{\eta|\xi_j|} < \infty, \quad j \in \mathcal{J}, \tag{3.4.19}$$

and

$$\lim_{\xi\to\pm\infty} \sum_{j\in\mathcal{J}} \|A_j(\xi) - A_j^\pm\|_{\mathscr{M}_n(\mathbb{C})} e^{\eta|\xi_j|} = 0. \tag{3.4.20}$$

Note that if we define the map $\mathcal{A}$ as

$$\begin{array}{rccl} \mathcal{A}: & \mathbb{R} & \longrightarrow & L^1_\eta(\mathbb{R}, \mathscr{M}_n(\mathbb{C})) \times \ell^1_\eta(\mathscr{M}_n(\mathbb{C})) \\ & \xi & \longmapsto & \mathcal{A}(\xi) = \Big(\mathcal{K}(\ \cdot\ ;\xi), (A_j(\xi))_{j\in\mathcal{J}}\Big) \end{array} \tag{3.4.21}$$

then, when Hypotheses 3.4.1 and 3.4.2 are satisfied, $\mathcal{A} \in \mathscr{C}^1(\mathbb{R}, L^1_\eta(\mathbb{R}, \mathscr{M}_n(\mathbb{C})) \times \ell^1_\eta(\mathscr{M}_n(\mathbb{C})))$ and is bounded. Here we have implicitly defined

$$\ell^1_\eta(\mathscr{M}_n(\mathbb{C})) = \left\{ (A_j)_{j\in\mathcal{J}} \in \mathscr{M}_n(\mathbb{C})^{\mathcal{J}} \mid \sum_{j\in\mathcal{J}} \|A_j\|_{\mathscr{M}_n(\mathbb{C})} e^{\eta|\xi_j|} < \infty \right\}.$$

Hypothesis 3.4.3. *We assume that for all* $\ell \in \mathbb{R}$

$$d^\pm(\mathbf{i}\ell) := \det\left(\mathbf{i}\ell\, \mathbb{I}_n - \widehat{\mathcal{K}^\pm}(\mathbf{i}\ell) - \sum_{j\in\mathcal{J}} A_j^\pm e^{-\mathbf{i}\ell\xi_j} \right) \neq 0 \tag{3.4.22}$$

where $\widehat{\mathcal{K}^\pm}$ *are the complex Fourier transforms of* $\mathcal{K}^\pm$ *defined by*

$$\widehat{\mathcal{K}^\pm}(\mathbf{i}\ell) = \int_{\mathbb{R}} \mathcal{K}^\pm(\xi) e^{-\mathbf{i}\ell\xi} d\xi.$$

Hypothesis 3.4.4. *We assume that, with the same* $\eta > 0$ *as in Hypotheses 3.4.1 and 3.4.2, the complex Fourier transforms*

$$\nu \longmapsto \widehat{\mathcal{K}^\pm}(\nu) + \sum_{j\in\mathcal{J}} A_j^\pm e^{-\nu\xi_j}$$

extend to bounded analytic functions in the strip $\mathcal{S}_\eta := \{\nu \in \mathbb{C} \mid |\Re(\nu)| < \eta\}$.

Main results. Consider Banach spaces $\mathcal{X}$ and $\mathcal{Y}$. We let $\mathscr{L}(\mathcal{X}, \mathcal{Y})$ denote the Banach space of bounded linear operators $\mathcal{T} : \mathcal{X} \to \mathcal{Y}$, and we denote the operator norm by $\|\mathcal{T}\|_{\mathscr{L}(\mathcal{X},\mathcal{Y})}$. We write rg $\mathcal{T}$ for the range of $\mathcal{T}$ and ker $\mathcal{T}$ for its kernel,

$$\mathrm{rg}\ \mathcal{T} := \{\mathcal{T}U \in \mathcal{Y}\ ;\ U \in \mathcal{X}\} \subset \mathcal{Y}, \qquad \ker \mathcal{T} := \{U \in \mathcal{X}\ ;\ \mathcal{T}U = 0\} \subset \mathcal{X}.$$

Let us recall that a bounded operator $\mathcal{T} : \mathcal{X} \to \mathcal{Y}$ is a Fredholm operator if

(i) its kernel ker $\mathcal{T}$ is finite-dimensional;

(ii) its range rg $\mathcal{T}$ is closed; and

(iii) rg $\mathcal{T}$ has finite codimension.

For such an operator, the integer

$$\mathrm{ind}\ \mathcal{T} := \dim(\ker \mathcal{T}) - \mathrm{codim}\,(\mathrm{rg}\ \mathcal{T})$$

is called the Fredholm index of $\mathcal{T}$.

We can now state our main results. The first theorem states the Fredholm property of the nonlocal operator $\mathcal{T}_{\mathcal{A}}$ while the second gives a characterization of the Fredholm index via the spectral flow.

Theorem 3.4.1 (The Fredholm Alternative)**.** *Suppose that Hypotheses 3.4.1, 3.4.2, and 3.4.3 are satisfied. Then the operator $\mathcal{T}_{\mathcal{A}} : \mathcal{W} \to \mathcal{H}$ is Fredholm. Furthermore, the Fredholm index of $\mathcal{T}_{\mathcal{A}}$ depends only on the limiting operators $\mathcal{A}^{\pm}$, the limits of $\mathcal{A}(\xi)$ as $\xi \to \pm\infty$. We denote $\iota(\mathcal{A}^-, \mathcal{A}^+)$ the Fredholm index* ind $\mathcal{T}_{\mathcal{A}}$.

Corollary 3.4.1 (Cocycle property)**.** *Suppose that $\mathcal{A}^0, \mathcal{A}^1$ and $\mathcal{A}^2$ are hyperbolic constant coefficient operators in $L^1_\eta(\mathbb{R}, \mathscr{M}_n(\mathbb{C})) \times \ell^1_\eta(\mathscr{M}_n(\mathbb{C}))$, then we have*

$$\iota(\mathcal{A}^0, \mathcal{A}^1) + \iota(\mathcal{A}^1, \mathcal{A}^2) = \iota(\mathcal{A}^0, \mathcal{A}^2).$$

Theorem 3.4.2 (Spectral Flow Theorem)**.** *Assume that Hypotheses 3.4.1, 3.4.2, 3.4.3, and 3.4.4 are satisfied and suppose, further, that there are only finitely many values of $\xi_0 \in \mathbb{R}$ for which $\mathcal{A}(\xi_0)$ is not hyperbolic. Then the Fredholm index of $\mathcal{T}_{\mathcal{A}}$*

$$\iota(\mathcal{A}^-, \mathcal{A}^+) = -\mathrm{cross}(\mathcal{A}) \tag{3.4.23}$$

is the net number of roots, counted with multiplicity, of the characteristic equation

$$d^\xi(\nu) := \det\left(\nu \mathbb{I}_n - \widehat{\mathcal{K}}_\xi(\nu) - \sum_{j\in\mathcal{J}} A_j(\xi) e^{-\nu\xi_j}\right) = 0, \tag{3.4.24}$$

which cross the imaginary axis from left to right as ξ is increased from $-\infty$ to $+\infty$.

Note that similar results are available in the literature when the interaction kernel is a finite sum of Dirac delta measures. In particular, the interaction kernel has finite range in that case. Such interaction kernels arise in the study of lattice dynamical systems. Mallet-Paret established Fredholm properties and showed how to compute the Fredholm index via a spectral flow [154]. His methods are reminiscent of Robbin & Salamon's work [171], who established similar results for operators $\frac{\mathrm{d}}{\mathrm{d}\xi}+A(\xi)$ where $A(\xi)$ is self-adjoint but does not necessarily generate a semi-group. For the operators studied in [154], Fredholm properties are in fact equivalent to the existence of exponential dichotomies for an appropriate formulation of (3.4.10) as an infinite- dimensional evolution problem [111, 156]. Our approach extends Mallet-Paret's results [154] to infinite-range kernels. We do not know if a dynamical systems formulation in the spirit of [111, 156] is possible. Our methods blend some of the tools in [171] with techniques from [154].

The proof of Theorem 3.4.1 relies crucially on an abstract lemma that we now present and whose proof can be found in [184].

Lemma 3.4.1 (Abstract Closed Range Lemma)**.** *Suppose that $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ are Banach spaces, that $\mathcal{T}:\mathcal{X}\to\mathcal{Y}$ is a bounded linear operator, and that $\mathcal{R}:\mathcal{X}\to\mathcal{Z}$ is a compact linear operator. Assume that there exists a constant $c>0$ such that*

$$\|U\|_{\mathcal{X}}\leq c\left(\|\mathcal{T}U\|_{\mathcal{Y}}+\|\mathcal{R}U\|_{\mathcal{Z}}\right),\quad \forall U\in\mathcal{X}.$$

Then T has closed range and finite-dimensional kernel.

This lemma can be used as follows to prove that $\mathcal{T}_{\mathcal{A}}$ has closed range and finite-dimensional kernel. For each $T>0$, we define $\mathcal{H}(T)=L^2([-T,T],\mathbb{C}^n)$ and $\mathcal{W}(T)=H^1([-T,T],\mathbb{C}^n)$. It is easy to see that the inclusion $\mathcal{W}(T)\hookrightarrow\mathcal{H}(T)$ defines a compact operator such that the restriction operator

$$\begin{aligned}\mathcal{R}:\mathcal{W}&\to\mathcal{H}(T)\\ U&\mapsto U_{[-T,T]}\end{aligned}$$

is a compact linear operator and $\|\mathcal{R}U\|_{\mathcal{H}(T)}=\|U\|_{\mathcal{H}(T)}$. Following ideas from [171], one can prove (see [86, Lemma 3.1]) that there exist constants $c>0$ and $T>0$ such that

$$\|U\|_{\mathcal{W}}\leq c\left(\|U\|_{\mathcal{H}(T)}+\|\mathcal{T}_{\mathcal{A}}U\|_{\mathcal{H}}\right)\tag{3.4.25}$$

for every $U\in\mathcal{W}$. As a consequence, applying the abstract closed range Lemma 3.4.1, we deduce that $\mathcal{T}_{\mathcal{A}}$ has closed range and finite-dimensional kernel. Actually, we can repeat the argument to the formal adjoint of $\mathcal{T}_{\mathcal{A}}$ to show that is also possesses closed range and finite-dimensional kernel which in turn implies that $\mathcal{T}_{\mathcal{A}}$ is Fredholm.

To conclude the proof of Theorem 3.4.1, one needs to prove that the Fredholm index depends only on the limiting operators $\mathcal{A}^{\pm}$. Thus, let us consider two families of operators $\mathcal{A}_0(\xi)$ and $\mathcal{A}_1(\xi)$ that satisfy Hypotheses 3.4.1, 3.4.2 and 3.4.3 with coefficients

$$\mathcal{A}_0(\xi)=\Big(\mathcal{K}_0(\ \cdot\ ;\xi),\left(A_{j,0}(\xi)\right)_{j\in\mathcal{J}}\Big),\quad \mathcal{A}_1(\xi)=\Big(\mathcal{K}_1(\ \cdot\ ;\xi),\left(A_{j,1}(\xi)\right)_{j\in\mathcal{J}}\Big)$$

and the same shifts ξ_j. We assume that the limiting operators at $\pm\infty$ are equal, that is,

$$\mathcal{A}_0^\pm = \mathcal{A}_1^\pm,$$

where

$$\mathcal{A}_\sigma^\pm = \left(\mathcal{K}_\sigma^\pm, \left(A_{j,\sigma}^\pm\right)_{j\in\mathcal{J}}\right) = \lim_{\xi\to\pm\xi} \mathcal{A}_\sigma(\xi), \quad \sigma = 0, 1.$$

For $0 \le \sigma \le 1$, we define $\mathcal{A}_\sigma(\xi) = (1-\sigma)\mathcal{A}_0(\xi) + \sigma\mathcal{A}_1(\xi)$. Then for each such σ, $\mathcal{A}_\sigma$ satisfies Hypotheses 3.4.1, 3.4.2 and 3.4.3 and $\mathcal{T}_{\mathcal{A}_\sigma}$ is a Fredholm operator and $\mathcal{T}_{\mathcal{A}_\sigma}$ varies continuously in $\mathscr{L}(\mathcal{W}, \mathcal{H})$ with σ. Thus the Fredholm index of $\mathcal{T}_{\mathcal{A}_\sigma}$ is independent of σ and only depends on the limiting operators $\mathcal{A}^\pm$.

Remarks 3.4.1. *The proof immediately generalizes to a set-up where $\mathcal{H}$ and $\mathcal{W}$ are L^p-based, with $1 < p < \infty$, with the exception of invertibility of the asymptotic, constant-coefficient operators, where we used Fourier transform as an isomorphism. On the other hand, analyticity of the Fourier multiplier shows that the inverse is in fact represented by a convolution with an exponentially localized kernel, which gives a bounded inverse in L^p, so that our theorem holds in L^p-based spaces as well.*

The proof of Theorem 3.4.2 is quite technical and relies on several ingredients. Here, we only present a key proposition which allows to better understand the relationship between crossing and Fredholm index in the case of constant coefficient operators. First, let us define the characteristic equation associated to the constant coefficient system (3.4.9):

$$d^0(\nu) := \det \Delta_{\mathcal{A}^0}(\nu) = 0, \tag{3.4.26}$$

where

$$\Delta_{\mathcal{A}^0}(\nu) = \nu\, \mathbb{I}_n - \widehat{\mathcal{K}^0}(\nu) - \sum_{j\in\mathcal{J}} A_j^0 e^{-\nu\xi_j}, \quad \nu \in \mathbb{C}. \tag{3.4.27}$$

Then, let us introduce the map $\Sigma_\gamma : L^1_\eta(\mathbb{R}, \mathscr{M}_n(\mathbb{C})) \times \ell^1_\eta(\mathscr{M}_n(\mathbb{C})) \to L^1_\eta(\mathbb{R}, \mathscr{M}_n(\mathbb{C})) \times \ell^1_\eta(\mathscr{M}_n(\mathbb{C}))$, defined for each $\gamma \in \mathbb{R}$ by

$$\Sigma_\gamma \cdot \mathcal{A}^0 = \Sigma_\gamma \cdot \left(\mathcal{K}^0, (A_j^0)_{j\in\mathcal{J}}\right) := \left(\mathcal{K}_\gamma^0, (A_{j,\gamma}^0)_{j\in\mathcal{J}}\right),$$

where

$$\mathcal{K}_\gamma^0(\zeta) = \mathcal{K}^0(\zeta)e^{\gamma\zeta}, \quad \forall\, \zeta \in \mathbb{R}, \quad A_{1,\gamma}^0 = A_1^0 + \gamma, \quad A_{j,\gamma}^0 = A_j^0 e^{\gamma\xi_j}, \quad \forall\, j \neq 1.$$

This transformation Σ_γ arises from a change of variables $V(\xi) = e^{\gamma\xi}U(\xi)$ in (3.4.9) with constant coefficient $\mathcal{A}^0 = \left(\mathcal{K}^0, \left(A_j^0\right)_{j\in\mathcal{J}}\right)$. One can then easily check that

$$\Delta_{\Sigma_\gamma\cdot\mathcal{A}^0}(\nu) = \Delta_{\mathcal{A}^0}(\nu - \gamma), \quad \nu \in \mathbb{C},$$

so that Σ_γ shifts all eigenvalues to the right by an amount of γ.

Proposition 3.4.1. *Suppose that $\nu = \mathbf{i}\ell_0$, with $\ell_0 \in \mathbb{R}$, is a simple root of the characteristic equation (3.4.26) associated to $\mathcal{A}^0$, and suppose that there are no other roots with $\Re\lambda = 0$. Then for $\gamma \in \mathbb{R}$, $0 < |\gamma| < \eta$ sufficiently small, we have that*

$$\iota(\Sigma_{-\gamma}\cdot\mathcal{A}^0, \Sigma_\gamma\cdot\mathcal{A}^0) = -\mathrm{sign}(\gamma). \tag{3.4.28}$$

Proof. Without loss of generality, we suppose that $\gamma > 0$ is small enough so that $\nu = \mathbf{i}\ell_0$ is the only root of $\det(\Delta_{\mathcal{A}^0}(\nu)) = 0$ in the strip $|\Re(\nu)| \leq \gamma < \eta$. We need to show that $\mathcal{T}_{\mathcal{A}_0}$ is Fredholm with index -1 in

$$L^2_\gamma(\mathbb{R}, \mathbb{C}^n) = \left\{ U : \mathbb{R} \to \mathbb{C} \mid \left\| U(\cdot) e^{\gamma|\cdot|} \right\|_{L^2(\mathbb{R},\mathbb{C}^n)} < \infty \right\}.$$

To see this, we give a factorization of $\mathcal{T}_{\mathcal{A}_0}$ of the form

$$\mathcal{T}_{\mathcal{A}_0} = \mathcal{B}_1 \cdot \mathcal{B}_2$$

so that $\mathcal{B}_1$ is Fredholm with index -1 on $L^2_\gamma(\mathbb{R}, \mathbb{C}^n)$ and $\mathcal{B}_2$ is bounded invertible. We construct $\mathcal{B}_1$ and $\mathcal{B}_2$ based on the Fourier symbol of $\mathcal{T}_{\mathcal{A}_0}$ as follows. As $\nu = \mathbf{i}\ell_0$ is a simple root of $\det(\Delta_{\mathcal{A}^0}(\nu)) = 0$, there exist two nonzero complex vectors $p \in \mathbb{C}^n$ and $q \in \mathbb{C}^n$ such that

$$\Delta_{\mathcal{A}^0}(\mathbf{i}\ell_0)p = 0, \quad \Delta_{\mathcal{A}^0}(\mathbf{i}\ell_0)^* q = 0, \text{ and } \langle p, q \rangle_{\mathbb{C}^n} = 1.$$

There exist two invertible matrices $P \in \mathscr{M}_n(\mathbb{C})$ and $A_1 \in \mathscr{M}_{n-1}(\mathbb{C})$, independent of ν, such that the following holds

$$P^{-1}\Delta_{\mathcal{A}^0}(\nu)P = \begin{pmatrix} 0 & 0_{1,n-1} \\ 0_{n-1,1} & A_1 \end{pmatrix} + \begin{pmatrix} a_0(\nu - \mathbf{i}\ell_0) & \mathcal{O}(\nu - \mathbf{i}\ell_0)_{1,n-1} \\ \mathcal{O}(\nu - \mathbf{i}\ell_0)_{n-1,1} & \mathcal{O}(\nu - \mathbf{i}\ell_0)_{n-1,n-1} \end{pmatrix}, \text{ as } \nu \to \mathbf{i}\ell_0,$$

with $a_0 = \langle \partial_\nu \Delta_{\mathcal{A}^0}(\mathbf{i}\ell_0)p, q \rangle_{\mathbb{C}^n} \neq 0$. We can then define the matrix $A(\nu) \in \mathscr{M}_n(\mathbb{C})$ via

$$A(\nu) := \begin{pmatrix} \frac{\nu+\omega}{\nu-\mathbf{i}\ell_0} & 0_{1,n-1} \\ 0_{n-1,1} & \mathbb{I}_{n-1} \end{pmatrix} P^{-1}\Delta_{\mathcal{A}^0}(\nu)P,$$

with $\omega > \eta$ a fixed real number. A straightforward computation shows that $A(\nu)$ is invertible for all $|\Re(\nu)| \leq \gamma$. Furthermore, the following equality holds true

$$\Delta_{\mathcal{A}^0}(\nu) = P \begin{pmatrix} \frac{\nu-\mathbf{i}\ell_0}{\nu+\omega} & 0_{1,n-1} \\ 0_{n-1,1} & \mathbb{I}_{n-1} \end{pmatrix} A(\nu)P^{-1}$$

for all ν in the strip $|\Re(\nu)| \leq \gamma$. We can now define $\mathcal{B}_1$ and $\mathcal{B}_2$ through their Fourier symbol as

$$\begin{aligned} \widehat{\mathcal{B}}_1(\nu) &= P \begin{pmatrix} \frac{\nu-\mathbf{i}\ell_0}{\nu+\omega} & 0_{1,n-1} \\ 0_{n-1,1} & \mathbb{I}_{n-1} \end{pmatrix} P^{-1}, \\ \widehat{\mathcal{B}}_2(\nu) &= PA(\nu)P^{-1}, \end{aligned}$$

such that $\Delta_{\mathcal{A}^0}(\nu) = \widehat{\mathcal{B}}_1(\nu)\widehat{\mathcal{B}}_2(\nu)$ for all $|\Re(\nu)| \leq \gamma$. Note that analyticity of $\widehat{\mathcal{B}}_1(\nu)$ and $\widehat{\mathcal{B}}_2(\nu)$ in $|\Re(\nu)| \leq \gamma$ implies that $\mathcal{B}_1 : L^2_\gamma(\mathbb{R}, \mathbb{C}^n) \to L^2_\gamma(\mathbb{R}, \mathbb{C}^n)$ and $\mathcal{B}_2 : H^1_\gamma(\mathbb{R}, \mathbb{C}^n) \to L^2_\gamma(\mathbb{R}, \mathbb{C}^n)$, together with $\mathcal{T}_{\mathcal{A}_0} = \mathcal{B}_1 \cdot \mathcal{B}_2$. Since we factored the unique root of $\det \Delta_{\mathcal{A}_0}(\nu) = 0$ into $\widehat{\mathcal{B}}_1(\nu)$, $\widehat{\mathcal{B}}_2(\nu)$ is invertible in the strip $|\Re(\nu)| \leq \gamma$. Therefore, $\mathcal{B}_2$ is actually an isomorphism from $H^1_\gamma(\mathbb{R}, \mathbb{C}^n)$ to $L^2_\gamma(\mathbb{R}, \mathbb{C}^n)$. Inspecting the explicit form of $\widehat{\mathcal{B}}_1(\nu)$ shows that $\mathcal{B}_1$ is conjugate to

$$\begin{pmatrix} \left(\frac{\mathrm{d}}{\mathrm{d}\xi} - \mathbf{i}\ell_0\right)\left(\omega + \frac{\mathrm{d}}{\mathrm{d}\xi}\right)^{-1} & 0_{1,n-1} \\ 0_{n-1,1} & \mathbb{I}_{n-1} \end{pmatrix},$$

which is Fredholm index -1 on $L^2_\gamma(\mathbb{R}, \mathbb{C}^n)$. This completes the proof of the proposition. ■

The core of the proof of Theorem 3.4.2 is to show that for general continuously varying one-parameter family of constant coefficient operators $\mathcal{A}^\rho$, $\rho \in \mathbb{R}$, with hyperbolic limit operators and only finitely many values of the parameters ρ for which $\mathcal{A}^\rho$ is not hyperbolic it is actually sufficient to consider the case of $\mathscr{C}^1$ families with only simple crossings. Here, a crossing is a real number ρ_j for which $\mathcal{A}^{\rho_j}$ is not hyperbolic. A crossing ρ_j is simple if there is precisely one simple root of $d_{\rho_j}(\nu_*)$ located on the imaginary axis, and if this root crosses the imaginary axis with non-vanishing speed as ρ passes through ρ_j. One then uses the above Proposition 3.4.1 and the cocycle property from Corollary 3.4.1 to conclude.

Application: Edge bifurcations and the nonlocal Gap Lemma. We conclude this section by showing how our methods can be used to study eigenvalue problems near the edge of the essential spectrum. Motivated most recently by questions on stability of coherent structures, such as solitons in dispersive equations and viscous shock profiles, there has been significant interest in studying spectra of operators near the edge of the essential spectrum. In the original works [100, 134], a Wronskian-type function that tracks eigenvalues and multiplicities via its roots was extended into the essential spectrum, exploiting the fact that coefficients of the linearized problem converge exponentially as $|x| \to \infty$. While Wronskians are usually finite-dimensional, extensions are sometimes possible to infinite-dimensional systems, using exponential dichotomies and Lyapunov-Schmidt reduction to obtain reduced Wronskians [104, 163].

Gap Lemma type arguments had been used routinely in the theory of Schrödinger operators, providing extensions of scattering coefficients into and across the continuous spectrum. One is often interested in tracking how eigenvalues may emerge out of the essential spectrum when parameters are varied. It was observed early that small localized traps inserted into a free Schrödinger equation will create bound states in dimensions $n \leq 2$; see [185]. The bound state corresponds to an eigenvalue emerging from the edge of the continuous spectrum.

We show here how a result analogous to [185] can be proved for nonlocal eigenvalue problems. We therefore consider the system

$$\mathcal{T}(\lambda,\epsilon)\cdot U := \frac{\mathrm{d}}{\mathrm{d}\xi}U + \left(\mathcal{K} + \epsilon\widetilde{\mathcal{K}}_\xi\right) * U - \lambda B U = 0, \quad U \in \mathbb{R}^n. \tag{3.4.29}$$

Here, $\mathcal{K}, \widetilde{\mathcal{K}}_\xi \in L^1_{\eta_0}(\mathbb{R}, \mathscr{M}_n(\mathbb{R}))$, $B \in \mathscr{M}_n(\mathbb{R})$, and $\widetilde{\mathcal{K}}_\xi \underset{\xi\to\pm\infty}{\longrightarrow} 0$ in $L^1_{\eta_0}(\mathbb{R}, \mathscr{M}_n(\mathbb{R}))$ such that there exist constants $C > 0$ and $\delta > 0$ with

$$\left\|\widetilde{\mathcal{K}}(\zeta;\xi)\right\|_n \leq C e^{-\delta|\xi|}, \quad \forall \zeta \in \mathbb{R}.$$

We think of (3.4.29) as coming from a higher-order differential operator such as ∂_ξ^2, including nonlocal terms, after rewriting the eigenvalue problem as a first-order system of (nonlocal) differential equations in ξ.

Proposition 3.4.2. *We assume that the dispersion relation*

$$d(\nu,\lambda) = \det\left(\nu\mathbb{I}_n + \widehat{\mathcal{K}}(\nu) - \lambda B\right)$$

is diffusive *near* $\lambda = 0$*:*

1. $d(0,0) = d_\nu(0,0) = 0$;
2. $d_{\nu\nu}(0,0) \cdot d_\lambda(0,0) < 0$; *and*
3. $d(\mathbf{i}\ell, 0) \neq 0$ *for all* $\ell \in \mathbb{R}$, $\ell \neq 0$.

We also assume that the localized perturbation is generic:

$$M := \frac{\left\langle \widetilde{\mathcal{K}}_\xi * e_0, e_0^* \right\rangle_{L^2(\mathbb{R},\mathbb{R}^n)}}{\left\langle 2(\mathbb{I}_n + \partial_\nu \widehat{\mathcal{K}}(0))e_1 + \partial_{\nu\nu}\widehat{\mathcal{K}}(0)e_0, e_0^* \right\rangle_{\mathbb{R}^n}} \sqrt{-\frac{d_{\nu\nu}(0,0)}{2d_\lambda(0,0)}} \neq 0,$$

where the nonzero complex vectors e_0, e_0^* *and* e_1 *are defined through*

$$\widehat{\mathcal{K}}(0)e_0 = 0, \quad \widehat{\mathcal{K}}^t(0)e_0^* = 0, \text{ and } \left(\mathbb{I}_n + \partial_\nu \widehat{\mathcal{K}}(0)\right) e_0 + \widehat{\mathcal{K}}(0)e_1 = 0.$$

Then there exists $\epsilon_0 > 0$, *such that for all* $0 < M\epsilon < \epsilon_0$ *there exist* $0 \neq U_\epsilon \in H^1(\mathbb{R},\mathbb{R}^n)$ *and* $\lambda_*(\epsilon) > 0$ *so that*

$$\mathcal{T}(\lambda_*(\epsilon), \epsilon) \cdot U_\epsilon = 0.$$

We also have the asymptotic expansion:

$$\lim_{\epsilon \to 0^+} \frac{\lambda_*(\epsilon)}{\epsilon^2} = M^2. \tag{3.4.30}$$

We prepare the proof of this proposition by reformulating the eigenvalue problem as a nonlinear equation that can be solved with the implicit function theorem near a trivial solution. We first introduce $\lambda = \gamma^2$, so that the dispersion relation has local analytic roots $\gamma \longmapsto \nu_\pm(\gamma) \in \mathbb{C}$. Expanding $d(\nu, \gamma^2)$ in γ^2, we arrive at the expansion

$$d(\nu, \gamma^2) = \nu^2 \frac{d_{\nu\nu}(0,0)}{2} + \gamma^2 d_\lambda(0,0) + \mathcal{O}\left(|\nu|^3 + |\gamma|^3\right),$$

so that to leading order we have

$$\nu_\pm(\gamma) = \pm\sqrt{-\frac{2d_\lambda(0,0)}{d_{\nu\nu}(0,0)}}\gamma + \mathcal{O}(\gamma^2).$$

Associated with these roots can be analytic vectors in the kernel, $\gamma \longmapsto e_\pm(\gamma) \in \mathbb{C}^n$, with

$$\left(\nu_\pm(\gamma)\mathbb{I}_n + \widehat{\mathcal{K}}(\nu_\pm(\gamma)) - \gamma^2 B\right) e_\pm(\gamma) = 0, \tag{3.4.31}$$

and $e_0 = e_\pm(0) \neq 0$ solves $\widehat{\mathcal{K}}(0)e_0 = 0$.

Furthermore, there exists $\eta_* > 0$ such that for each fixed η with $0 < \eta < \eta_*$, the linear operator $\mathcal{L}$

$$\mathcal{L} : U \longmapsto \frac{\mathrm{d}}{\mathrm{d}\xi} U + \mathcal{K} * U,$$

defined on $L^2_\eta(\mathbb{R},\mathbb{R}^n)$, is Fredholm with index -2 and has trivial null space. Indeed, from the above properties, we see that

$$d(\nu, 0) = \det\left(\nu\mathbb{I}_n + \widehat{\mathcal{K}}(\nu)\right) = \nu^2 \widetilde{d}(\nu), \quad \widetilde{d}(0) \neq 0,$$

with $d(\mathbf{i}\ell,0)\neq 0$ for all $\ell\in\mathbb{R}$, $\ell\neq 0$. This implies that $\nu=0$ is a root with multiplicity 2 and all other roots have nonzero real part. Thus the Fredholm index of $\mathcal{L}$ is -2 and it is straightforward to check that the kernel of $\mathcal{L}$ in the exponentially weighted space $L^2_\eta(\mathbb{R},\mathbb{R}^n)$ is trivial. Thus the kernel of the L^2-adjoint $\mathcal{L}^*$ of $\mathcal{L}$ considered on $L^2_{-\eta}(\mathbb{R},\mathbb{R}^n)$ is two-dimensional. Here, the adjoint $\mathcal{L}^*$ is given via

$$\mathcal{L}^*: U\longmapsto -\frac{\mathrm{d}}{\mathrm{d}\xi}U+\mathcal{K}^t_-*U,$$

where $\mathcal{K}^t_-(\xi)=\mathcal{K}^t(-\xi)$ for all $\xi\in\mathbb{R}$. Note that

$$\det\left(\widehat{\mathcal{L}^*}(\nu)\right)=\det\left(-\nu\mathbb{I}_n+\widehat{\mathcal{K}}^t(-\nu)\right)=d(-\nu,0)=\nu^2\widetilde{d}(-\nu),$$

so that there exists $e_0^*\in\mathbb{R}^n$ with $\widehat{\mathcal{K}}^t(0)e_0^*=0$ and thus $\mathcal{L}^*(e_0^*)=0$. As $d_\nu(0,0)=0$, the following scalar product vanishes:

$$\left\langle(\mathbb{I}_n+\partial_\nu\widehat{\mathcal{K}}(0))e_0,e_0^*\right\rangle_{\mathbb{R}^n}=0, \tag{3.4.32}$$

which ensures the existence of $e_1^*\in\mathbb{R}^n$ so that

$$-\left(\mathbb{I}_n+\partial_\nu\widehat{\mathcal{K}}^t(0)\right)e_0^*+\widehat{\mathcal{K}}^t(0)e_1^*=0. \tag{3.4.33}$$

Indeed, the above equation can be solved if $\left\langle\left(\mathbb{I}_n+\partial_\nu\widehat{\mathcal{K}}^t(0)\right)e_0^*,e_0\right\rangle_{\mathbb{R}^n}=0$, which holds true because of (3.4.32). We now claim that $\xi e_0^*+e_1^*$ belongs to the kernel of $\mathcal{L}^*$:

$$\begin{aligned}\mathcal{L}^*\left(\xi e_0^*+e_1^*\right)&=\left[-e_0^*+\mathcal{K}^t_-*(\xi e_0^*)\right]+\widehat{\mathcal{K}}^t(0)e_1^*\\&=\left[-e_0^*-\partial_\nu\widehat{\mathcal{K}}^t(0)e_0^*\right]+\widehat{\mathcal{K}}^t(0)e_1^*\\&=0.\end{aligned}$$

Summarizing, the kernel of $\mathcal{L}^*$, considered on $L^2_{-\eta}(\mathbb{R},\mathbb{R}^n)$, is spanned by the functions e_0^* and $\xi e_0^*+e_1^*$.

In the same way, we also define $e_1\in\mathbb{R}^n$ via

$$\left(\mathbb{I}_n+\partial_\nu\widehat{\mathcal{K}}(0)\right)e_0+\widehat{\mathcal{K}}(0)e_1=0. \tag{3.4.34}$$

Furthermore, differentiating (3.4.31) with respect to γ and evaluating at $\gamma=0$ we obtain

$$\pm\sqrt{-\frac{2d_\lambda(0,0)}{d_{\nu\nu}(0,0)}}\left(\mathbb{I}_n+\partial_\nu\widehat{\mathcal{K}}(0)\right)e_0+\widehat{\mathcal{K}}(0)e'_\pm(0)=0.$$

We see from the above equation and (3.4.34) that $e'_\pm(0)=\pm\sqrt{-\frac{2d_\lambda(0,0)}{d_{\nu\nu}(0,0)}}e_1$. Moreover, combining equations (3.4.33) and (3.4.34) we have the equality

$$\left\langle(\mathbb{I}_n+\partial_\nu\widehat{\mathcal{K}}(0))e_1,e_0^*\right\rangle_{\mathbb{R}^n}=-\left\langle(\mathbb{I}_n+\partial_\nu\widehat{\mathcal{K}}(0))e_0,e_1^*\right\rangle_{\mathbb{R}^n}. \tag{3.4.35}$$

The fact that $d_{\nu\nu}(0,0)\neq 0$ ensures that the following quantity is not vanishing:

$$\left\langle(\mathbb{I}_n+\partial_\nu\widehat{\mathcal{K}}(0))e_1,e_0^*\right\rangle_{\mathbb{R}^n}+\frac{1}{2}\left\langle\partial_{\nu\nu}\widehat{\mathcal{K}}(0)e_0,e_0^*\right\rangle_{\mathbb{R}^n}\neq 0. \tag{3.4.36}$$

To find solutions of the eigenvalue problem (3.4.29), for small ϵ, we make the following ansatz

$$U(\xi) = a_+ e_+(\gamma)\chi_+(\xi)e^{\nu_+(\gamma)\xi} + a_- e_-(\gamma)\chi_-(\xi)e^{\nu_-(\gamma)\xi} + w(\xi), \tag{3.4.37}$$

where $a_+, a_- \in \mathbb{R}$ and $w \in L^2_\eta(\mathbb{R}, \mathbb{R}^n)$. Here $\chi_+(\xi) = \frac{1+\rho(\xi)}{2}$, where $\rho \in \mathscr{C}^\infty(\mathbb{R})$ is a smooth even function satisfying $\rho(\xi) = -1$ for all $\xi \leq -1$, $\rho(\xi) = 1$ for all $\xi \geq 1$ and $\chi_-(\xi) = 1 - \chi_+(\xi)$. Substituting the ansatz into (3.4.29), we obtain an equation of the form

$$\mathcal{F}(a, \gamma, w; \epsilon) = 0, \quad \mathcal{F}(\ \cdot\ ; \epsilon) : \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{D}(\mathcal{L}) \longrightarrow L^2_\eta(\mathbb{R}, \mathbb{R}^n) \tag{3.4.38}$$

for $\mathbf{a} = (a_+, a_-)$. We have that $\mathcal{F}((1,1), 0, 0; 0) = 0$. For small enough η, exploiting the localization of $\widetilde{\mathcal{K}}_\xi$, we have that $\mathcal{F}$ is a smooth map. Its linearization at $(\mathbf{a}, \gamma, w) = (\mathbf{1}, 0, 0)$ (here for convenience we have denoted $\mathbf{1} = (1,1)$) is given by

$$\begin{aligned}
&\mathcal{F}_w(\mathbf{1}, 0, 0; 0) = \mathcal{L},\\
&\mathcal{F}_{a_\pm}(\mathbf{1}, 0, 0; 0) = \mathcal{L}\left(\chi_\pm e_0\right),\\
&\mathcal{F}_\gamma(\mathbf{1}, 0, 0; 0) = \sqrt{-\frac{2d_\lambda(0,0)}{d_{\nu\nu}(0,0)}}\left[\mathcal{L}\left(\chi_+ e_1\right) + \mathcal{L}\left(\xi\chi_+ e_0\right)\right] - \sqrt{-\frac{2d_\lambda(0,0)}{d_{\nu\nu}(0,0)}}\left[\mathcal{L}\left(\chi_- e_1\right) + \mathcal{L}\left(\xi\chi_- e_0\right)\right]
\end{aligned}$$

where $\mathcal{F}_{\mathbf{a}}(\mathbf{1}, 0, 0; 0)$ and $\mathcal{F}_\gamma(\mathbf{1}, 0, 0; 0)$ lie in $L^2_\eta(\mathbb{R}, \mathbb{R}^n)$. Simple computations show that

$$\begin{aligned}
\left\langle \mathcal{F}_{a_-}(\mathbf{1}, 0, 0; 0), e_0^* \right\rangle_{L^2(\mathbb{R},\mathbb{R}^n)} &= 0,\\
\left\langle \mathcal{F}_{a_-}(\mathbf{1}, 0, 0; 0), e_1^* + \xi e_0^* \right\rangle_{L^2(\mathbb{R},\mathbb{R}^n)} &\neq 0,\\
\langle \mathcal{F}_\gamma(\mathbf{1}, 0, 0; 0), e_0^* \rangle_{L^2(\mathbb{R},\mathbb{R}^n)} &\neq 0.
\end{aligned}$$

Thus $\mathcal{F}_{a_-,\gamma}(0;0)$ span the cokernel of $\mathcal{L}$, which implies that the operator

$$\begin{array}{rcll}
\mathcal{F}_{a_-,\gamma,w}(\mathbf{1}, 0, 0; 0): & \mathbb{R} \times \mathbb{R} \times L^2_\eta(\mathbb{R}, \mathbb{R}^n) & \longrightarrow & L^2_\eta(\mathbb{R}, \mathbb{R}^n)\\
& (a_-, \gamma, w) & \longmapsto & \mathcal{F}_{a_-}(\mathbf{1}, 0, 0; 0)a_- + \mathcal{F}_\gamma(\mathbf{1}, 0, 0; 0)\gamma + \mathcal{F}_w(\mathbf{1}, 0, 0; 0)w
\end{array}$$

is invertible, as a Fredholm index 0 operator that is onto. As a consequence, we can solve (3.4.38) using the implicit function theorem and obtain a unique solution (a_-, γ, w) as a function of (a_+, ϵ). First, the asymptotic expansion (3.4.30) follows directly by noticing that, to leading order in ϵ, we have

$$\gamma \left\langle \mathcal{F}_\gamma(\mathbf{1}, 0, 0; 0), e_0^* \right\rangle_{L^2(\mathbb{R},\mathbb{R}^n)} + \epsilon \left\langle \widetilde{\mathcal{K}}_\xi * e_0, e_0^* \right\rangle_{L^2(\mathbb{R},\mathbb{R}^n)} + \mathcal{O}(\epsilon^2) = 0.$$

Here, we have used the fact that $\left\langle \mathcal{F}_{a_-}(\mathbf{1}, 0, 0; 0), e_0^* \right\rangle_{L^2(\mathbb{R},\mathbb{R}^n)} = \langle \mathcal{L}e_0, e_0^* \rangle_{L^2(\mathbb{R},\mathbb{R}^n)} = 0$. Our above computations lead to

$$\langle \mathcal{F}_\gamma(\mathbf{1}, 0, 0; 0), e_0^* \rangle_{L^2(\mathbb{R},\mathbb{R}^n)} = 2\sqrt{-\frac{2d_\lambda(0,0)}{d_{\nu\nu}(0,0)}}\left\langle (\mathbb{I}_n + \partial_\nu \widehat{\mathcal{K}}(0))e_1 + \frac{1}{2}\partial_{\nu\nu}\widehat{\mathcal{K}}(0)e_0, e_0^* \right\rangle_{\mathbb{R}^n} \neq 0.$$

This gives the desired expansion (3.4.30) and implies that $\gamma = -M\epsilon + \mathcal{O}(\epsilon^2)$ is of negative sign for $M\epsilon > 0$. In order to find have an eigenvalue $\lambda_*(\epsilon) > 0$ for (3.4.29), we need to check that $U_\epsilon(\xi)$ given in the ansatz (3.4.37) belongs to $L^2(\mathbb{R},\mathbb{R}^n)$. For small $M\epsilon > 0$, we have that $\nu_\pm(\gamma) = \mp\sqrt{-\frac{2d_\lambda(0,0)}{d_{\nu\nu}(0,0)}M\epsilon} + \mathcal{O}(\epsilon^2)$, such that $\mp\Re(\nu_\pm(\gamma)) > 0$ and U_ϵ is exponentially localized. Since for $\lambda > 0$, there are no roots $\nu \in \mathbf{i}\mathbb{R}$, we know that $\mathcal{T}(\lambda,\epsilon)$ is Fredholm with index zero. Together, this implies that $\mathcal{T}(\lambda,\epsilon)$ possesses a kernel for $\lambda = \lambda_*(\epsilon)$. This completes the proof of Proposition 3.4.2.

Remarks 3.4.2. *Following [168, Prop. 5.11], one can show uniqueness and simplicity of the eigenvalue $\lambda_*(\epsilon)$ for $M\epsilon > 0$. Also, the analysis here gives a natural extension of the eigenvalue concept into the essential spectrum: for $M\epsilon < 0$, we can track the eigenvalue $\lambda_*(\epsilon)$ in smooth fashion as a resonance pole, that is, a function with particular prescribed exponential growth. In this sense, our method here provides an alternative to the Gap Lemma [100, 134], where this possibility of tracking eigenvalues into the essential spectrum was the main objective.*

3.5 Existence of traveling pulse for the FitzHugh-Nagumo equations

We consider the nonlocal reaction-diffusion equation (1.0.3) supplemented by a linear adaptation mechanism

$$\begin{cases} \partial_t \mathbf{u}(t,x) = -\mathbf{u}(t,x) + \int_{\mathbb{R}} \mathcal{K}(x-y)\mathbf{u}(t,y)\mathrm{d}y + f(\mathbf{u}(t,x)) - \mathbf{v}(t,x), \\ \partial_t \mathbf{v}(t,x) = \epsilon(\mathbf{u}(t,x) - \gamma\mathbf{v}(t,x)). \end{cases} \tag{3.5.1}$$

Such a system is often referred to as the spatially extended FitzHugh-Nagumo equations [129] and it has been used to model the propagation of nerve impulse [41, 112]. We are interested in traveling pulse solutions of (3.5.1). These are stationary profiles $(\mathbf{u}(\xi),\mathbf{v}(\xi))$ of (3.5.1) in a comoving frame $\xi = x - ct$ that are localized so that $(\mathbf{u}(\xi),\mathbf{v}(\xi)) \to 0$ as $\xi \to \pm\infty$ and that satisfy

$$\begin{cases} -c\dfrac{\mathrm{d}}{\mathrm{d}\xi}\mathbf{u}(\xi) = -\mathbf{u}(\xi) + \mathcal{K} * \mathbf{u}\,(\xi) + f(\mathbf{u}(\xi)) - \mathbf{v}(\xi), \\ -c\dfrac{\mathrm{d}}{\mathrm{d}\xi}\mathbf{v}(\xi) = \epsilon(\mathbf{u}(\xi) - \gamma\mathbf{v}(\xi)). \end{cases} \tag{3.5.2}$$

Here, $c > 0$ is the wave speed that needs to be determined as part of the problem and $0 < \epsilon \ll 1$ is a small but fixed parameter. Rigorous approaches to the existence of traveling pulses in equations of the form (3.5.1) with local diffusion $\partial_x^2\mathbf{u}$ instead of $-\mathbf{u} + \mathcal{K} * \mathbf{u}$ have been based on singular perturbation methods along the lines of the proof of Theorem 3.1.1 from Section 3.1. In that local setting, one realizes traveling pulses as homoclinic solutions to the origin in a first-order ODE system. The small parameter ϵ introduces a singularly perturbed structure into the problem which allows one to find such a homoclinic orbit by tracking stable and unstable manifolds along fast intersections and slow, normally hyberbolic manifolds [41, 112, 131]. This approach has been successfully applied in many other contexts with slow-fast like structures, with higher- or even infinite-dimensional slow-fast ODEs; see for instance [126, 129, 183, 193].

Our first assumption concerns the nonlinearity, which we assume to be of bistable type.

Hypothesis 3.5.1. *The nonlinearity f is a $\mathscr{C}^\infty$-smooth function with $f(0) = f(1) = 0$, $f'(0) < 0$ and $f'(1) < 0$. Moreover, we assume that $\gamma > 0$ is small enough so that $f(\gamma v) \neq v$. Lastly, we*

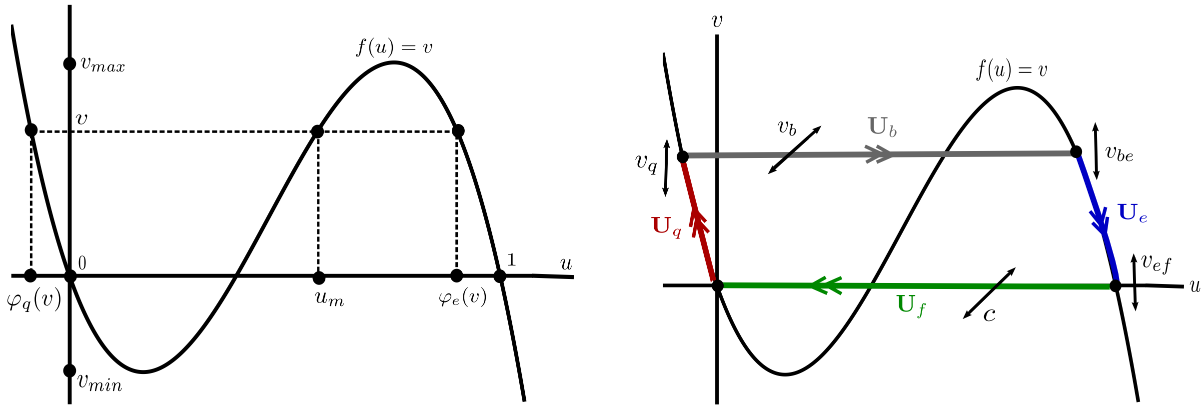


Figure 3.5: Illustration of the assumptions on the nonlinearity f, left. To the right, the singular pulse, consisting of the quiescent part $\mathbf{U}_q$ on the left branch of the slow manifold, the back $\mathbf{U}_b$ connecting to the excited branch, the excitatory part $\mathbf{U}_e$, and the front solution $\mathbf{U}_f$. The five parameters $(v_q, v_b, v_{be}, v_{ef}, c)$ encode take-off and touch-down points, and ensure invertibility of the linearization at the singular solution.

assume that $f(u) - v$ is of bistable type for $v \in (v_{min}, v_{max})$, fixed, that is, it possesses precisely three nondegenerate zeroes.

The assumptions on f are illustrated in Figure 3.5. We denote the left and right zeroes of $f(u) - v$ by $u_q = \varphi_q(v)$ and $u_e = \varphi_e(v)$ and denote by I_q and I_e the ranges of φ_q and φ_e.

Our second assumption concerns the convolution kernel $\mathcal{K}$. For any $\eta \in \mathbb{R}$, we define the space of exponentially weighted functions on the real line equipped with its usual norm

$$L^1_\eta := \left\{ \mathbf{u} : \mathbb{R} \to \mathbb{R} \mid \int_{\mathbb{R}} e^{\eta|\xi|} |\mathbf{u}(\xi)| d\xi < \infty \right\}.$$

We also write $\delta(\xi)$ for the Dirac distribution with $\int \delta = 1$.

Hypothesis 3.5.2. *We suppose that the kernel $\mathcal{K}$ can be written as a sum $\mathcal{K}_{cont} + \mathcal{K}_{disc}$ with the following properties:*

- *There exists $\eta_0 > 0$ such that $\mathcal{K}_{cont} \in L^1_{\eta_0}$;*
- *$\mathcal{K}_{disc} = \sum_{j \in \mathbb{Z}} a_j \delta(\xi - \xi_j)$, and $\sum_j |a_j| e^{\eta_0 |\xi_j|} < \infty$;*
- *the Fourier transform $\widehat{\mathcal{K}}(\mathbf{i}\ell)$ of $\mathcal{K}$ satisfies $\widehat{\mathcal{K}}(0) = 1$ and $\widehat{\mathcal{K}}(\mathbf{i}\ell) - 1 < 0$ for $\ell \neq 0$.*

The first two assumptions, on regularity and on localization, mimic the assumptions the previous section. The assumption $\widehat{\mathcal{K}}(0) = \int \mathcal{K} = 1$ is merely a normalization condition and can be achieved by scaling and redefining f. The last assumption can be slightly relaxed to

$$\widehat{\mathcal{K}}(\mathbf{i}\ell) - 1 - f'(u) < 0, \text{ for all } \ell \neq 0, \; u \in [\varphi_q(v_*), 0] \cup [\varphi_e(v_*), 1],$$

where v_* is defined in Hypothesis 3.5.3, below. Our assumptions do cover typical exponential or Gaussian kernels, as well as infinite-range pointwise interactions. A few comments on the last assumption are in order. Exponential localization guarantees that

$$\widehat{\mathcal{K}}(z) = \int_{\mathbb{R}} \mathcal{K}(x)e^{-zx}dx$$

is analytic in a strip $|\Re(z)| < \eta_*$. Values of the characteristic function

$$\Delta_{u,c}(z) = zc - 1 + \widehat{\mathcal{K}}(z) + f'(u),$$

determine the spectrum of the linearization at a constant state u. Our assumption then guarantees that constant states with $f'(u) < 0$ do not possess zero spectrum, also in spaces with exponential weights $|\eta| < \eta_*$ sufficiently small.

The last assumption refers to the u-system with $\mathbf{v} \equiv const$. Consider therefore

$$-c_*\frac{\mathrm{d}}{\mathrm{d}\xi}\mathbf{u}(\xi) = -\mathbf{u}(\xi) + \mathcal{K} * \mathbf{u}(\xi) + f(\mathbf{u}(\xi)) - v_0, \tag{3.5.3}$$

and the corresponding linearized operator

$$\mathcal{L}(\mathbf{u}_*)\mathbf{u}(\xi) = c_*\frac{\mathrm{d}}{\mathrm{d}\xi}\mathbf{u}(\xi) - \mathbf{u}(\xi) + \mathcal{K} * \mathbf{u}(\xi) + f'(\mathbf{u}_*(\xi))\mathbf{u}(\xi). \tag{3.5.4}$$

Hypothesis 3.5.3. *We assume that there exists non-degenerate front and back solutions with equal speed. More precisely, there exists $c_*, v_* > 0$ such that (3.5.3) possesses a **front** solution $\mathbf{u}_f$ and a **back** solution $\mathbf{u}_b$ with equal speed $c = c_*$, and v-values $v = 0$ and $0 < v = v_* < v_{max}$, respectively, that satisfy the limits*

$$\lim_{\xi\to-\infty}\mathbf{u}_f(\xi) = 1, \quad \lim_{\xi\to+\infty}\mathbf{u}_f(\xi) = 0,$$
$$\lim_{\xi\to-\infty}\mathbf{u}_b(\xi) = \varphi_q(v_*), \quad \lim_{\xi\to+\infty}\mathbf{u}_b(\xi) = \varphi_e(v_*).$$

Moreover, the operators $\mathcal{L}(\mathbf{u}_f)$ and $\mathcal{L}(\mathbf{u}_b)$ each possess an algebraically simple eigenvalue $\lambda = 0$.

We remark that both linearized operators are automatically Fredholm of index zero from Theorem 3.4.1, so that the algebraic multiplicity of the eigenvalue $\lambda = 0$ is finite. Since the derivatives of front and back profile contribute to the kernel, multiplicity is at least one.

While Hypothesis 3.5.1 and 3.5.2 are direct assumptions on nonlinearity and kernel, Hypothesis 3.5.3 is an indirect assumption on both. For positive and even kernels, existence and stability can be established using comparison principles and monotonicity arguments; see for instance Theorem 1.2.5 and [12, 16, 44] for the specific case where $f(u) = u(1-u)(u-a)$, with $0 < a < \frac{1}{2}$. Note that in that case, fronts are in fact monotone, a property that is however not needed in our construction. On the other hand, the set of hypotheses 3.5.1, 3.5.2 and 3.5.3 forms open conditions on nonlinearity and kernel: non-degenerate fronts can readily be seen to persist under small perturbations, using for instance a variation of the methods presented in our proof.

We can now state our main result.

Theorem 3.5.1. *Consider the nonlocal FitzHugh-Nagumo equation (3.5.1) and suppose that Hypotheses 3.5.1, 3.5.2 and 3.5.3 are satisfied; then for every sufficiently small $\epsilon > 0$, there* *exist functions* $\mathbf{u}_\epsilon, \mathbf{v}_\epsilon \in \mathscr{C}^1(\mathbb{R},\mathbb{R})$ *and a wave speed* $c(\epsilon) > 0$ *that depends smoothly on* $\epsilon > 0$ *with* $c(0) = c_*$, *such that*

$$(\mathbf{u}(x,t), \mathbf{v}(x,t)) = (\mathbf{u}_\epsilon(x - c(\epsilon)t), \mathbf{v}_\epsilon(x - c(\epsilon)t)) \tag{3.5.5}$$

is a traveling wave solution of (3.5.1) *that satisfies the limits*

$$\lim_{\xi\to\pm\infty} (\mathbf{u}_\epsilon(\xi), \mathbf{v}_\epsilon(\xi)) = (0,0). \tag{3.5.6}$$

Together with the discussion after Hypothesis 3.5.3, we can state the following somewhat more explicit result.

Corollary 3.5.1. *The nonlocal FitzHugh-Nagumo equation,* $f(u) = u(1-u)(u-a)$, $0 < a < \frac{1}{2}$, γ *sufficiently small,* $\mathcal{K}, \mathcal{K}' \in L^1_{\eta_0}$, $\mathcal{K}$ *even, positive, with* $\int \mathcal{K} = 1$, *possesses a traveling pulse solution.*

Our approach is self-contained, roughly replacing subtle results on exponential dichotomies [111, 156, 165] with crude Fredholm theory. Given the basic simplicity, we believe that our approach should cover a variety of different solution types and different media. For instance, one can readily see how to prove the existence of periodic wave trains in excitable or oscillatory regimes, or front solutions in bistable regimes. In analogy to the case of discrete media [128], we expect different phenomena when $c_* = 0$, that is, for $a \sim 1/2$ in the cubic case, or for the slow pulse [15, 146]. Since the convolution operator does not regularize, compactly supported and discontinuous solutions can occur.

Our proof of Theorem 3.5.1 can be roughly divided into four main parts that can be outlined as follows.

Step 1: Slow manifolds. In a first step, we shall construct invariant slow manifolds for nonlocal differential equations of the form (3.5.2) for $0 < \epsilon \ll 1$ and $c > 0$. Proving the persistence of invariant slow manifolds in the context of singularly perturbed ODEs was originally shown using graph transform [90]. Later, an alternative proof based on variation of constant formulas and exponential dichotomies for differential equations with slowly varying coefficients was given [173]. This latter approach was extended to ill-posed, forward-backward equations in [128, 178]. Our approach completely renounces the concept of a phase space while picking up the main ingredients from the dynamical systems proofs: we modify nonlinearities outside a fixed neighborhood, construct an approximate trial solution, linearize at this “almost solution”, and find a linear convolution type operator with slowly varying coefficients. We invert this operator by constructing suitable local approximate inverses and conclude the proof by setting up a Newton iteration scheme. We will see that the solution on the slow manifold satisfies a scalar ordinary differential equation, with leading order given by an expression equivalent to the one formally derived in [166].

Step 2: The singular solution. We construct a singular solution using front and back solutions from Hypothesis 3.5.3, together with pieces of slow manifolds from Step 1. We glue those solutions using appropriately positioned partitions of unity. Using partitions of unity instead of the matching procedure in cross-sections to the flow, common in dynamical systems approaches, is a second key difference of our approach. It allows us to avoid the notion of a phase space. Schematically, the solution is formed by gluing together a quiescent part $\mathbf{U}_q$ on the left branch of the slow manifold to a back solution $\mathbf{U}_b$, then to an excitatory part $\mathbf{U}_e$ on the right branch of the slow manifold, then to a front solution $\mathbf{U}_f$ as shown in Figure 3.5. On each solution piece, we allow for a correction $\mathbf{W}$. See also Figure 3.6 for a detailed picture.

Step 3: Linearizing and counting parameters. In order to allow for weak interaction between the different corrections to solutions, we use function spaces with appropriately centered exponential weights. The weights, at the same time, encode the facts that solution pieces lie in either strong stable or unstable manifolds, or, in a more subtle way, the Exchange Lemma that tracks inclination of manifolds transverse to stable foliation forward with a flow [39, 131, 132]. Our setup can be viewed as a version of [39], without phase space, in the simplest setting of a one-dimensional slow manifold.

Linearizing at the different solution pieces, we find Fredholm operators with negative index. Roughly speaking, uniform exponential localization of perturbations does not allow corrections in the slow direction. In addition, the linearizations at back and front contribute one-dimensional cokernels, each. In the dynamical systems proofs, matching in cross-sections is accomplished by exploiting

- free variables in stable and unstable manifolds;
- auxiliary parameters, in our case c;
- variations of touchdown and takeoff points on the slow manifolds.

We mimic precisely this idea, pairing the negative index Fredholm operators with suitable additional parameters, so that parameter derivatives span cokernels. A more detailed description is encoded in Figure 3.5. We associate to the quiescent part $\mathbf{U}_q$ the takeoff parameter $v_q \approx v_*$, which encodes the base point of the stable foliation that contains the back. We associate to the excitatory part $\mathbf{U}_e$ touchdown and takeoff parameters $v_{be} \approx v_*$ and $v_{ef} \approx 0$ that will compensate for the mismatched between the back and front parts. Finally we assign to the back $\mathbf{U}_b$ the separate touchdown parameter $v_b \approx v_*$ and to the front $\mathbf{U}_f$ the wave speed $c \approx c_*$. These two parameters effectively compensate for cokernels of front and back linearizations.

Step 4: Errors and fixed point argument. Our last step will be to use a fixed point argument to solve an equation of the form

$$\mathcal{F}_\epsilon(\mathbf{W}, (v_q, v_b, v_{be}, v_{ef}, c)) = 0,$$

that is obtained by substituting our Ansatz directly into the system (3.5.2). More precisely, we will show that

(i) $\|\mathcal{F}_\epsilon(\mathbf{0}, (v_*, v_*, v_*, 0, c_*))\| \to 0$ as $\epsilon \to 0$ in a suitable norm;

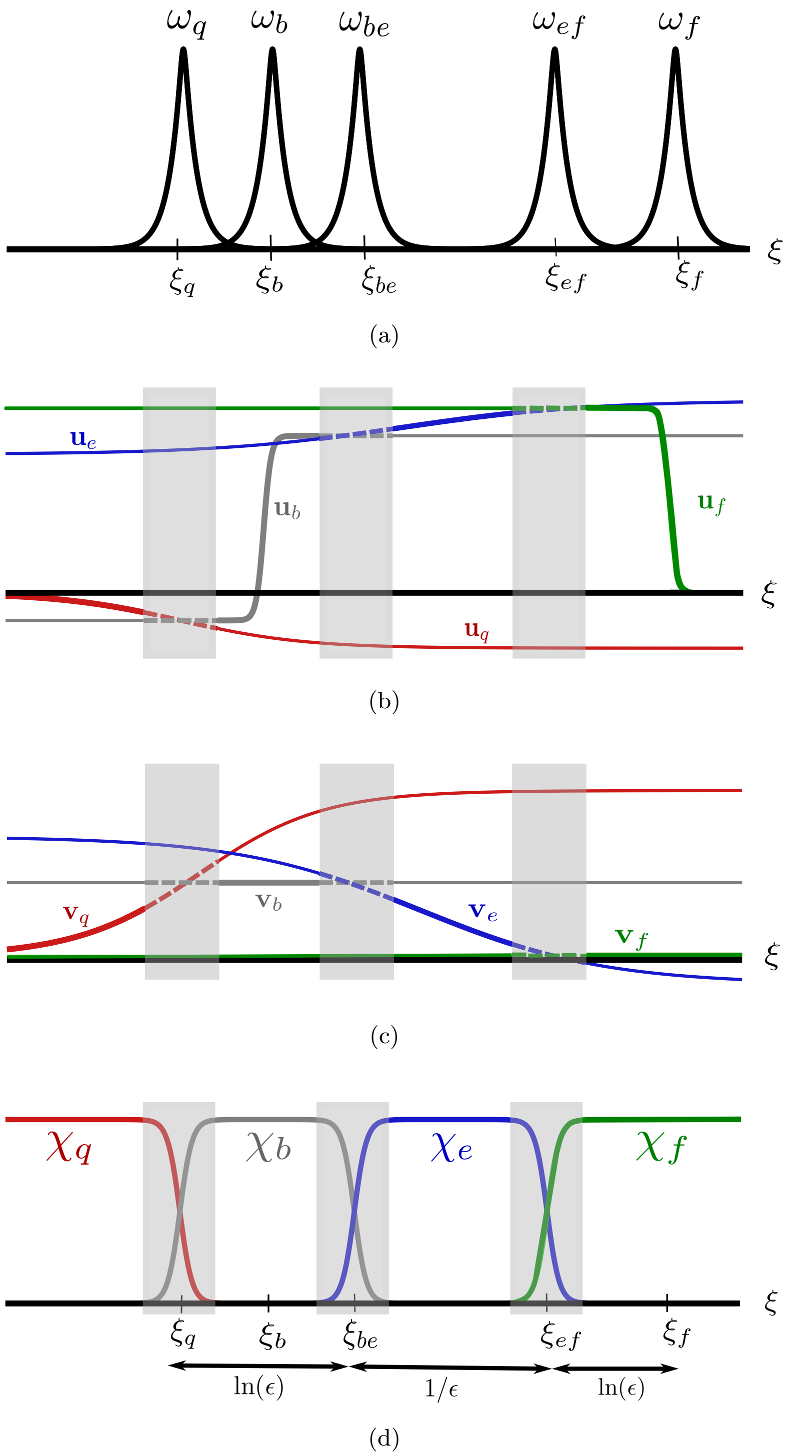


Figure 3.6: Schematic description of the Ansatz solution (3.5.22). Envelopes ω_j for corrections $\mathbf{W}_j$ as imposed by the weights ω_j^{-1} and $\mathbf{u}_j$-components of the different parts of the Ansatz (3.5.22). Profiles χ_j of the partition of unity as defined in (3.5.21).

(ii) $D_{(\mathbf{W},\lambda)}\mathcal{F}_\epsilon(\mathbf{0},(v_*,v_*,v_*,0,c_*))$ is invertible with bounded inverse uniformly in $0<\epsilon\ll 1$;

(iii) $\mathcal{F}_\epsilon$ possesses a unique zero on suitable Banach spaces using a Newton iteration argument.

Here, (ii) follows from Step 3 and (iii) is a simple fixed point iteration. Errors (i) are controlled due to the careful choice of Ansatz and a sequence of commutator estimates between convolution kernels and linear or nonlinear operators.

Now that the proof of Theorem 3.5.1 has been sketched, we present in the remaining of this section the construction of the Ansatz from Step 2 and illustrated in Figure 3.6. To do so, we need first to construct slow manifolds for (3.5.2).

3.5.1 Persistence of slow manifolds

In this section, we prove existence of solutions near the quiescent and the excited branch of $f(u)=v$,

$$\mathcal{M}_q := \{(\varphi_q(v),v)\}, \qquad \mathcal{M}_e := \{(\varphi_e(v),v)\}.$$

We follow the ideas used in the construction of slow manifolds in dynamical systems and use a cut-off function to modify the slow flow outside a neighborhood that is relevant for our construction. We emphasize however that, due to the infinite-range coupling, the concept of solution defined locally in time is not applicable. In other words, the fact that we are modifying the equation outside of a neighborood will create error terms for all ξ.

We use a simple modification of (3.5.2), multiplying the right-hand side of the v-equation by a cut-off function $\Theta(v)$ as shown in Figure 3.7. The modified equation now reads

$$\begin{cases} -c\dfrac{\mathrm{d}}{\mathrm{d}\xi}\mathbf{u}(\xi) = -\mathbf{u}(\xi)+\mathcal{K}*\mathbf{u}(\xi)+f(\mathbf{u}(\xi))-\mathbf{v}(\xi), \\ -c\dfrac{\mathrm{d}}{\mathrm{d}\xi}\mathbf{v}(\xi) = \epsilon(\mathbf{u}(\xi)-\gamma\mathbf{v}(\xi))\Theta(\mathbf{v}(\xi)). \end{cases} \tag{3.5.7}$$

Formally, this introduces two equilibria on the slow manifold, with the effect that the solution on the slow manifold is expected to be a simple heteroclinic orbit. In order to exhibit the slow flow, we rescale space by introducing $\zeta=\epsilon\xi$ so that (3.5.7) becomes

$$\begin{cases} -\epsilon c\dfrac{\mathrm{d}}{\mathrm{d}\zeta}\mathbf{u}(\zeta) = -\mathbf{u}(\zeta)+\mathcal{K}_\epsilon*\mathbf{u}(\zeta)+f(\mathbf{u}(\zeta))-\mathbf{v}(\zeta), \\ -c\dfrac{\mathrm{d}}{\mathrm{d}\zeta}\mathbf{v}(\zeta) = (\mathbf{u}(\zeta)-\gamma\mathbf{v}(\zeta))\Theta(\mathbf{v}(\zeta)), \end{cases} \tag{3.5.8}$$

where we have defined the rescaled kernel as $\mathcal{K}_\epsilon(\zeta):=\epsilon^{-1}\mathcal{K}(\epsilon^{-1}\zeta)$. At $\epsilon=0$, the slow system is given by

$$\begin{cases} 0 = f(\mathbf{u}(\zeta))-\mathbf{v}(\zeta), \\ -c\dfrac{\mathrm{d}}{\mathrm{d}\zeta}\mathbf{v}(\zeta) = (\mathbf{u}(\zeta)-\gamma\mathbf{v}(\zeta))\Theta(\mathbf{v}(\zeta)), \end{cases} \tag{3.5.9}$$

since formally, $\mathcal{K}_\epsilon\to\delta$, the Dirac distribution. Now, for each $c>0$, there exists a heteroclinic solution $(\varphi_q(\mathbf{v}_{h,q}),\mathbf{v}_{h,q})$ to (3.5.9) on the quiescent slow manifold $\mathcal{M}_q$, connecting the rest state $(0,0)$ to $(\varphi_q(v_+),v_+)$ for which the profile $\mathbf{v}_{h,q}\in\mathscr{C}^\infty(\mathbb{R},\mathbb{R})$ satisfies

$$-c\frac{\mathrm{d}}{\mathrm{d}\zeta}\mathbf{v} = (\varphi_q(\mathbf{v})-\gamma\mathbf{v})\Theta(\mathbf{v}), \quad (\varphi_q(\mathbf{v}),\mathbf{v})\in\mathcal{M}_q \tag{3.5.10}$$

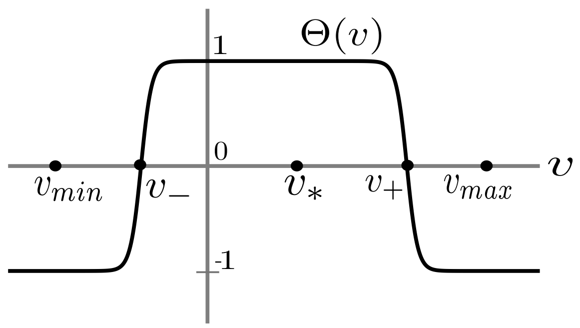


Figure 3.7: The definition of the cut-off function $\Theta(v)$.

with limits

$$\lim_{\zeta\to-\infty}\mathbf{v}_{h,q}(\zeta) = 0 \text{ and } \lim_{\zeta\to+\infty}\mathbf{v}_{h,q}(\zeta) = v_+. \tag{3.5.11}$$

We normalize the solution so that $\mathbf{v}_{h,q}(0) = v_*$. Furthermore, for each $c > 0$, there also exists a heteroclinic solution $(\varphi_e(\mathbf{v}_{h,e}), \mathbf{v}_{h,e}) \in \mathcal{M}_e$ connecting the rest state $(\varphi_e(v_+), v_+)$ to $(\varphi_e(v_-), v_-)$ on the excitatory slow manifold $\mathcal{M}_e$ for which the profile $\mathbf{v}_{h,e} \in \mathscr{C}^\infty(\mathbb{R},\mathbb{R})$ satisfies

$$-c\frac{\mathrm{d}}{\mathrm{d}\zeta}\mathbf{v} = (\varphi_e(\mathbf{v}) - \gamma\mathbf{v})\Theta(\mathbf{v}), \quad (\varphi_e(\mathbf{v}), \mathbf{v}) \in \mathcal{M}_e \tag{3.5.12}$$

with limits

$$\lim_{\zeta\to-\infty}\mathbf{v}_{h,e}(\zeta) = v_+ \text{ and } \lim_{\zeta\to+\infty}\mathbf{v}_{h,e}(\zeta) = v_-. \tag{3.5.13}$$

We normalize this solution so that $\mathbf{v}_{h,e}(0) = 0$.

One then needs to show that these two heteroclinic solutions persist for $0 < \epsilon \ll 1$ using a fixed point argument for nonlocal differential evolution equations with slowly varying coefficients. We give formal statements of the main result; a schematic picture of these heteroclinics relative to the singular pulse is shown in Figure 3.8.

Proposition 3.5.1 (Quiescent slow manifold)**.** *For every sufficiently small $\epsilon > 0$ and any $c > 0$, there exist functions $\mathbf{u}_q, \mathbf{v}_q \in \mathscr{C}^\infty(\mathbb{R},\mathbb{R})$ such that*

$$(\mathbf{u}_q(\epsilon\xi), \mathbf{v}_q(\epsilon\xi)) \tag{3.5.14}$$

is a heteroclinic solution of (3.5.7) *that satisfies the limits*

$$\lim_{\zeta\to-\infty}(\mathbf{u}_q(\zeta), \mathbf{v}_q(\zeta)) = (0,0) \text{ and } \lim_{\zeta\to+\infty}(\mathbf{u}_q(\zeta), \mathbf{v}_q(\zeta)) = (\varphi_q(v_+), v_+). \tag{3.5.15}$$

Up to translation, this solution is locally unique and depends smoothly on ϵ and c.

Proposition 3.5.2 (Excitatory slow manifold)**.** *For every sufficiently small $\epsilon > 0$ and any $c > 0$, there exist functions $\mathbf{u}_e, \mathbf{v}_e \in \mathscr{C}^\infty(\mathbb{R},\mathbb{R})$ such that*

$$(\mathbf{u}_e(\epsilon\xi), \mathbf{v}_e(\epsilon\xi)) \tag{3.5.16}$$

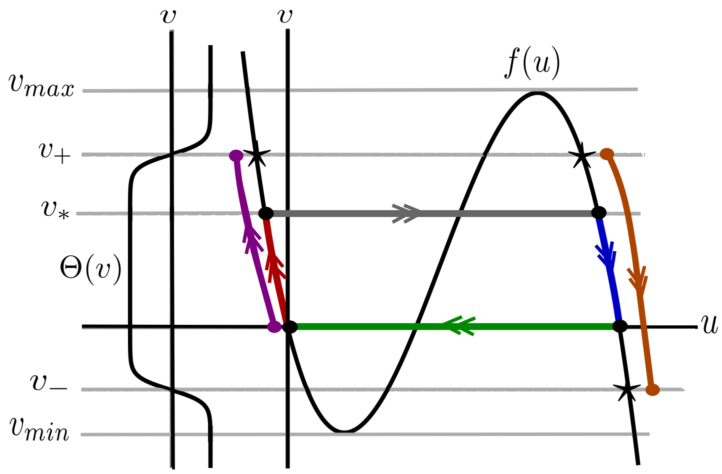


Figure 3.8: Heteroclinics from Proposition 3.5.1 (left,purple) and from Proposition 3.5.2 (right,orange). The upper limit v_+ is induced by the cut-off Θ which is superimposed on the v-axis.

is a heteroclinic solution of (3.5.7) *that satisfies the limits*

$$\lim_{\zeta\to-\infty}(\mathbf{u}_e(\zeta),\mathbf{v}_e(\zeta))=(\varphi_e(u_+),u_+) \text{ and } \lim_{\zeta\to+\infty}(\mathbf{u}_e(\zeta),\mathbf{v}_e(\zeta))=(\varphi_e(v_-),v_-). \tag{3.5.17}$$

Up to translation, this solution is locally unique and depends smoothly on ϵ *and* c*.*

We remark that this construction of slow manifolds in nonlocal equations is somewhat general but comes with some caveats. First, the construction is simple here, since the slow manifold is one-dimensional and hence consists of a single trajectory, only. As a consequence, smoothness of slow manifolds is trivial, here. Second, the solutions are *not* solutions for the original system, without the modifier Θ, since the equation has infinite-range interaction in time. In other words, the modified piece of the trajectory influences the solution even where the solution takes values in the unmodified range. However, in Step 4 of the proof Theorem 3.5.1, we exploit that the error terms stemming from this modification are exponentially small due to the exponential localization of the kernel.

We also note that monotonicity of $\mathbf{v}_q$ (and similarly $\mathbf{v}_e$) implies that $\mathbf{v}_q$ solves a simple first-order differential equation, the "reduced equation" on the slow manifold. Again, this equation depends, even locally, on the modifier Θ. From our construction, below, one can easily see that the leading-order vector field in ϵ is just the one given in (3.5.12).

As already explained, the strategy for the proof of Propositions 3.5.1 and 3.5.2 is to construct an approximate trial solution based on the limiting system, (3.5.9) linearize at this "almost solution", and find a linear convolution type operator with slowly varying coefficients. Now that we have proved the persistence of slow manifolds we can present the construction of the singular pulse.

3.5.2 The Ansatz

In the following, we present a decomposition of the solution into the singular pulse and corrections, separated using cut-off functions and exponentially localized weights. A schematic illustration of this procedure is shown in Figure 3.6.

We write $\mathbf{U}_f = (\mathbf{u}_f, 0)$ where $\mathbf{u}_f \in \mathscr{C}^\infty(\mathbb{R}, \mathbb{R})$ is the front solution from Hypothesis 3.5.3, solving

$$-c_* \frac{\mathrm{d}}{\mathrm{d}\xi}\mathbf{u}(\xi) = -\mathbf{u}(\xi) + \int_{\mathbb{R}} \mathcal{K}(\xi - \xi')\mathbf{u}(\xi')d\xi' + f(\mathbf{u}(\xi)), \tag{3.5.18}$$

with

$$\lim_{\xi \to -\infty} \mathbf{u}_f(\xi) = 1, \ \lim_{\xi \to +\infty} \mathbf{u}_f(\xi) = 0, \text{ and } \mathbf{u}_f(0) = \frac{1}{2}.$$

Similarly, we set $\mathbf{U}_b = (\mathbf{u}_b, v_b\mathbf{1})$ where $v_b \in (v_* - \delta_b, v_* + \delta_b) \subset (v_{min}, v_{max})$ is a free parameter and $\mathbf{u}_b \in \mathscr{C}^\infty(\mathbb{R}, \mathbb{R})$ is the solution of

$$-c_b \frac{\mathrm{d}}{\mathrm{d}\xi}\mathbf{u}(\xi) = -\mathbf{u}(\xi) + \int_{\mathbb{R}} \mathcal{K}(\xi - \xi')\mathbf{u}(\xi')d\xi' + f(\mathbf{u}(\xi)) - v_b \tag{3.5.19}$$

with limits

$$\lim_{\xi \to -\infty} \mathbf{u}_b(\xi) = \varphi_q(v_b), \ \lim_{\xi \to +\infty} \mathbf{u}_b(\xi) = \varphi_e(v_b) \text{ and } \mathbf{u}_b(0) = \left(\varphi_e(v_b) - \varphi_q(v_b)\right)/2.$$

Again, this solution is obtained from Hypothesis 3.5.3 for $v_b = v_*$. Using the implicit function theorem and simplicity of the zero eigenvalue, we can find the profile $\mathbf{u}_b \in \mathscr{C}^\infty$ and the wave speed c as smooth functions of $v_b \sim v_*$.

Using Proposition 3.5.1 and 3.5.2, we define $\mathbf{U}_q = (\mathbf{u}_q, \mathbf{v}_q)$ and $\mathbf{U}_e = (\mathbf{u}_e, \mathbf{v}_e)$ where the heteroclinic solutions $(\mathbf{u}_q(\epsilon\xi), \mathbf{v}_q(\epsilon\xi))$ and $(\mathbf{u}_e(\epsilon\xi), \mathbf{v}_e(\epsilon\xi))$ solve

$$\begin{cases} -c\dfrac{\mathrm{d}}{\mathrm{d}\xi}\mathbf{u}(\xi) = -\mathbf{u}(\xi) + \displaystyle\int_{\mathbb{R}} \mathcal{K}(\xi - \xi')\mathbf{u}(\xi')d\xi' + f(\mathbf{u}(\xi)) - \mathbf{v}(\xi), \\ -c\dfrac{\mathrm{d}}{\mathrm{d}\xi}\mathbf{v}(\xi) = \epsilon(\mathbf{u}(\xi) - \gamma\mathbf{v}(\xi))\Theta(\mathbf{v}(\xi)), \end{cases} \tag{3.5.20}$$

with limits

$$\lim_{\zeta \to -\infty} (\mathbf{u}_q(\zeta), \mathbf{v}_q(\zeta)) = (0,0) \text{ and } \lim_{\zeta \to +\infty} (\mathbf{u}_q(\zeta), \mathbf{v}_q(\zeta)) = (\varphi_q(v_+), v_+),$$
$$\lim_{\zeta \to -\infty} (\mathbf{u}_e(\zeta), \mathbf{v}_e(\zeta)) = (\varphi_e(v_+), v_+) \text{ and } \lim_{\zeta \to +\infty} (\mathbf{u}_e(\zeta), \mathbf{v}_e(\zeta)) = (\varphi_e(v_-), v_-).$$

Let $\delta_q > 0$, $\delta_{be} > 0$ and $\delta_{ef} > 0$ be fixed such that $(v_* - \delta_q, v_* + \delta_q) \subset (v_{min}, v_{max})$, $(v_* - \delta_{be}, v_* + \delta_{be}) \subset (v_{min}, v_{max})$ and $(-\delta_{ef}, \delta_{ef}) \subset (v_{min}, v_{max})$. We introduce three parameters $v_q \in (v_* - \delta_q, v_* + \delta_q)$, $v_{be} \in (v_* - \delta_{be}, v_* + \delta_{be})$ and $v_{ef} \in (-\delta_{ef}, \delta_{ef})$. We normalize the solutions $\mathbf{U}_q$ and $\mathbf{U}_e$ by specifying their $\mathbf{v}$-value at $\xi = 0$ as $\mathbf{v}_q(0) = v_q$ and $\mathbf{v}_e(0) = v_{ef}$. Since the solutions in the slow manifold are monotone, by the implicit function theorem, we obtain two maps $\widetilde{\varphi}_j$ so that

$$\mathbf{u}_j(0) = \widetilde{\varphi}_j(\mathbf{v}_j(0), \epsilon, c), \quad j = q, e,$$

uniformly in the parameters. As a consequence, we have $(\mathbf{u}_q(0), \mathbf{v}_q(0)) = (\widetilde{\varphi}_q(v_q, \epsilon, c), v_q)$ and $(\mathbf{u}_e(0), \mathbf{v}_e(0)) = (\widetilde{\varphi}_e(v_{ef}, \epsilon, c), v_{ef})$. We also define $T(v_{be}, v_{ef}) > 0$ as the leading order time spent by $(\mathbf{u}_e, \mathbf{v}_e)$ on the excitatory slow manifold from $(\widetilde{\varphi}_e(v_{be}, \epsilon, c), v_{be})$ to $(\widetilde{\varphi}_e(v_{ef}, \epsilon, c), v_{ef})$. Note that $(v_{be}, v_{ef}) \longmapsto T(v_{be}, v_{ef})$ is a continuously differentiable function on $(v_* - \delta_{be}, v_* + \delta_{be}) \times (-\delta_{ef}, \delta_{ef})$.

We introduce a partition of unity through four $\mathscr{C}^\infty$-functions χ_j, $j \in \{q, b, e, f\}$, so that:

$$\chi_q(\xi) + \chi_b(\xi) + \chi_e(\xi) + \chi_f(\xi) = 1, \quad \forall \xi \in \mathbb{R}, \tag{3.5.21}$$

and

$$\chi_q(\xi) = \begin{cases} 0 & \xi \geq \xi_q + 1 \\ 1 & \xi \leq \xi_q - 1 \end{cases}, \quad \chi_b(\xi) = \begin{cases} 0 & \xi \leq \xi_q - 1 \text{ and } \xi \geq \xi_{be} + 1 \\ 1 & \xi_q + 1 \leq \xi \leq \xi_{be} - 1 \end{cases},$$
$$\chi_f(\xi) = \begin{cases} 0 & \xi \leq -1 \\ 1 & \xi \geq 1 \end{cases}, \quad \chi_e(\xi) = \begin{cases} 0 & \xi \leq \xi_{be} - 1 \text{ and } \xi \geq 1 \\ 1 & \xi_{be} + 1 \leq \xi \leq -1 \end{cases}.$$

The constants ξ_q and ξ_{be} are defined as

$$\xi_{be} = -\frac{T(v_{be}, v_{ef})}{\epsilon}, \qquad \xi_q = \xi_{be} + 2\eta_b \ln(\epsilon),$$

where $\eta_b > 0$ will be fixed later. We will rely on the exponentially weighted spaces H^1_η, L^2_η with

$$L^2_\eta = \left\{ \mathbf{u} : \mathbb{R} \to \mathbb{R} \mid \left\| e^{\eta|\xi|} \mathbf{u}(\xi) \right\|_{L^2} < +\infty \right\}, \qquad H^1_\eta = \left\{ \mathbf{u} \in L^2_\eta \mid \partial_\xi \mathbf{u} \in L^2_\eta \right\},$$

where $\eta > 0$ is sufficiently small.

To find a pulse solution, we use the following Ansatz

$$\begin{aligned} \mathbf{U}_a(\xi) = (\mathbf{u}_a(\xi), \mathbf{v}_a(\xi)) = {} & \mathbf{U}_q\left(\epsilon\left(\xi - \xi_q\right)\right)\chi_q(\xi) + \mathbf{U}_b(\xi - \xi_b)\chi_b(\xi) + \mathbf{U}_e(\epsilon\xi)\chi_e(\xi) \\ & + \mathbf{U}_f(\xi - \xi_f)\chi_f(\xi) + \mathbf{W}_q\left(\xi - \xi_q\right) + \mathbf{W}_b(\xi - \xi_b) + \mathbf{W}_e(\xi) \\ & + \mathbf{W}_f(\xi - \xi_f), \end{aligned} \tag{3.5.22}$$

where $\xi_b = \xi_{be} + \eta_b \ln(\epsilon)$, $\xi_f := -\eta_f \ln(\epsilon) > 0$, and $\mathbf{W}_j = (\mathbf{w}^u_j, \mathbf{w}^v_j)$, $\quad j \in J_{\mathbf{w}} := \{q, b, e, f\}$.

The $\mathbf{u}_a$ and $\mathbf{v}_a$ components of $\mathbf{U}_a$ are thus given by

$$\begin{aligned} \mathbf{u}_a(\xi) = {} & \mathbf{u}_q\left(\epsilon\left(\xi - \xi_q\right)\right)\chi_q(\xi) + \mathbf{u}_b(\xi - \xi_b)\chi_b(\xi) + \mathbf{u}_e(\epsilon\xi)\chi_e(\xi) + \mathbf{u}_f(\xi - \xi_f)\chi_f(\xi) \quad (3.5.23\text{a}) \\ & + \mathbf{w}^u_q\left(\xi - \xi_q\right) + \mathbf{w}^u_b(\xi - \xi_b) + \mathbf{w}^u_e(\xi) + \mathbf{w}^u_f(\xi - \xi_f), \\ \mathbf{v}_a(\xi) = {} & \mathbf{v}_q\left(\epsilon\left(\xi - \xi_q\right)\right)\chi_q(\xi) + v_b\chi_b(\xi) + \mathbf{v}_e(\epsilon\xi)\chi_e(\xi) + \mathbf{w}^v_q\left(\xi - \xi_q\right) + \mathbf{w}^v_b(\xi - \xi_b) \quad (3.5.23\text{b}) \\ & + \mathbf{w}^v_e(\xi) + \mathbf{w}^v_f(\xi - \xi_f). \end{aligned}$$

Remarks 3.5.1. *We retain five free parameters* $(c, v_q, v_b, v_{be}, v_{ef})$*;* $(\delta_q, \delta_b, \delta_{be}, \delta_{ef}, \eta_b, \eta_f)$ *are fixed during the proof of Theorem 3.5.1.*

The idea behind the Ansatz (3.5.22) is the following. We look for a solution that is approximately given by 4 trajectories (quiescent, back, excitatory and front parts) glued together. At each overlap between two pieces of trajectories, we use a cut-off function and introduce corrections $\mathbf{W}_j$ to compensate for mismatches. We have carefully adjusted the precise location of the gluing site which scales with the parameter ϵ (of order $-\ln(\epsilon)$ for back and front, and order $1/\epsilon$ along the slow excitatory manifold). The correction of the excitatory part $\mathbf{W}_e$ is commonly constructed using the Exchange Lemma in a dynamical systems based approach [132]. Those corrections are exponentially localized close to touchdown and takeoff points. Rather than encoding this localization at two diverging points 0 and ξ_{be}, with a varying family of weights, we prefer to again split $\mathbf{W}_e$ into $\mathbf{W}_{be}$ and $\mathbf{W}_{ef}$,

$$\mathbf{W}_e(\xi) = \mathbf{W}_{be}(\xi - \xi_{be}) + \mathbf{W}_{ef}(\xi).$$

Here, all corrections $\mathbf{W}_j$ are exponentially localized given by some associated weighted ω_j, see Figure 3.6.

With such an Ansatz, we substitute the expressions of $\mathbf{u}_a$ and $\mathbf{v}_a$ into (3.5.2) and obtain equations for the corrections $\mathbf{w}_j^{u,v}$. The crucial idea is then to split the equations into a weakly coupled system of 5 equations for the corrections $\mathbf{w}_j^{u,v}$ (one for each correction) so that one obtains a general system that can be written in compact form:

$$0 = \mathcal{L}_\epsilon\left(\mathbf{W}, \lambda - \lambda_*\right) + \mathcal{R}_\epsilon + \mathcal{N}_\epsilon(\mathbf{W}, \lambda - \lambda_*) := \mathcal{F}_\epsilon(\mathbf{W}, \lambda), \tag{3.5.24}$$

where we have set

$$\begin{aligned}
\mathbf{W} &:= \left(\left(\mathbf{w}_q^u, \mathbf{w}_q^v\right), \left(\mathbf{w}_b^u, \mathbf{w}_b^v\right), \left(\mathbf{w}_{be}^u, \mathbf{w}_{be}^v\right), \left(\mathbf{w}_{ef}^u, \mathbf{w}_{ef}^v\right), \left(\mathbf{w}_f^u, \mathbf{w}_f^v\right)\right) \in \mathcal{X} := \left(H_\eta^1 \times H_\eta^1\right)^5, \\
\lambda &:= \left(c, v_q, v_b, v_{be}, v_{ef}\right) \in \mathcal{V}, \\
\lambda_* &:= \left(c_*, v_*, v_*, v_*, 0\right).
\end{aligned}$$

Here, $\mathcal{L}_\epsilon$ represents all the linear terms, $\mathcal{R}_\epsilon$ collects all the error terms and $\mathcal{N}_\epsilon$ all the nonlinear terms. We define the nonlinear map $\mathcal{F}_\epsilon$ as follows

$$\mathcal{F}_\epsilon : \begin{array}{ccc} \mathcal{X} \times \mathcal{V} & \longrightarrow & \mathcal{Y} \\ (\mathbf{W}, \lambda) & \longmapsto & \mathcal{F}_\epsilon(\mathbf{W}, \lambda) \end{array} \tag{3.5.25}$$

where $\mathcal{X} := \left(H_\eta^1 \times H_\eta^1\right)^5$, $\mathcal{Y} := \left(L_\eta^2 \times L_\eta^2\right)^5$ and $\mathcal{V} := (c_* - \delta_c, c_* + \delta_c) \times (v_* - \delta_b, v_* + \delta_b) \times (v_* - \delta_b, v_* + \delta_b) \times (v_* - \delta_{be}, v_* + \delta_{be}) \times (-\delta_{ef}, \delta_{ef})$ is a neighborhood of $\lambda_* = (c_*, v_*, v_*, v_*, 0)$ in $\mathbb{R}^5$. The remaining part of the proof is to show that

1. the map $\mathcal{F}_\epsilon$ is well-defined from $\mathcal{X} \times \mathcal{V}$ to $\mathcal{Y}$ and is $\mathscr{C}^\infty$;

2. $\mathcal{R}_\epsilon = \mathcal{F}_\epsilon(\mathbf{0}, \lambda_*) \longrightarrow 0$ as $\epsilon \longrightarrow 0$;

3. $\mathcal{L}_\epsilon = D\mathcal{F}_\epsilon(\mathbf{0}, \lambda_*)$ can be decomposed in two parts:

$$\mathcal{L}_\epsilon = \mathcal{L}_\epsilon^i + \mathcal{L}_\epsilon^p, \tag{3.5.26}$$

where $\mathcal{L}_\epsilon^i$ is invertible with bounded inverse on suitable Banach spaces and $\mathcal{L}_\epsilon^p$ is an ϵ-perturbation: $\mathcal{L}_\epsilon^p \longrightarrow 0$ as $\epsilon \longrightarrow 0$. This is in this third step that we use Fredholm properties of nonlocal differential operators.

Then, to conclude the proof of Theorem 3.5.1, we use a fixed point iteration argument on the map $\mathcal{F}_\epsilon$ which gives the existence of $(\mathbf{W}(\epsilon), \lambda(\epsilon))$, solution of (3.5.24), in a neighborhood of $(\mathbf{0}, \lambda_*)$ for small values of $\epsilon > 0$.

3.5.3 Generalizations and limitations

We comment on several aspects of our main result, pointing out generalizations, limitations, and possible future work.

Uniqueness. The construction of the pulse ultimately relies on a contraction principle, which guarantees uniqueness (up to translation) in a small neighborhood. A description of this neighborhood is rather technical and uniqueness would presumably hold in a larger class, so that we refrain from adding in Theorem 3.5.1 a precise statement. Generally, dynamical systems approaches give stronger uniqueness results since the set of solutions is described pointwise in space, rather than in a function space. The pulse is certainly not unique in the set of bounded or even localized solutions. For instance, our methods should also give existence of periodic pulse trains or a slow pulse with speed $c = \mathcal{O}(\sqrt{\varepsilon})$, which are well understood in the local, PDE setting [146, 187].

Stability. One suspects that the pulses constructed here are spectrally and nonlinearly stable, similar to pulses in the PDE setting. Using similar methods as in the existence proof, one can show that the linearized problem does not possess eigenvalues in $\Re(\lambda) \geq -\delta$ for some $\delta > 0$ in a suitably weighted exponential space — except for possibly two eigenvalues in a neighborhood of the origin stemming from separate translations of front and back of the pulse. Since one of those two eigenvalues is pinned at $\lambda = 0$ by translation symmetry, the key step for proving stability then involves the expansion of the second eigenvalue; see [126]. It is not clear if the error estimates in our construction will suffice to obtain such an expansion.

Limitations — localization and regularity. Existence of front and back (see Theorem 1.2.5) does not require exponential localization of the kernel $\mathcal{K}$, while our construction does. Also, one expects existence results for front and back with somewhat singular kernels $\mathcal{K}$. In both cases, adapting our results might be challenging. The construction of the pulse essentially relies on a weak interaction between front and back. In the ODE construction, this is reflected in hyperbolicity of the slow manifold. Algebraic localized kernels introduce a second, competing mode of interaction, possibly destroying or destabilizing the pulse. On the other hand, singularity in the kernel may compete with the advection term $c\partial_\xi$ in the $\mathbf{v}$-component: results from the previous Section 3.3 give some evidence how such singularities may alter asymptotics.

Generalizations. Our approach was used, in a simpler context, to establish existence of shocks in non-local conservation laws (see [86]) and to track eigenvalues in the continuous spectrum via a Gap lemma construction for an Evans function (see the application of Section 3.4). Both applications basically rely on the construction of strong stable manifolds for nonlocal problems, a problem much simpler than the construction of slow manifolds and exchange lemmas required for the FitzHugh-Nagumo pulse; see the outline of our proof, below. On the other hand, we believe that one can mimic most ODE constructions, including higher-dimensional center-manifolds and normal forms using our approach. This is actually the subject of the forthcoming Section 3.6.

In a different direction, it would nevertheless be interesting to construct a dynamical-systems approach via exponential dichotomies to these infinite-delay forward-backward problems.

3.6 Center manifolds without a phase space

Center-manifold reductions have become a central tool to the analysis of dynamical systems. The very first results on center manifolds go back to the pioneering works of Pliss [167] and Kelley [136] in the finite-dimensional setting. In the simplest context, one studies differential equations in the vicinity of a non-hyperbolic equilibrium,

$$\frac{\mathrm{d}u}{\mathrm{d}t} = f(u) \in \mathbb{R}^n, \qquad f(0) = 0, \quad \mathrm{spec}(f'(0)) \cap \mathbf{i}\mathbb{R} \neq \emptyset.$$

The basic reduction establishes that the set of small bounded solutions $u(t)$, $t \in \mathbb{R}$, $\sup|u(t)| < \delta \ll 1$, is *pointwise* contained in a manifold, that is, $u(t) \in \mathcal{W}^{\mathrm{c}}$ for all t. This manifold is a subset of phase space, $\mathcal{W}^{\mathrm{c}} \subset \mathbb{R}^n$, contains the origin, $0 \in \mathcal{W}^{\mathrm{c}}$, and is tangent to $\mathcal{E}^{\mathrm{c}}$, the generalized eigenspace associated with purely imaginary eigenvalues of $f'(0)$. As a consequence, the flow on $\mathcal{W}^{\mathrm{c}}$ can be projected onto $\mathcal{E}^{\mathrm{c}}$, to yield a *reduced vector field*. The reduction to this lower-dimensional ODE then allows one to describe solutions qualitatively, even explicitly in some cases. Of course, the method applies to higher-order differential equation, which one simply writes as first-order equation in a canonical fashion. Extensions to infinite-dimensional dynamical systems were pursued soon after; see for instance [114].

Starting with the work of Kirchgässner [140], such reductions have been extended to systems with $u \in \mathcal{X}$, a Banach space, where the initial value problem is not well-posed: For most initial conditions u_0, there does not exist a local solution $u(t)$, $0 \le t < t_*$, say. Local solutions do exist however for all initial conditions on a finite-dimensional center-manifold, and much of the theory is quite analogous to the finite-dimensional case; see [191]. In these theories, one can typically split the phase space in infinite-dimensional linear spaces where solutions to the linearized equation either decay or grow, and a finite-dimensional center subspace. Such splittings are known as Wiener-Hopf factorizations and can be difficult to achieve in the case of forward-backward delay equations, where nevertheless center-manifold reductions are available [127].

Our point of view here is slightly more abstract, shedding the concept of a phase space in favor of a focus on small bounded trajectories. We perform a purely functional analytic reduction, based on Fredholm theory developed in Section 3.4 in the space of bounded trajectories (rather than the phase space). We parameterize the set of bounded solutions by the set of (weakly) bounded solutions to the linear equation, which is a finite-dimensional vector space, amenable to a variety of parameterizations. Only after this reduction, we derive a differential equation on this finite-dimensional vector space, whose solutions, when lifted to the set of bounded solutions to the nonlinear problem describe all small bounded solutions.

To be more precise, we focus on nonlocal equations of the form

$$\mathcal{T}u + \mathcal{F}(u) = 0, \qquad \mathcal{T}u = u + \mathcal{K} * u, \tag{3.6.1}$$

for $u:\mathbb{R}\to\mathbb{R}^n$, $n\geq 1$. Here, $\mathcal{K}*u$ stands for matrix convolution on $\mathbb{R}$,

$$(\mathcal{K}*u(x))_i=\sum_{j=1}^{n}\int_{\mathbb{R}}\mathcal{K}_{i,j}(x-y)u_j(y)\mathrm{d}y,\quad 1\leq i\leq n,$$

and $\mathcal{F}(u)$ encodes nonlinear terms, possibly also involving nonlocal interactions. A prototypical example arises when studying stationary or traveling-wave solutions to neural field equations (1.0.1). The idea is to go beyond the method which consists in focusing on kernels $\mathcal{K}$ with rational Fourier transform. In order to avoid unnecessary technicalities, we first present our main results in an informal way. The precise statements and hypotheses will be detailed in a second step. We thus assume that the matrix kernel $\mathcal{K}$ and nonlinear operator $\mathcal{F}$ satisfy the following informal assumptions:

- *Exponential Localization*: the interaction kernel $\mathcal{K}$ and its derivative $\mathcal{K}'$ are exponentially localized (see Hypothesis 3.6.1);
- *Smoothness and Invariance*: the nonlinear operator $\mathcal{F}$ is assumed to be sufficiently smooth and translation invariant $\mathcal{F}(u(\cdot+\xi))(\cdot)=\mathcal{F}(u(\cdot))(\cdot+\xi)$, with $\mathcal{F}(0)=0$, $D_u\mathcal{F}(0)=0$ (see Hypothesis 3.6.2).

Using Fourier transform, one can readily find the finite-dimensional space $\ker\mathcal{T}$ of solutions to $\mathcal{T}u=0$ with at most algebraic growth and construct a bounded projection $\mathcal{Q}$ onto this set, in a space of functions allowing for slow exponential growth.

Theorem 3.6.1. *Assume that the interaction kernel $\mathcal{K}$ and the nonlinear operator $\mathcal{F}$ satisfy Hypotheses 3.6.1 and 3.6.2. Then, there exists $\delta>0$ and a map $\Psi\in\mathscr{C}^k(\ker\mathcal{T},\ker\mathcal{Q})$ with $\Psi(0)=0$, $D_u\Psi(0)=0$, such that the manifold*

$$\mathcal{M}_0:=\{u_0+\Psi(u_0)\mid u_0\in\ker\mathcal{T}\}$$

contains the set of all bounded solutions of (3.6.1) with $\sup_{x\in\mathbb{R}}|u(x)|\leq\delta$.

We refer to $\mathcal{M}_0$ as a (global) center manifold for (3.6.1). Note however that points on $\mathcal{M}_0$ consist of *trajectories*, that is, of solutions $u(x)$, $x\in\mathbb{R}$, rather than of initial values to solutions, in the more common view of center manifolds. Also note that, according to the theorem, $\mathcal{M}_0$ only contains the set of bounded solutions, not all elements of $\mathcal{M}_0$ are necessarily bounded solutions. As is well known from the classical center manifold theorem, the set of bounded solutions may well be trivial, consisting of the point $u\equiv 0$, only, rather than being diffeomorphic to a finite-dimensional ball. It is therefore necessary to study the elements of $\mathcal{M}_0$ in more detail.

We will see in the proof that, as is common in the construction of center manifolds, we modify the nonlinearity $\mathcal{F}$ to $\mathcal{F}_\epsilon$ outside of a small ϵ-neighborhood, $\sup_x|u(x)|\leq\epsilon$, in the construction of $\mathcal{M}_0$. Therefore, *all* elements of $\mathcal{M}_0$ are in fact solutions, with possibly mild exponential growth, and to the modified equation $\mathcal{T}u+\mathcal{F}_\epsilon(u)=0$. The set of solutions to this equation is translation invariant and parameterized over $\ker\mathcal{T}$. The action of the shift on this set of solutions can therefore be pulled back to $\ker\mathcal{T}$, where it induces a flow with associated vector field as stated in the following result; see Section 3.6.3 and the diagrams there for more details on this idea, and our application in the following toy model and Section 3.6.4 for a constructive approach, computing Taylor jets.

Corollary 3.6.1. *Under the assumptions 3.6.1 and 3.6.2 of Theorem 3.6.1, any element $u = u_0 + \Psi(u_0)$ of $\mathcal{M}_0$, corresponds to a unique solution of a differential equation*

$$\frac{\mathrm{d}u_0}{\mathrm{d}x} = f(u_0) := \frac{\mathrm{d}}{\mathrm{d}x}\mathcal{Q}\left(u_0(\cdot + x) + \Psi(u_0(\cdot + x))\right)|_{x=0}, \tag{3.6.2}$$

on the linear vector space $u_0 \in \ker \mathcal{T}$. The Taylor jet of f can be computed from properties of $\mathcal{T}$ and $\mathcal{F}$, solving linear equations, only.

Note that the differentiation in (3.6.2) does not refer to differentiation of u_0, which of course is a function of x when viewed as an element of the kernel. We rather view $\ker \mathcal{T}$ as an abstract vector space on which we study the differential equation (3.6.2). Note also that we do not claim that every solution to (3.6.2) is a solution to (3.6.1) — this is true only for small solutions.

We will explain below how to actually compute the Taylor jet of f. Having access to (3.6.2) as a means of describing elements of $\mathcal{M}_0$, the abstract reduction Theorem 3.6.1 becomes very valuable: one simply studies the differential equation (3.6.2), or, to start with, the equation obtained from the leading order Taylor approximation, using traditional dynamical systems methods. Small bounded solutions obtained in this fashion will then correspond to solutions of the original nonlocal problem (3.6.1).

Application: a toy model. We will apply our main result later on but we already want to give a fairly trivial example of how to compute Taylor jets in practice, here. In fact, the procedure of deriving the reduced system (3.6.2) involves algebra that is somewhat different from the more commonly known algebra associated with Taylor jets in phase space and ordinary center manifolds. We consider a scalar nonlocal equation of the form,

$$u + \mathcal{K} * u - u^2 = 0, \tag{3.6.3}$$

where we suppose that $\mathcal{K}$ satisfies Hypothesis 3.6.1 for a given $\eta_0 > 0$ together with the assumptions that

$$\int_{\mathbb{R}} \mathcal{K}(x)\mathrm{d}x = -1, \quad \int_{\mathbb{R}} x\mathcal{K}(x)\mathrm{d}x = -\alpha^{-1} \neq 0, \quad \text{and} \quad d(\mathbf{i}\ell) = 1+\widehat{\mathcal{K}}(\mathbf{i}\ell) \neq 0 \text{ for all } \ell \in \mathbb{R}\backslash\{0\}.$$

As a consequence, $\mathcal{E}_0 = \ker \mathcal{T} = \{1\}$, the constant functions. A natural candidate for the projection onto the kernel is $(\mathcal{Q}u)(x) \equiv u(0) \in \mathcal{E}_0$, clearly defining a bounded projection on $H^1_{-\eta}$ onto $\mathcal{E}_0$ for any $0 < \eta < \eta_0$. Furthermore, the nonlinear operator $\mathcal{F}(u) = -u^2$ is a Nemytskii operator and satisfies Hypothesis 3.6.2. Our main result, Theorem 3.6.1, then implies existence of a center manifold $\mathcal{M}_0$, and any small bounded solutions of (3.6.3) can be written as

$$u = u_0 + \Psi(u_0),$$

where $u_0 := A \cdot 1 \in \mathcal{E}_0$. As the map Ψ is $\mathscr{C}^k$ for any $k \geq 2$, we can look for its Taylor expansion near 0, and using the properties $\Psi(0) = D_u\Psi(0) = 0$, we obtain

$$\Psi(u_0) = A^2\Psi_2 + A^3\Psi_3 + \mathcal{O}(A^4).$$

Inserting this ansatz into the nonlocal equation (3.6.3) and identifying terms of order A^2, we obtain that Ψ_2 should satisfy

$$\mathcal{T}\Psi_2 = 1, \text{ with } \mathcal{Q}(\Psi_2) = 0.$$

Using that $\int x\mathcal{K}(x)\mathrm{d}x \neq 0$, we obtain that $\Psi_2(x) = \alpha x$, for all $x \in \mathbb{R}$. At cubic order, we find that

$$\mathcal{T}\Psi_3 = 2\Psi_2, \text{ with } \mathcal{Q}(\Psi_3) = 0.$$

We look for solution Ψ_3 that can be written as $\Psi_3(x) = \beta_2 x^2 + \beta_1 x$, which leads to the compatibility conditions

$$\begin{aligned} \beta_2 \int_{\mathbb{R}} \mathcal{K}(y)y^2\mathrm{d}y + \frac{\beta_1}{\alpha} &= 0, \\ 2\frac{\beta_2}{\alpha} &= 2\alpha, \end{aligned}$$

such that $\beta_2 = \alpha^2$ and $\beta_1 := -\kappa_2\alpha^3$, where $\kappa_2 := \int_{\mathbb{R}} \mathcal{K}(y)y^2\mathrm{d}y$.

Finally, we construct the reduced vector field as stated in Corollary 3.6.1; see the proof in Section 3.6.3 and diagrams there for details on the abstract concepts. Invariance of the set of bounded solutions by translations constitutes an action of the group $\mathbb{R}$. Using our parameterization of the set of bounded solutions over the kernel, this action can be pulled back to an action of the shift on the kernel. With the Taylor expansion of the representation of our bounded solutions over the kernel,

$$u(x) = A + \alpha x A^2 + (\alpha^2 x^2 - \kappa_2\alpha^3 x)A^3 + \mathcal{O}_x(A^4), \tag{3.6.4}$$

we obtain the Taylor expansion of the action of the shift on the kernel, parameterized by $A \in \mathbb{R}$. We therefore shift $u(x)$ from (3.6.4), and then invert $\mathrm{id} + \Psi$ explicitly as $(\mathrm{id} + \Psi)^{-1} = \mathcal{Q}$, to find

$$\begin{aligned} \varphi_x(A) &= \mathcal{Q}\left[A + \alpha(\cdot + x)A^2 + \left(\alpha^2(\cdot + x)^2 - \kappa_2\alpha^3(\cdot + x)\right)A^3 + \mathcal{O}_{(\cdot + x)}(A^4)\right] \\ &= A + \alpha x A^2 + \left(\alpha^2 x^2 - \kappa_2\alpha^3 x\right)A^3 + \mathcal{O}_x(A^4). \end{aligned}$$

Differentiating this action of the shift, that is, computing the derivative of a flow at time $x = 0$, we obtain the vector field that generates the flow as stated in (3.6.2),

$$\frac{\mathrm{d}\varphi_x}{\mathrm{d}x}|_{x=0} = \alpha A^2 - \kappa_2\alpha^3 A^3 + \mathcal{O}(A^4),$$

thus giving the Taylor expansion of the reduced differential equation up to third order through

$$\frac{\mathrm{d}A}{\mathrm{d}x} = \alpha A^2 - \kappa_2\alpha^3 A^3 + \mathcal{O}(A^4).$$

Note that, absent further parameters, the reduced differential equation, here, does not possess any non-trivial bounded solutions. In other words, the center manifold here yields a uniqueness result for small bounded solutions, in a class of sufficiently smooth functions. Adding parameters, one would find the typical heteroclinic trajectories in a saddle-node bifurcation.

3.6.1 Functional-analytic setup and main assumptions

In this subsection, we introduce function spaces and state our main hypotheses 3.6.1 and 3.6.2.

Function spaces. For $\eta \in \mathbb{R}$, $1 \leq p \leq \infty$, we define the weighted space $L^p_\eta(\mathbb{R}, \mathbb{R}^n)$, or simply L^p_η, when $n = 1$, through

$$L^p_\eta(\mathbb{R}, \mathbb{R}^n) := \left\{ u \in L^p_{\text{loc}}(\mathbb{R}, \mathbb{R}^n) \ : \ \omega_\eta u \in L^p(\mathbb{R}, \mathbb{R}^n) \right\},$$

where ω_η is a $\mathscr{C}^\infty$ function defined as

$$\omega_\eta(x) = \begin{cases} e^{\eta x} & \text{for} \quad x \geq 1, \\ e^{-\eta x} & \text{for} \quad x \leq -1 \end{cases}, \quad \omega_\eta > 0 \text{ on } [-1, 1].$$

We also use the standard Sobolev spaces $W^{k,p}(\mathbb{R}, \mathbb{R}^n)$, or simply $W^{k,p}$ when $n = 1$, for $k \geq 0$ and $1 \leq p \leq \infty$:

$$W^{k,p}(\mathbb{R}, \mathbb{R}^n) := \left\{ u \in L^p(\mathbb{R}, \mathbb{R}^n) \ : \ \partial_x^\alpha u \in L^p(\mathbb{R}, \mathbb{R}^n), \quad 1 \leq \alpha \leq k \right\},$$

with norm

$$\|u\|_{W^{k,p}(\mathbb{R},\mathbb{R}^n)} = \begin{cases} \left(\sum_{\alpha \leq k} \|\partial_x^\alpha u\|^p_{L^p(\mathbb{R},\mathbb{R}^n)} \right)^{\frac{1}{p}}, & 1 \leq p < \infty \\ \max_{\alpha \leq k} \|\partial_x^\alpha u\|_{L^\infty(\mathbb{R},\mathbb{R}^n)}, & p = \infty. \end{cases}$$

We denote by $H^k(\mathbb{R}, \mathbb{R}^n)$ the Sobolev space $W^{k,2}(\mathbb{R}, \mathbb{R}^n)$ and use the weighted spaces $W^{k,p}_\eta(\mathbb{R}, \mathbb{R}^n)$ and $H^k_\eta(\mathbb{R}, \mathbb{R}^n)$ the weighted Sobolev spaces defined through

$$W^{k,p}_\eta(\mathbb{R}, \mathbb{R}^n) := \left\{ u \in L^p_{\text{loc}}(\mathbb{R}, \mathbb{R}^n) \ : \ \omega_\eta \partial_x^\alpha u \in L^p(\mathbb{R}, \mathbb{R}^n), \quad 0 \leq \alpha \leq k \right\},$$

with norm

$$\|u\|_{W^{k,p}_\eta(\mathbb{R},\mathbb{R}^n)} = \begin{cases} \left(\sum_{\alpha \leq k} \|\omega_\eta \partial_x^\alpha u\|^p_{L^p(\mathbb{R},\mathbb{R}^n)} \right)^{\frac{1}{p}}, & 1 \leq p < \infty \\ \max_{\alpha \leq k} \|\omega_\eta \partial_x^\alpha u\|_{L^\infty(\mathbb{R},\mathbb{R}^n)}, & p = \infty, \end{cases}$$

and $H^k_\eta(\mathbb{R}, \mathbb{R}^n) := W^{k,2}_\eta(\mathbb{R}, \mathbb{R}^n)$.

Assumptions on the linear part. We require that the convolution kernel is exponentially localized and smooth in the following sense.

Hypothesis 3.6.1. *We assume that there exists $\eta_0 > 0$ such that $\mathcal{K}_{i,j} \in W^{1,1}_{\eta_0}(\mathbb{R})$ for all $1 \leq i, j \leq n$.*

We define the complex Fourier transform $\widehat{\mathcal{K}}(\nu)$ of $\mathcal{K}$ as

$$\widehat{\mathcal{K}}(\nu) = \int_{\mathbb{R}} \mathcal{K}(x) e^{-\nu x} \mathrm{d}x, \tag{3.6.5}$$

for all $\nu \in \mathbb{C}$ where the above integral is well-defined. Note that because each component of the matrix kernel $\mathcal{K}$ belongs to $L^1_{\eta_0}$, the Fourier transform $\widehat{\mathcal{K}}(\nu)$ is analytic in the strip $\mathscr{S}_{\eta_0} := \{\nu \in \mathbb{C} \mid |\Re(\nu)| < \eta_0\}$. Pure exponential solutions of the linearized equation can be detected as roots of the characteristic equation

$$d(\nu) := \det\left(I_n + \widehat{\mathcal{K}}(\nu)\right) = 0, \tag{3.6.6}$$

where the left-hand side $d(\nu)$ is an analytic function in the strip $\mathscr{S}_{\eta_0}$ and has isolated roots on the imaginary axis, when counted with multiplicity. Moreover, since $\mathcal{K}' \in L^1_{\eta_0}$ (component-wise), we have $|\widehat{\mathcal{K}}(\ell + \eta)| \underset{\ell\to\pm\infty}{\longrightarrow} 0$, for $|\eta| < \eta_0$, such that the number of roots of d on the imaginary axis, counted with multiplicity, is finite. Throughout, we will assume that the number of roots is not zero, in which case our results would be trivial.

We consider $\mathcal{T}$ as a bounded operator on $H^1_{-\eta}(\mathbb{R},\mathbb{R}^n)$, $0 < \eta < \eta_0$, slightly abusing notation and not making the dependence of $\mathcal{T}$ on η explicit. With the natural bounded inclusion $\iota^{\eta,\eta'}$, $\eta < \eta'$, one finds $\mathcal{T}\iota^{\eta,\eta'} = \iota^{\eta,\eta'}\mathcal{T}$. Now, by finiteness of the number of roots of d, we can choose $\eta_1 > 0$, small, such that $d(\nu)$ does not vanish in $0 < |\Im\nu| \le \eta_1$. We will then find that the kernel $\mathcal{E}_0$ of $\mathcal{T}$ is independent of η for $0 < \eta < \eta_1$ in the sense that $\iota^{\eta,\eta'}$ provides isomorphisms between kernels for η and η', $0 < \eta < \eta' < \eta_1$. The dimension of $\mathcal{E}_0$ is given by the sum of multiplicities of roots $\nu \in \mathbb{R}$ of $d(\nu)$, with a basis of the form $p(x)\mathrm{e}^{\nu x}$, p a vector-valued polynomial of degree at most $m-1$ when ν is a root of d of order m. We also need a bounded projection

$$\mathcal{Q} : H^1_{-\eta}(\mathbb{R},\mathbb{R}^n) \to H^1_{-\eta}(\mathbb{R},\mathbb{R}^n), \qquad \mathcal{Q}^2 = \mathcal{Q}, \qquad \mathrm{rg}\,(\mathcal{Q}) = \mathcal{E}_0 = \ker\mathcal{T}, \tag{3.6.7}$$

with a continuous extension to $L^2_{-\eta}(\mathbb{R},\mathbb{R}^n)$. Again, we require $\mathcal{Q}\iota^{\eta,\eta'} = \iota^{\eta,\eta'}\mathcal{Q}$, a possible choice being the $L^2_{\eta_1}(\mathbb{R},\mathbb{R}^n)$-orthonormal projection.

Assumptions on the nonlinear part. A common approach to the construction of center manifolds is to modify the nonlinearity outside of a small neighborhood of the origin. We therefore first define a pointwise, smooth cut-off function $\bar{\chi} : \mathbb{R}^n \to \mathbb{R}$, with

$$\bar{\chi}(u) = \begin{cases} 1 & \text{for} \quad \|u\| \le 1 \\ 0 & \text{for} \quad \|u\| \ge 2 \end{cases}, \qquad \bar{\chi}(u) \in [0,1],$$

and then a cut-off operator χ_ϵ, mapping measurable functions $u : \mathbb{R} \to \mathbb{R}^n$ into $L^\infty(\mathbb{R},\mathbb{R}^n)$,

$$\chi_\epsilon(u)(x) = \bar{\chi}(u(x)/\epsilon) \cdot u(x).$$

Lastly, formally define the family of translation operators τ_ξ, $\xi \in \mathbb{R}$,

$$(\tau_\xi \cdot u)(x) := u(x - \xi),$$

the canonical representation of the group $\mathbb{R}$ on functions over $\mathbb{R}$. Slightly abusing notation, we will use the same symbol τ_ξ for the action on various function spaces. Note that τ_ξ will be bounded for ξ fixed on all spaces introduced above. We define the modified nonlinearities

$$\mathcal{F}^\varepsilon := \mathcal{F} \circ \chi_\epsilon. \tag{3.6.8}$$

Hypothesis 3.6.2. *We assume that there exists $k \geq 2$ and $\eta_0 > 0$ such that for all $\epsilon > 0$, sufficiently small, the following properties hold.*

1. $\mathcal{F} \in \mathscr{C}^k(\mathcal{V}, W^{1,\infty}(\mathbb{R}, \mathbb{R}^n))$, *for some small neighborhood* $0 \in \mathcal{V} \subset W^{1,\infty}(\mathbb{R}, \mathbb{R}^n)$, *and* $\mathcal{F}(0) = 0$, $D_u\mathcal{F}(0) = 0$;
2. $\mathcal{F}$ *commutes with translations,* $\mathcal{F} \circ \tau_\xi = \tau_\xi \circ \mathcal{F}$ *for all* $\xi \in \mathbb{R}$;
3. $\mathcal{F}^\epsilon : H^1_{-\zeta}(\mathbb{R}, \mathbb{R}^n) \longrightarrow H^1_{-\eta}(\mathbb{R}, \mathbb{R}^n)$ *is* $\mathscr{C}^k$ *for all nonnegative pairs* (ζ, η) *such that* $0 < k\zeta < \eta < \eta_0$, $D^j\mathcal{F}^\epsilon(u) : (H^1_{-\zeta}(\mathbb{R}, \mathbb{R}^n))^j \longrightarrow H^1_{-\eta}(\mathbb{R}, \mathbb{R}^n)$ *is bounded for* $0 < j\zeta \leq \eta < \eta_0$, $0 \leq j \leq k$ *and Lipschitz in* u *for* $1 \leq j \leq k-1$.

Note that $\mathcal{F}^\epsilon$ commutes with τ_ξ since $\mathcal{F}$ and χ_ϵ do. The first condition is the common condition, guaranteeing that $\mathcal{T}$ is actually the linearization at an equilibrium $u \equiv 0$, that is, at a solution invariant under translations τ_ξ. The second condition puts us in the scenario of an autonomous dynamical system. The last condition on the modified nonlinearity is a technical condition, known from the proofs of smoothness of center manifolds in ODEs [192], that will imply smoothness of our center-manifold.

3.6.2 Main results — precise statements

We are now in a position to state a precise version of Theorem 3.6.1 and Corollary 3.6.1. We are interested in system (3.6.1) and its modified variant,

$$\mathcal{T}u + \mathcal{F}(u) = 0, \tag{3.6.9}$$

$$\mathcal{T}u + \mathcal{F}^\epsilon(u) = 0. \tag{3.6.10}$$

Theorem 3.6.2 (Center manifolds and reduced vector fields)**.** *Consider equations* (3.6.9) *and* (3.6.10) *with Hypothesis 3.6.1 on the linear convolution operator* $\mathcal{K}$ *and Hypothesis 3.6.2 on the nonlinearity* $\mathcal{F}$. *Recall the definitions of the kernel* $\mathcal{E}_0$ *and the projection* $\mathcal{Q}$ *on* $H^1_{-\eta}(\mathbb{R}, \mathbb{R}^n)$, (3.6.7). *Then there exists a cut-off radius* ϵ, *a weight* $\delta > 0$, *and a map*

$$\Psi : \ker \mathcal{T} \subset H^1_{-\delta}(\mathbb{R}, \mathbb{R}^n) \to \ker \mathcal{Q} \subset H^1_{-\delta}(\mathbb{R}, \mathbb{R}^n),$$

with graph

$$\mathcal{M}_0 := \{u_0 + \Psi(u_0) \mid u_0 \in \ker \mathcal{T}\} \subset H^1_{-\delta}(\mathbb{R}, \mathbb{R}^n),$$

such that the following properties hold:

1. (smoothness) $\Psi \in \mathscr{C}^k$, *with* k *specified in Hypothesis 3.6.2;*
2. (tangency) $\Psi(0) = 0$, $D_u\Psi(0) = 0$;
3. (global reduction) $\mathcal{M}_0$ *consists precisely of the solutions* $u \in H^1_{-\delta}(\mathbb{R}, \mathbb{R}^n)$ *of the modified equation* (3.6.10);
4. (local reduction) *any solution* $u \in H^1_{-\delta}(\mathbb{R}, \mathbb{R}^n)$ *of the original equation* (3.6.9) *with* $\sup_{x\in\mathbb{R}} |u(x)| \leq \epsilon$ *is contained in* $\mathcal{M}_0$;

5. (translation invariance) *the shift* $\tau_\xi, \xi \in \mathbb{R}$ *acts on* $\mathcal{M}_0$ *and induces a flow* $\Phi_\xi : \mathcal{E}_0 \to \mathcal{E}_0$ *through* $\Phi_\xi = \mathcal{Q} \circ \tau_\xi \circ \Psi$*;*

6. (reduced vector field) *the reduced flow* $\Phi_\xi(u_0)$ *is of class* $\mathscr{C}^k$ *in* u_0, ξ *and generated by a reduced vector field* f *of class* $\mathscr{C}^{k-1}$ *on the finite-dimensional vector space* $\mathcal{E}_0$*.*

In particular, small solutions on $t \in \mathbb{R}$ *to* $v' = f(v)$ *on* $\mathcal{E}_0$ *are in one-to-one correspondence with small bounded solutions of* (3.6.9)*.*

Higher regularity. Completely analogous formulations of our main result are possible in spaces with higher regularity, $H^m_\eta(\mathbb{R}, \mathbb{R}^n)$, changing simply the assumptions on the nonlinearity, which will typically require higher regularity of pointwise nonlinearities. Moreover, one then concludes that small bounded solutions are in fact smooth in x, which one can, however, also conclude after using bootstrap arguments in the equation.

Parameters. In the context of bifurcation theory, one usually deals with parameter dependent problems. One then hopes to find center manifolds and reduced equations that depend smoothly on parameters. We therefore consider

$$u + \mathcal{K} * u + \mathcal{F}(u, \mu) = 0, \tag{3.6.11}$$

for $u : \mathbb{R} \to \mathbb{R}^n$, $n \geq 1$, $\mu \in \mathbb{R}^d$, $d \geq 1$, and the nonlinear operator $\mathcal{F}$ is defined in a neighborhood of $(u, \mu) = (0, 0)$. Again, we can define $\mathcal{F}^\epsilon = \mathcal{F} \circ (\chi_\epsilon, \mathrm{id})$, cutting off in the $u-$variable, only, leading to

$$u + \mathcal{K} * u + \mathcal{F}^\epsilon(u, \mu) = 0, \tag{3.6.12}$$

We then require a μ-dependent version of Hypothesis 3.6.2.

Hypothesis 3.6.3. *We assume that there exists* $k \geq 2$ *and* $\eta_0 > 0$ *such that for all* $\epsilon > 0$*, sufficiently small, the following properties hold.*

1. $\mathcal{F} \in \mathscr{C}^k(\mathcal{V}_u \times \mathcal{V}_\mu, W^{1,\infty}(\mathbb{R}, \mathbb{R}^n))$*, for some small neighborhoods* $0 \in \mathcal{V}_u \subset W^{1,\infty}(\mathbb{R}, \mathbb{R}^n)$*,* $0 \in \mathcal{V}_\mu \subset \mathbb{R}^d$*, and* $\mathcal{F}(0, 0) = 0$*,* $D_u\mathcal{F}(0, 0) = 0$*;*

2. $\mathcal{F}$ *commutes with translations for all* μ*,* $\mathcal{F} \circ \tau_\xi = \tau_\xi \circ \mathcal{F}$ *for all* $\xi \in \mathbb{R}$*;*

3. $\mathcal{F}^\epsilon : H^1_{-\zeta}(\mathbb{R}, \mathbb{R}^n) \times \mathcal{V}_\mu \longrightarrow H^1_{-\eta}(\mathbb{R}, \mathbb{R}^n)$ *is* $\mathscr{C}^k$ *for all nonnegative pairs* (ζ, η) *such that* $0 < k\zeta < \eta < \eta_0$*,* $D^j\mathcal{F}^\epsilon(u, \mu) : (H^1_{-\zeta}(\mathbb{R}, \mathbb{R}^n))^j \longrightarrow H^1_{-\eta}(\mathbb{R}, \mathbb{R}^n)$ *is bounded for* $0 < j\zeta \leq \eta < \eta_0$*,* $0 \leq j \leq k$ *and Lipschitz in* u *for* $1 \leq j \leq k - 1$*, uniformly in* $\mu \in \mathcal{V}_\mu$*.*

The analogue of the center manifold Theorem 3.6.1 for the parameter-dependent nonlocal equation (3.6.11) is the following result.

Theorem 3.6.3 (Parameter-Dependent Center Manifold)**.** *Consider equations* (3.6.11) *and* (3.6.12) *with Hypothesis 3.6.1 on the linear convolution operator* $\mathcal{K}$ *and with Hypothesis 3.6.3 on the nonlinearity* $\mathcal{F}$*. Recall the definition of kernel* $\mathcal{E}_0$ *and projection* $\mathcal{Q}$ *on* $H^1_{-\eta}(\mathbb{R}, \mathbb{R}^n)$*,* (3.6.7)*.*

Then, possibly shrinking the neighborhood $\mathcal{V}_\mu$, there exist a cut-off radius ϵ, a weight $\delta > 0$, and a map

$$\Psi : \ker \mathcal{T} \times \mathcal{V}_\mu \subset H^1_{-\delta}(\mathbb{R}, \mathbb{R}^n) \times \mathbb{R}^d \to \ker \mathcal{Q} \subset H^1_{-\delta}(\mathbb{R}, \mathbb{R}^n),$$

with graph

$$\mathcal{M}_0 := \{(u_0 + \Psi(u_0, \mu), \mu) \mid u_0 \in \ker \mathcal{T},\ \mu \in \mathcal{V}_\mu\} \subset H^1_{-\delta}(\mathbb{R}, \mathbb{R}^n),$$

such that the following properties hold:

1. (smoothness) $\Psi \in \mathscr{C}^k$, *with k specified in Hypothesis 3.6.3;*
2. (tangency) $\Psi(0, 0) = 0$, $D_u\Psi(0, 0) = 0$*;*
3. (global reduction) $\mathcal{M}_0$ *consists precisely of the pairs (u, μ), such that $u \in H^1_{-\delta}(\mathbb{R}, \mathbb{R}^n)$ is a solution of the modified equation (3.6.12) for this value of μ;*
4. (local reduction) *any pair (u, μ) such that u is a solution $u \in H^1_{-\delta}(\mathbb{R}, \mathbb{R}^n)$ of the original equation (3.6.11) with $\sup_{x\in\mathbb{R}} |u(x)| \le \epsilon$ for this value of μ is contained in $\mathcal{M}_0$;*
5. (translation invariance) *the shift $\tau_\xi, \xi \in \mathbb{R}$ acts on the u-component of $\mathcal{M}_0$ and induces a μ-dependent flow $\Phi_\xi : \mathcal{E}_0 \to \mathcal{E}_0$ through $\Phi_\xi = \mathcal{Q} \circ \tau_\xi \circ \Psi$;*
6. (reduced vector field) *the reduced flow $\Phi_\xi(u_0; \mu)$ is of class $\mathscr{C}^k$ in u_0, ξ, μ and generated by a reduced parameter-dependent vector field f of class $\mathscr{C}^{k-1}$ on the finite-dimensional vector space $\mathcal{E}_0$.*

In particular, small solutions on $t \in \mathbb{R}$ to $v' = f(v; \mu)$ on $\mathcal{E}_0$ are in one-to-one correspondence with small bounded solutions of (3.6.11).

Symmetries and reversibility We can also cover the cases of equations possessing symmetries in addition to translation invariance. The aim is to show that such symmetries are inherited by the reduced equation. Generally speaking, we have an action of the direct product $G = \mathbf{O}(n) \times (\mathbb{R} \times \mathbb{Z}_2)$ on spaces of functions over the real line with values in $\mathbb{R}^n$, where $\mathbf{O}(n)$ is the group of orthogonal $n \times n$-matrices, and the action is defined through

$$((\rho, \tau_\xi, \kappa) \cdot u)(x) = \rho \cdot u(\kappa(x - \xi)).$$

Here, $\kappa x = -x$ when κ is the nontrivial element of $\mathbb{Z}_2$. Note that χ_ϵ commutes with the action of the full group $\mathbf{O}(n) \times (\mathbb{R} \times \mathbb{Z}_2)$.

There is a subgroup $\Gamma \subset G$ that contains the pure translations, $\mathrm{id} \times \mathbb{R} \times \mathrm{id} \subset \Gamma$, such that (3.6.1) is invariant under Γ, that is,

$$\gamma \circ \mathcal{T} = \mathcal{T}\gamma, \qquad \gamma \circ \mathcal{F} = \mathcal{F}\gamma, \qquad \text{for all } \gamma \in \Gamma.$$

We say the equation is reversible if $\Gamma \not\subset \mathbf{O}(n) \times \mathbb{R} \times \mathrm{id}$, that is, if the group of symmetries contains a reflection. We call $\Gamma_\mathrm{e} := \Gamma \cap (\mathbf{O}(n) \times \mathbb{R} \times \mathrm{id})$ the equivariant part and $\Gamma_\mathrm{r} := \Gamma \setminus \Gamma_\mathrm{e}$ the reversible part of the symmetries Γ.

We remark that the equivariance properties of $\mathcal{Q}$ are concerned with symmetries in $\mathbf{O}(n) \times (\{0\} \times \mathbb{Z}_2)$, since the action of the shift on the kernel is induced through the projection itself, hence automatically respects the symmetry. We obtain the following result.

Theorem 3.6.4 (Equivariant Center Manifold)**.** *Assume that the above Hypotheses 3.6.1, 3.6.2 and 3.6.2 are satisfied. Then reduced center manifold $\mathcal{M}_0 = \mathrm{graph}\,(\Psi)$ and vector field f from Theorem 3.6.2 respect the symmetry, that is,*

1. *$\mathcal{E}_0$ is invariant under Γ and $\mathcal{Q}$ can be chosen to commute with all $\gamma \in \Gamma$;*
2. *Ψ commutes with the action of Γ, $\mathcal{M}_0$ is invariant under the action of Γ;*
3. *f commutes with the equivariant part, $f \circ \gamma_1 = \gamma_1 \circ f$ for $\gamma = (\gamma_1, \tau_\xi, \mathrm{id}) \in \Gamma_\mathrm{e}$, and anti-commutes with the reversible part of the symmetries, $f \circ \gamma_1 = -\gamma_1 \circ f$ for $\gamma = (\gamma_1, \tau_\xi, \kappa) \in \Gamma_\mathrm{r}$.*

Analogous results hold for the parameter-dependent equation (3.6.11).

3.6.3 Sketch of the proof of the main results

In this subsection, we present the main ingredients regarding the proof of Theorem 3.6.1 and Corollary 3.6.1.

Existence of a center manifold. The key ideas of the proof of Theorem 3.6.1 are as follows. First, we consider the linearization

$$\mathcal{T} : H^1_{-\eta}(\mathbb{R}, \mathbb{R}^n) \longrightarrow H^1_{-\eta}(\mathbb{R}, \mathbb{R}^n), \qquad \mathcal{T}u = u + \mathcal{K} * u, \qquad 0 < \eta \ll 1, \tag{3.6.13}$$

with associated characteristic equation $d(\nu) := \det(\widehat{\mathcal{T}}(\nu))$. Using the Fredholm properties developed in Section 3.4, one can show that the operator $\mathcal{T}$ defined in (3.6.13) is Fredholm of index M and onto, where M is the sum of the multiplicities of roots of $d(\nu)$ on $\nu \in \mathbf{i}\mathbb{R}$. We then augment equation (3.6.1) with the "initial condition", $\mathcal{Q}(u) = u_0$, for a given parameter $u_0 \in \mathcal{E}_0$, which leads us to consider the "bordered" operator

$$\begin{array}{rrcl} \widetilde{\mathcal{T}} : & H^1_{-\eta}(\mathbb{R}, \mathbb{R}^n) & \longrightarrow & H^1_{-\eta}(\mathbb{R}, \mathbb{R}^n) \times \mathcal{E}_0 \\ & u & \longmapsto & (\mathcal{T}(u), \mathcal{Q}(u))\,. \end{array} \tag{3.6.14}$$

Since we are adding finitely many dimensions to the range, Fredholm bordering implies that $\widetilde{\mathcal{T}}$ is Fredholm, of index 0. As a consequence, for any $0 < \eta < \eta_0$, $\widetilde{\mathcal{T}}$ defined in (3.6.14) is invertible with bounded inverse,

$$\|\widetilde{\mathcal{T}}^{-1}\|_{H^1_{-\eta}(\mathbb{R},\mathbb{R}^n) \to H^1_{-\eta}(\mathbb{R},\mathbb{R}^n) \times \mathcal{E}_0} \leq C(\eta), \tag{3.6.15}$$

with $C(\eta) < \infty$ continuous for $0 < \eta < \eta_0$.

We now rewrite equations (3.6.1) together with (3.6.14), using the modified nonlinearity $\mathcal{F}^\epsilon$ instead of $\mathcal{F}$, into a more compact form

$$\widetilde{\mathcal{T}}(u) + \widetilde{\mathcal{F}^\epsilon}(u; u_0) = 0, \tag{3.6.16}$$

where

$$\widetilde{\mathcal{F}^\epsilon}(u;u_0) = (\mathcal{F}^\epsilon(u), -u_0).$$

Applying $\widetilde{\mathcal{T}}^{-1}$ to equation (3.6.16), we obtain an equation of the form

$$u = -\widetilde{\mathcal{T}}^{-1}\left(\widetilde{\mathcal{F}}^\epsilon(u;u_0)\right) := \mathcal{S}^\epsilon(u;u_0), \tag{3.6.17}$$

for any $u_0 \in \mathcal{E}_0$. We view (3.6.17) as a fixed point equation with parameter u_0 and establish that that $\mathcal{S}^\epsilon(\cdot;u_0)$ is a contraction map on $H^1_{-\eta}(\mathbb{R},\mathbb{R}^n)$. From the definition of $\mathcal{F}^\epsilon$ and the fact that $\mathcal{F}^\epsilon(0) = D\mathcal{F}^\epsilon(0) = 0$ with $\mathcal{F}^\epsilon$ of class $\mathscr{C}^k$ for $k \geq 2$ on $W^{1,\infty}(\mathbb{R},\mathbb{R}^n)$, one obtains the following estimates as $\epsilon \to 0$,

$$\delta_0(\epsilon) := \sup_{u \in H^1_{-\eta}(\mathbb{R},\mathbb{R}^n)} \|\mathcal{F}^\epsilon(u)\|_{H^1_{-\eta}(\mathbb{R},\mathbb{R}^n)} = \mathcal{O}(\epsilon^2), \tag{3.6.18a}$$

$$\delta_1(\epsilon) := \mathrm{Lip}_{H^1_{-\eta}(\mathbb{R},\mathbb{R}^n)}(\mathcal{F}^\epsilon) = \mathcal{O}(\epsilon). \tag{3.6.18b}$$

Indeed, by definition, we have $\mathcal{F}^\epsilon(u)(x) = \mathcal{F}(u)(x)$ whenever $\|u(x)\| \leq \epsilon$ and $\mathcal{F}^\epsilon(u)(x) = 0$ whenever $\|u(x)\| \geq 2\epsilon$. Using the fact that H^1 functions are also continuous functions, we obtain the desired estimates by further noticing that $\mathcal{F}^\epsilon(u)$ is superlinear near $u = 0$. In turn, these estimates imply

$$\|\mathcal{S}^\epsilon(u;u_0)\|_{H^1_{-\eta}(\mathbb{R},\mathbb{R}^n)} \leq C(\eta)\left(\delta_0(\epsilon) + \|u_0\|_{H^1_{-\eta}(\mathbb{R},\mathbb{R}^n)}\right),$$
$$\|\mathcal{S}^\epsilon(u;u_0) - \mathcal{S}^\epsilon(v;u_0)\|_{H^1_{-\eta}(\mathbb{R},\mathbb{R}^n)} \leq C(\eta)\delta_1(\epsilon)\|u-v\|_{H^1_{-\eta}(\mathbb{R},\mathbb{R}^n)},$$

for all $u, v \in H^1_{-\eta}(\mathbb{R},\mathbb{R}^n)$ and $u_0 \in \mathcal{E}_0$. Let $\bar{\eta} \in (0,\eta_0)$ and $\tilde{\eta} \in (0,\bar{\eta}/k)$, then, for sufficiently small ϵ, we have

$$C(\eta)\delta_1(\epsilon) < 1, \quad \forall \eta \in [\tilde{\eta},\bar{\eta}].$$

As a consequence, there exists a unique fixed point $u = \Phi(u_0) \in H^1_{-\eta}(\mathbb{R},\mathbb{R}^n)$. From Lipshitz continuity of the fixed point iteration, we conclude that Φ is a Lipschitz map, and $\Phi(0) = 0$ by uniqueness of the fixed point. For each $\eta \in [\tilde{\eta},\bar{\eta}]$, this defines a continuous map $\Psi : \mathcal{E}_0 \to \ker \mathcal{Q} \subset H^1_{-\eta}(\mathbb{R},\mathbb{R}^n)$ so that

$$u = \Phi(u_0) := u_0 + \Psi(u_0).$$

The most delicate part of the remaining of the proof is to show that for each p with $1 \leq p \leq k$ and for each $\eta \in (p\tilde{\eta},\bar{\eta}]$ that $\Psi : \mathcal{E}_0 \to H^1_{-\eta}(\mathbb{R},\mathbb{R}^n)$ is of class $\mathscr{C}^p$. This actually follows from our careful assumptions on the kernel matrix $\mathcal{K}$ and the nonlinear operator $\mathcal{F}$. The main ingredient is an application of the contraction mapping theorem on scales of Banach spaces as presented in [192] (see [88, Appendix A] for the details).

Smoothness of the reduced flow and reduced vector fields. In this paragraph, we establish that the flow on the center manifold is smooth such that we can obtain the reduced ordinary differential equation (3.6.2) simply through differentiating the flow at time zero. Consider the action of the shift operator on functions, defined through

$$\begin{array}{llll} \mathbb{R} \times H^1_{-\eta}(\mathbb{R},\mathbb{R}^n) & \longrightarrow & H^1_{-\eta}(\mathbb{R},\mathbb{R}^n) \\ (x,u) & \longmapsto & \phi(x,u) := u(\cdot + x), \end{array} \tag{3.6.19}$$

for any $0 < \eta < \eta_0$. We briefly write $\phi_x := \phi(x, \cdot) : H^1_{-\eta}(\mathbb{R}, \mathbb{R}^n) \longrightarrow L^2_{-\eta}(\mathbb{R}, \mathbb{R}^n)$. Clearly, ϕ_x is bounded linear. Therefore, and by translation invariance of the original equation, ϕ_x maps bounded solutions to bounded solutions. The following commutative diagram shows how this action of the shift induces a flow on the kernel $\mathcal{E}_0$,

$$
\begin{tikzcd}
\mathcal{E}_0 \arrow[r, "\mathrm{id}+\Psi"] \arrow[d, "\varphi_x"'] \arrow[dr, phantom, "\circlearrowright"] & H^1_{-\eta}(\mathbb{R}, \mathbb{R}^n) \arrow[d, "\phi_x"] \\
\mathcal{E}_0 \arrow[r, shift left, "\mathrm{id}+\Psi"] & H^1_{-\eta}(\mathbb{R}, \mathbb{R}^n) \arrow[l, shift left, "\mathcal{Q}"]
\end{tikzcd}
\quad
\begin{tikzcd}
\mathcal{E}_0 \arrow[r, "\mathrm{id}+\Psi"] \arrow[d, "\varphi_x"'] \arrow[dr, phantom, "\circlearrowright"] & H^1_{-\eta}(\mathbb{R}, \mathbb{R}^n) \arrow[d, "\iota \circ \phi_x"] \\
\mathcal{E}_0 \arrow[r, shift left, "\iota\circ(\mathrm{id}+\Psi)"] & L^2_{-\eta}(\mathbb{R}, \mathbb{R}^n) \arrow[l, shift left, "\widetilde{\mathcal{Q}}"]
\end{tikzcd}
\quad
\begin{tikzcd}
\mathcal{E}_0 \arrow[r, "\iota\circ(\mathrm{id}+\Psi)"] \arrow[d, "\varphi_x"'] \arrow[dr, phantom, "\circlearrowright"] & L^2_{-\eta}(\mathbb{R}, \mathbb{R}^n) \arrow[d, "\iota \circ \phi_x \circ \iota^{-1}"] \\
\mathcal{E}_0 \arrow[r, shift left, "\iota\circ(\mathrm{id}+\Psi)"] & L^2_{-\eta}(\mathbb{R}, \mathbb{R}^n) \arrow[l, shift left, "\widetilde{\mathcal{Q}}"]
\end{tikzcd}
$$

The left diagram, $\mathrm{id} + \Psi$ denotes the parameterization of bounded solutions over the kernel. On the right, ϕ_x denotes the shift which is pulled back to the kernel via the projection $\mathcal{Q}$, the inverse of $\mathrm{id} + \Psi$. The right diagram views the bounded solutions as elements of $L^2_{-\eta}(\mathbb{R}, \mathbb{R}^n)$, by composing the parameterization $\mathrm{id} + \Psi$ with the embedding $\iota : H^1_{-\eta}(\mathbb{R}, \mathbb{R}^n) \to L^2_{-\eta}(\mathbb{R}, \mathbb{R}^n)$. The inverse of the parameterization is the extension of the projection $\mathcal{Q}$ to $L^2_{-\eta}(\mathbb{R}, \mathbb{R}^n)$. The induced flow on the kernel $\mathcal{E}_0$ is naturally the same as in the left diagram. In the center diagram, we view the shift as a map from $H^1_{-\eta}(\mathbb{R}, \mathbb{R}^n)$ into $L^2_{-\eta}(\mathbb{R}, \mathbb{R}^n)$. Clearly, $\iota \circ \Phi_x$ is continuously differentiable in x, with derivative given by the bounded linear map $\frac{\mathrm{d}y}{\mathrm{d}x}$. Since $\widetilde{\mathcal{Q}}$ is a bounded projection on $L^2_{-\eta}(\mathbb{R}, \mathbb{R}^n)$, we find that

$$\varphi_x := \widetilde{Q}\phi_x \circ (\mathrm{id} + \Psi),$$

is continuously differentiable in x. From Theorem 3.6.1 we know that Ψ is a $\mathscr{C}^k$ map from $\mathcal{E}_0$ to $H^1_{-\eta}(\mathbb{R}, \mathbb{R}^n)$. Therefore, the map $x \mapsto \varphi_x$ inherits the regularity properties of ϕ, from which we deduce that $\frac{\mathrm{d}\varphi_x}{\mathrm{d}x}|_{x=0}$ is a $\mathscr{C}^k$ vector field on $\mathcal{E}_0$,

$$\frac{\mathrm{d}\varphi_x}{\mathrm{d}x}|_{x=0} =: f(u_0). \tag{3.6.20}$$

Conversely, solutions to $\frac{\mathrm{d}u}{\mathrm{d}x} = f(u)$, $u(0) = u_0$ yield trajectories $\varphi_x(u_0)$ and solutions to the nonlocal equation $(\mathrm{id} + \Psi)(\varphi_x(u_0))$.

3.6.4 Slowly varying traveling waves in neural field equations

We now present a somehow nontrivial example that is concerned with slowly varying traveling waves in a system of n coupled neural field equations,

$$\partial_t \mathbf{u}(t, x) = -D\mathbf{u}(t, x) + \int_{\mathbb{R}} \mathscr{K}(x - y) F(\mathbf{u}(t, y), \mu) \mathrm{d}y, \quad (t, x) \in (0, \infty) \times \mathbb{R}, \tag{3.6.21}$$

for $\mathbf{u} : \mathbb{R} \to \mathbb{R}^n$, $n \geq 1$, and $\mu \geq 0$, where $D = \mathrm{diag}(d_j)$ is a diagonal matrix with positives entries $d_j > 0$ for all $j = 1 \cdots n$. Throughout the sequel, we will assume that $\mathscr{K}$ is a Gaussian matrix kernel in the sense that for all $1 \leq i, j \leq n$, there exists $a_{i,j} > 0$, such that $\mathscr{K}_{i,j}(x) = \exp(-a_{i,j} x^2)$

for all $x \in \mathbb{R}$, and thus $\mathscr{K}$ satisfies Hypothesis 3.6.1 for all $\eta_0 > 0$. We also suppose that the nonlinear operator $\mathbf{u} \mapsto \mathscr{K} * F(\mathbf{u}, \mu)$ verifies Hypothesis 3.6.3 and that $\mathbf{u} \mapsto F(\mathbf{u}, \mu)$ is odd. Although this last assumption on the oddness of the nonlinearity is not required for the analysis and could be removed, it simplifies the subsequent computations of the reduced vector field on the center manifold.

Spatially homogeneous states of (3.6.21) are solutions of the kinetic equation on $\mathbb{R}^n$

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = -D\mathbf{u} + \mathscr{K}_0 F(\mathbf{u}, \mu), \tag{3.6.22}$$

where the matrix $\mathscr{K}_0$ is defined through $\mathscr{K}_0 := \int_{\mathbb{R}} \mathscr{K}(x)\mathrm{d}x$. In a neighborhood of $(\mathbf{u}, \mu) = (0, 0)$, we assume that the dynamics of (3.6.22) can be reduced to a one-dimensional center manifold with a vector field

$$\frac{\mathrm{d}z}{\mathrm{d}t} = g(z, \mu), \quad z \in \mathbb{R}.$$

We suppose that the resulting bifurcation is a supercritical pitchfork bifurcation.

Hypothesis 3.6.4 (Supercritical pitchfork bifurcation)**.** *The reduced vector field on the one-dimensional center manifold is odd in z for all μ close to zero and*

$$g(z, \mu) = z\left(\alpha\mu - \beta z^2\right) + \mathcal{O}\left(|z|\left(|\mu| + z^2\right)^2\right), \text{ as } (z, \mu) \to (0, 0)$$

with $\alpha > 0$ and $\beta > 0$.

Traveling wave solutions of (3.6.21) are stationary solutions of the following system of equations

$$\partial_t \mathbf{u} = c\partial_\xi \mathbf{u} - D\mathbf{u} + \mathscr{K} * F(\mathbf{u}, \mu), \tag{3.6.23}$$

where $\xi = x - ct$ for some constant $c \in \mathbb{R}$. Steady states of (3.6.23) are thus solutions of the following nonlocal system

$$0 = \mathbf{u} + \mathscr{G}_c * F(\mathbf{u}, \mu), \tag{3.6.24}$$

where we set $\mathscr{G}_c = \left(c\frac{\mathrm{d}}{\mathrm{d}\xi} I_n - D\right)^{-1} \mathscr{K}$. It is important to note that $c \mapsto \mathscr{G}_c$ is a smooth operator from $W^{1,\infty}(\mathbb{R}, \mathbb{R}^n)$ to itself because of the Gaussian nature of $\mathscr{K}$. From now on, we will assume that there is a dependence between c and μ by imposing that $c = \epsilon c_*$, $\mu = \epsilon^2$ for $\epsilon \geq 0$ and some $c_* \in \mathbb{R}$ independent of ϵ. Such a scaling is motivated by an analogous study [142] for systems of reaction-diffusion equations. It is also useful to note that in the limit $c \to 0$, we have $\mathscr{G}_0 = -D^{-1}\mathscr{K}$.

The linearization of (3.6.24) about the trivial state $\mathbf{u} = 0$ leads to the linear operator

$$\mathcal{T}_\epsilon \mathbf{u} := \mathbf{u} + \mathscr{G}_{\epsilon c_*} * \mathrm{D}_\mathbf{u} F\left(0, \epsilon^2\right).$$

We define the linear characteristic equation $d(\nu, \epsilon)$ as

$$d(\nu, \epsilon) := \det\left(\widehat{\mathcal{T}_\epsilon}(\nu)\right) = \det\left(I_n + \widehat{\mathscr{G}_{\epsilon c_*}}(\nu)\mathrm{D}_\mathbf{u} F\left(0, \epsilon^2\right)\right), \text{ for } (\nu, \epsilon) \in \mathbb{C} \times \mathbb{R}^+.$$

We make the following hypotheses on the characteristic equation.

Hypothesis 3.6.5 (Homogeneous instability)**.** *We assume that the characteristic equation $d(\nu,\epsilon)$ satisfies:*

- $d(0,0)=\partial_\nu d(0,0)=0$ *with* $\partial_{\nu\nu}d(0,0)\neq 0$*;*
- $d(\mathbf{i}\ell,0)\neq 0$ *for all* $\ell\neq 0$.

Notation. As $d(0,0)=0$, there exists $\mathbf{e}_0,\mathbf{e}_0^*\in\mathbb{R}^n$ such that

$$\begin{aligned}\widehat{\mathscr{T}_0}(0)\mathbf{e}_0=\mathbf{e}_0+\widehat{\mathscr{G}_0}(0)\mathrm{D}_{\mathbf{u}}F(0,0)\mathbf{e}_0=0,\\ \widehat{\mathscr{T}_0}(0)^{\mathbf{T}}\mathbf{e}_0^*=\mathbf{e}_0^*+\mathrm{D}_{\mathbf{u}}F(0,0)^{\mathbf{T}}\widehat{\mathscr{G}_0}(0)^{\mathbf{T}}\mathbf{e}_0^*=0,\\ \langle\mathbf{e}_0,\mathbf{e}_0^*\rangle=1,\end{aligned}$$

where $\langle\cdot,\cdot\rangle$ denotes the standard inner product on $\mathbb{R}^n$ given by

$$\langle\mathbf{u},\mathbf{v}\rangle=\sum_{k=1}^n u_k v_k,\ \text{ for any } \mathbf{u}=(u_k)_{k=1}^n\in\mathbb{R}^n \text{ and } \mathbf{v}=(v_k)_{k=1}^n\in\mathbb{R}^n.$$

Note that $\widehat{\mathscr{G}_0}(0)=-D^{-1}\mathscr{K}_0$, together with

$$\begin{aligned}\alpha&=\langle\mathscr{K}_0\mathrm{D}_{\mathbf{u},\mu}F(0,0)\mathbf{e}_0,\mathbf{e}_0^*\rangle>0,\\ \beta&=-\frac{1}{6}\langle\mathscr{K}_0\mathrm{D}_{\mathbf{u},\mathbf{u},\mathbf{u}}F(0,0)\left[\mathbf{e}_0,\mathbf{e}_0,\mathbf{e}_0\right],\mathbf{e}_0^*\rangle>0,\end{aligned}$$

where α and β are the coefficients appearing in the Taylor expansion of $g(z,\mu)$.

Symmetries. As in the previous section, in addition to the translation equivariance, equation (3.6.24) possesses two other symmetries, that we denote $\mathbf{S}_1$ and $\mathbf{S}_2$ respectively and act on functions as

$$\mathbf{S}_1u(\xi):=u(-\xi),\quad\text{and}\quad\mathbf{S}_2u(\xi):=-u(\xi),\quad\forall\xi\in\mathbb{R}.$$

The first symmetry is a consequence of the fact that each element of the matrix kernel $\mathscr{K}$ is a symmetric function, whereas the second symmetry results from the odd symmetry of the nonlinear operator F with respect to its first argument. Finally, let us remark that the conditions on the dispersion relation ensures that the kernel $\mathcal{E}_0$ of $\mathscr{T}_0$ is given by

$$\mathcal{E}_0=\mathrm{Span}\left\{\mathbf{e}_0,\xi\mathbf{e}_0\right\}\subset H^1_{-\eta}(\mathbb{R},\mathbb{R}^n),$$

for all $0<\eta<\eta_0$ and any fixed $\eta_0>0$. As a consequence, any functions $\mathbf{u}_0\in\mathcal{E}_0$, can be decomposed as

$$\mathbf{u}_0=A\mathbf{e}_0+B\mathbf{e}_1,\tag{3.6.25}$$

for $(A,B)\in\mathbb{R}^2$ and $\mathbf{e}_1(\xi):=\xi\mathbf{e}_0$. We remark that the actions of $\mathbf{S}_{1,2}$ on $\mathbf{u}_0$ are given by

$$\begin{aligned}\mathbf{S}_1\mathbf{u}_0&=A\mathbf{e}_0-B\mathbf{e}_1,\\ \mathbf{S}_2\mathbf{u}_0&=-A\mathbf{e}_0-B\mathbf{e}_1.\end{aligned}$$

We identify the action of $\mathbf{S}_{1,2}$ on the couple (A,B) as

$$\begin{aligned}\mathbf{S}_1\cdot(A,B)&=(A,-B),\\ \mathbf{S}_2\cdot(A,B)&=(-A,-B).\end{aligned}$$

Projection $\mathcal{Q}$. We now define the projection $\mathcal{Q}$ from $H^2_{-\eta}(\mathbb{R},\mathbb{R}^n) \to \mathcal{E}_0$. Note that by Sobolev embedding we have $H^2(\mathbb{R},\mathbb{R}^n) \subset \mathscr{C}^1(\mathbb{R},\mathbb{R}^n)$, and thus we can take linear combinations of $\mathbf{u}(0)$ and $\mathbf{u}'(0)$. We define the projection $\mathcal{Q} : H^2_{-\eta}(\mathbb{R},\mathbb{R}^n) \to \mathcal{E}_0$ through

$$\mathcal{Q}(\mathbf{u}) := (\mathbf{u}(0), \mathbf{e}_0^*)\, \mathbf{e}_0 + \left(\mathbf{u}'(0), \mathbf{e}_0^*\right) \mathbf{e}_1. \tag{3.6.26}$$

Center manifold theorem. We apply the parameter-dependent center manifold theorem with symmetries to system (3.6.24), to obtain the existence of neighborhoods $\mathcal{U}_\mathbf{u}$, $\mathcal{U}_0$ of $(0,0)$ in $\mathcal{E}_0 \times (0,+\infty)$ and a map $\Psi \in \mathscr{C}^k(\mathcal{U}_\mathbf{u} \times \mathcal{U}_0, \ker \mathcal{Q})$ with $\Psi(0,0) = 0$, $D_\mathbf{u}\Psi(0,0) = 0$, which commutes with $\mathbf{S}_{1,2}$, and such that for all $\epsilon \in \mathcal{U}_0$ the manifold

$$\mathcal{M}_0(\epsilon) := \{\mathbf{u}_0 + \Psi(\mathbf{u}_0,\epsilon) \mid \mathbf{u}_0 \in \mathcal{U}_\mathbf{u}\}$$

contains the set of all bounded solutions of (3.6.24). From now on, we write

$$\Psi(\mathbf{u}_0,\epsilon) = \Psi(A,B,\epsilon), \qquad \text{for } \mathbf{u}_0 = A\mathbf{e}_0 + B\mathbf{e}_1.$$

The fact that Ψ should commute with $\mathbf{S}_2$ implies that

$$\mathbf{S}_2\Psi(A,B,\epsilon) = \Psi(\mathbf{S}_2 \cdot (A,B),\epsilon),$$

which yields

$$-\Psi(A,B,\epsilon) = \Psi(-A,-B,\epsilon).$$

Thus, there will not be any quadratic term in the Taylor expansion of Ψ. From now on, we write

$$\Psi(A,B,\epsilon) = \sum_{\substack{l_1,l_2,r\geq 0\\ l_1+l_2+r>1}} A^{l_1}B^{l_1}\epsilon^r\Psi_{l_1,l_2,r},$$

the Taylor expansion of Ψ. Our next task is to compute the lower order terms of this expansion.

Terms of order $\mathcal{O}(\epsilon A)$ and $\mathcal{O}(\epsilon B)$. We first start by computing the linear leading order terms in ϵ in the above Taylor expansion of Ψ. The function $\Psi_{1,0,1}$ is the solution to the equation

$$\mathcal{T}_0\Psi_{1,0,1} - c_* D^{-2}\frac{\mathrm{d}}{\mathrm{d}\xi}\left[\mathscr{K} * (\mathrm{D}_\mathbf{u}F(0,0)\mathbf{e}_0)\right] = 0, \text{ with } \Psi_{1,0,1} \in \ker \mathcal{Q}.$$

A trivial computation shows that $\frac{\mathrm{d}}{\mathrm{d}\xi}\left[\mathscr{K} * (\mathrm{D}_\mathbf{u}F(0,0)\mathbf{e}_0)\right] = 0$, such that $\Psi_{1,0,1} \in \ker \mathcal{Q} \cap \ker \mathcal{T}_0$ and thus

$$\Psi_{1,0,1} = 0.$$

On the other hand, we have that $\Psi_{0,1,1}$ is the solution to the equation

$$\mathcal{T}_0\Psi_{0,1,1} - c_* D^{-2}\frac{\mathrm{d}}{\mathrm{d}\xi}\left[\mathscr{K} * (\mathrm{D}_\mathbf{u}F(0,0)\mathbf{e}_1)\right] = 0, \text{ with } \Psi_{0,1,1} \in \ker \mathcal{Q}.$$

First we note that, $\frac{\mathrm{d}}{\mathrm{d}\xi}\left[\mathscr{K} * (\mathrm{D}_\mathbf{u}F(0,0)\mathbf{e}_1)\right] = \mathscr{K}_0\mathrm{D}_\mathbf{u}F(0,0)\mathbf{e}_0$ and we look for solutions of the form

$$\Psi_{0,1,1}(\xi) = \gamma_0\xi^2\mathbf{e}_0 + \psi_{0,1,1}, \text{ with } \psi_{0,1,1} \in \mathcal{E}_0.$$

We then find that

$$-\gamma_0 D^{-1}\int_{\mathbb{R}} y^2\mathscr{K}(y)\mathrm{D}_{\mathbf{u}}F(0,0)\mathbf{e}_0\mathrm{d}y - c_* D^{-2}\mathscr{K}_0\mathrm{D}_{\mathbf{u}}F(0,0)\mathbf{e}_0 = 0,$$

such that

$$\gamma_0 = -\frac{c_*}{\kappa_2}, \quad \kappa_2 := \int_{\mathbb{R}} y^2\langle\mathscr{K}(y)\mathrm{D}_{\mathbf{u}}F(0,0)\mathbf{e}_0, \mathbf{e}_0^*\rangle\mathrm{d}y;$$

here, we used the fact that $\mathbf{e}_0 = D^{-1}\mathscr{K}_0\mathrm{D}_{\mathbf{u}}F(0,0)\mathbf{e}_0$ and $\langle\mathbf{e}_0, \mathbf{e}_0^*\rangle = 1$. Note that $\kappa_2 \neq 0$ as $\partial_{\nu\nu}d(0,0) \neq 0$ from our hypothesis on the characteristic equation. Finally, as $\mathcal{Q}(\Psi_{0,1,1}) = \psi_{0,1,1}$ and $\Psi_{0,1,1} \in \ker\mathcal{Q}$, we necessarily have $\psi_{0,1,1} = 0$.

Terms of order $\mathcal{O}(A^3)$. The function $\Psi_{3,0,0}$ solves

$$\mathcal{T}_0\Psi_{3,0,0} - D^{-1}\mathscr{K} * \left(\frac{1}{6}\mathrm{D}_{\mathbf{u},\mathbf{u},\mathbf{u}}F(0,0)[\mathbf{e}_0,\mathbf{e}_0,\mathbf{e}_0]\right) = 0, \text{ with } \Psi_{3,0,0} \in \ker\mathcal{Q}.$$

We find that

$$\Psi_{3,0,0}(\xi) = \beta_0\xi^2\mathbf{e}_0,$$

where β_0 is given by

$$\beta_0 = -\frac{1}{6}\frac{\langle\mathscr{K}_0\mathrm{D}_{\mathbf{u},\mathbf{u},\mathbf{u}}F(0,0)[\mathbf{e}_0,\mathbf{e}_0,\mathbf{e}_0], \mathbf{e}_0^*\rangle}{\kappa_2} = \frac{\beta}{\kappa_2}.$$

Terms of order $\mathcal{O}(\epsilon^2 A)$. The function $\Psi_{1,0,2}$ is solution of the equation

$$\mathcal{T}_0\Psi_{1,0,2} - D^{-1}\mathscr{K} * (\mathrm{D}_{\mathbf{u},\mu}F(0,0)\mathbf{e}_0) = 0, \text{ with } \Psi_{1,0,2} \in \ker\mathcal{Q}.$$

We find

$$\Psi_{1,0,2}(\xi) = \alpha_0\xi^2\mathbf{e}_0,$$

where α_0 is given by

$$\alpha_0 = -\frac{\langle\mathscr{K}_0\mathrm{D}_{\mathbf{u},\mu}(0,0)\mathbf{e}_0, \mathbf{e}_0^*\rangle}{\kappa_2} = -\frac{\alpha}{\kappa_2}.$$

The reduced vector field. The reduced vector field will be of the form

$$\frac{\mathrm{d}A}{\mathrm{d}\xi} = f(A,B,\epsilon), \tag{3.6.27a}$$

$$\frac{\mathrm{d}B}{\mathrm{d}\xi} = g(A,B,\epsilon), \tag{3.6.27b}$$

where f and g are obtained by computing

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\mathcal{Q}\left(\Phi(\mathbf{u}_0(\cdot+\xi))\right)|_{\xi=0} = (f,g).$$

Note that we slightly abused notation as we identify elements in $\mathcal{E}_0$ with their components on the basis $\{\mathbf{e}_0,\mathbf{e}_1\}$. We also remark that $\Phi(\mathbf{u}_0) = \mathbf{u}_0 + \Psi(\mathbf{u}_0,\epsilon)$, such that $\mathcal{Q}\left(\Phi(\mathbf{u}_0(\cdot+\xi))\right) = \mathcal{Q}\left(\mathbf{u}_0(\cdot+\xi)\right) + \mathcal{Q}\left(\Psi(\mathbf{u}_0(\cdot+\xi),\epsilon\right)$ where

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\mathcal{Q}\left(\mathbf{u}_0(\cdot+\xi)\right)|_{\xi=0} = (B,0).$$

Furthermore, we also have that

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\mathcal{Q}\left((\cdot+\xi)^2\mathbf{e}_0\right)|_{\xi=0}=(0,2).$$

Collecting all terms, we obtain the system

$$\frac{\mathrm{d}A}{\mathrm{d}\xi}=B+\mathcal{O}\left(\epsilon(|A|+|B|)+(|A|+|B|)^3\right), \tag{3.6.28a}$$

$$\frac{\mathrm{d}B}{\mathrm{d}\xi}=2\epsilon\gamma_0 B+2\alpha_0\epsilon^2 A+2\beta_0 A^3+\mathcal{O}\left(|B|(\epsilon^2+|B|^2+|A|^2)\right). \tag{3.6.28b}$$

We now rescale space with $\zeta=\epsilon\xi$, and the amplitudes $A=\epsilon\hat{A}$, $B=\epsilon^2\hat{B}$ to obtain a new system

$$\frac{\mathrm{d}\hat{A}}{\mathrm{d}\zeta}=\hat{B}+\mathcal{O}(\epsilon), \tag{3.6.29a}$$

$$\frac{\mathrm{d}\hat{B}}{\mathrm{d}\zeta}=\frac{2}{\kappa_2}\left(-c_*\hat{B}-\hat{A}\left[\alpha-\beta\hat{A}^2\right]\right)+\mathcal{O}(\epsilon). \tag{3.6.29b}$$

From now on we suppose that

$$\kappa:=\frac{\kappa_2}{2}=\frac{1}{2}\int_{\mathbb{R}}y^2\langle\mathscr{K}(y)\mathrm{D}_{\mathbf{u}}F(0,0)\mathbf{e}_0,\mathbf{e}_0^*\rangle\mathrm{d}y>0,$$

and formally set $\epsilon=0$ in (3.6.29) to obtain the second order ordinary differential equation

$$\kappa\frac{\mathrm{d}^2\hat{A}}{\mathrm{d}\zeta^2}+c_*\frac{\mathrm{d}\hat{A}}{\mathrm{d}\zeta}+\hat{A}\left[\alpha-\beta\hat{A}^2\right]=0. \tag{3.6.30}$$

We know that such an equation admits monotone front solutions for any $|c_*|\geq 2\sqrt{\kappa\alpha}$ connecting the state $\hat{A}=0$ to the state $\hat{A}=\sqrt{\alpha/\beta}$ (see [92, 143]). Note that for $\epsilon>0$, there exists a unique saddle-point $\mathbf{a}(\epsilon):=(\sqrt{\alpha/\beta}+\mathcal{O}(\epsilon),0)$. Then it follows from perturbative arguments [63, 142] that system (3.6.29) has front solutions connecting $(0,0)$ with $\mathbf{a}(\epsilon)$. Monotonicity in the tails can be established for speeds $|c_*|>2\sqrt{\kappa\alpha}+\mathcal{O}(\epsilon)$. We denote by u_* the front solution of equation (3.6.30) connecting 0 to $\sqrt{\alpha/\beta}$, In our initial problem, we thus have thus shown the existence of slowly varying front solutions of (3.6.21) of the form

$$\mathbf{u}(t,x)=\epsilon u_*(\epsilon(x-\epsilon c_* t))\mathbf{e}_0+\mathcal{O}(\epsilon^2),$$

for all $t,x\in\mathbb{R}$, with monotone tails for $|c_*|\geq 2\sqrt{\kappa\alpha}+\mathcal{O}(\epsilon)$ and u_* solution of (3.6.30).

Chapter 4

Propagation phenomena in general reaction-diffusion equations

In this chapter, we present elements of answer to the question raised in the introduction regarding propagation phenomena in reaction-diffusion equations along three complementary directions. First, we study spreading speeds in reaction-diffusion equations, that is we analyze how compactly supported initial conditions spread into an unstable state and characterize the asymptotic speed of propagation. Here, we are interested in the case where some nonlinear coupling terms are present in the system which induce a *resonant* spreading speed, see Section 4.1. In some cases, it is hopeless to precisely characterize spreading speeds, and thus one needs to rely on another point of view. Fortunately, in many reaction-diffusion systems, it is often the case that compactly supported initial conditions eventually converge to a traveling front. In these cases, one approach is to consider the speed selection problem as a front selection problem and identify fronts which are consistent with selection from compactly supported initial data. This is precisely the analysis conducted in Section 4.2 where we investigate the bifurcation to locked fronts in two-component reaction-diffusion systems. Once traveling fronts have been shown to exist, it is natural to ask if whether or not they are stable under some perturbations. In the last Section 4.3, we present two results pertaining at the stability of traveling fronts for reaction-diffusion equations. In both cases the boundary of the essential spectrum of the linearized operator around the traveling front touches the imaginary axis resulting in an algebraic decay of the perturbations in some well-chosen functional spaces.

4.1 Spreading speeds in reaction-diffusion equations

We are interested in spreading speeds in spatially extended systems when more than one scalar mode participates in the instability. As a particular example, we are interested in systems possessing a homogeneous steady state that is unstable with respect to both homogeneous perturbations and perturbations near a fixed nonzero wavelength (a homogeneous-Turing instability). Dynamics of such systems can be captured well by amplitude equations for weak instabilities, a real scalar amplitude equation for the homogeneous mode, and a complex scalar equation for the Turing mode. When the unstable steady state is perturbed, a competition between modes ensues and a fundamental question is to determine which of these modes prevails. For localized perturbations this process is governed by the formation of a traveling invasion front. A defining characteristic of this front is its speed. Predictions of this spreading speed usually come with predictions for eigenmodes and eigenfrequencies in the leading edge, which ultimately allow one to predict selected modes and patterns formed in the wake of the invasion process.

Spreading speeds for scalar equations are reasonably well understood, in particular in the case when speeds are linearly determined. Criteria for the speed can be readily calculated and proofs for the invasion speed can be obtained using comparison principles. For systems, some results are available for particular structures, such as competitive or cooperative systems, which again allow for the use of comparison principles. Leaving this restrictive class, we are interested in predictions for spreading speeds based on general properties of the linearization, in particular the linear dispersion relation. In essence, these predictions characterize spreading speeds through marginal stability criteria, based on pinched double roots, that is, on linear coupling of modes.

The principle objective of this study is to derive criteria for spreading speeds that incorporate the possibility of nonlinear mode coupling. Before stating and corroborating such criteria for general systems, far from onset of instabilities, we now motivate the effect in a simple pair of amplitude equations that one would derive near a homogeneous–Turing instability [56],

$$\left\{ \begin{array}{l} \partial_t U = d_U \partial_x^2 U + (a_1 + a_2|A|^2)U + a_3 U^3 + a_4|A|^2 \\ \partial_t A = d_A \partial_x^2 A + (b_1 + b_2 U + b_3 U^2)A + b_4 A|A|^2, \end{array} \right. \tag{4.1.1}$$

where $U(t,x) \in \mathbb{R}$ represents the amplitude of the homogeneous perturbation, $A(t,x) \in \mathbb{C}$ represents the amplitude of the Turing mode and the real coefficients a_j and b_j are determined from the particular system being studied. In the following, we restrict our considerations to the simplest case where $A \in \mathbb{R}$. In (4.1.1), the zero solution is unstable for $a_1, b_1 > 0$ and the linearization is diagonal, reducing to two uncoupled scalar equations,

$$\left\{ \begin{array}{l} \partial_t U = d_U \partial_x^2 U + a_1 U \\ \partial_t A = d_A \partial_x^2 A + b_1 A, \end{array} \right. \tag{4.1.2}$$

Localized initial conditions in these scalar equations grow and spread spatially. The resulting speeds of propagation are the usual linear spreading speeds, $s_U = 2\sqrt{d_A a_1}$ for the homogeneous mode and $s_A = 2\sqrt{d_A b_1}$ for the Turing mode. The larger of these two spreading speeds is a natural candidate for the spreading speed in the original system. We are interested in cases where the nonlinear interaction of modes can create a faster spreading speed $s_{AU} > \max\{s_A, s_U\}$. In (4.1.1) the most relevant interaction is generated by the coupling term $a_4|A|^2$. This term can be interpreted as quadratic interactions of the Turing mode driving the U equation as a spatio-temporal inhomogeneity. Our main findings point to precise parameter regions where this acceleration through interaction is possible. Interestingly, the accelerated spreading speed s_{AU} is *independent of the strength of the nonlinearity*, but rather reliant only on the mere presence of a quadratic coupling term, $a_4 \neq 0$.

Of course, there are many examples of situations where the nonlinearity significantly amplifies growth and leads to spreading faster than the linear spreading speed, a situation which is mostly observed in subcritical instabilities and referred to as "pushed", nonlinear invasion, rather than "pulled", linear invasion. This does not occur in the parameter regimes that we study here. We emphasize that the invasion fronts of interest here are inherently linear – their speeds can be determined entirely from linear information and the nonlinearity is required only to provide the requisite coupling. In this way, the acceleration mechanism we observe here is fundamentally different from those leading to pushed fronts.

4.1.1 Review of invasion speed theory

We briefly review invasion speed theory, in particular linear criteria for the speed. Our focus here is rather narrow and we emphasize those features pertinent for the results obtained in the remainder of this section. We point the interested reader to [190] for a more in depth review and general treatment.

Localized perturbations of an unstable steady state grow in time and spread spatially. The spreading process is mediated by invasion fronts that propagate into the unstable state and select a secondary state in their wake. A defining feature of these fronts is the speed at which they propagate. Invasion fronts can be loosely characterized as either pulled or pushed. Pulled fronts are driven by the instability of the unstable state ahead of the front interface and their speed can be calculated from the linearization of the system about this state. On the other hand, the growth of perturbations can sometimes be enhanced by nonlinear effects such that the speed is determined by the nonlinearity. Fronts of this variety are commonly referred to as pushed.

Determining the speed of pulled fronts involves calculating the *linear spreading speed*. In words, the linear spreading speed is the critical speed at which a moving observer witnesses a transition from pointwise exponential stability to pointwise exponential instability. At any speed faster than the linear spreading speed the observer outruns the instability while at slower speeds the instability outruns the observer. In this way, linear spreading speeds are related to the notion of absolute and convective instabilities, see for example [24, 34, 36, 46, 57, 120, 124, 177]. Mathematically, the linear spreading speed associated to an unstable state can be determined by locating pinched double roots of the dispersion relation.

To be precise, consider a scalar partial differential equation

$$\partial_t u = \mathcal{L}u + N(u), \quad N(u) = N_2[u,u] + \mathcal{O}(|u|^3), \quad \widehat{\mathcal{L}u}(\ell) = A(\mathbf{i}\ell)\hat{u}(\ell),$$

with $\max_\ell \Re A(\mathbf{i}\ell) > 0$, that is, $u \equiv 0$ is unstable. Associated to this unstable state is a dispersion relation, $D(\lambda,\nu) = A(\nu) - \lambda$. Simple roots of the dispersion relation dictate the temporal evolution $e^{\lambda t}$, $\lambda \in \mathbb{C}$ of modes $e^{\nu x}$, with $\nu \in \mathbb{C}$. Double roots (λ,ν) correspond to a "double" mode with spatio-temporal behavior $e^{\lambda t+\nu x}$. Such double roots, together with a pinching condition, give rise to a singularity of the Green's function and therefore induce spatio-temporal behavior $e^{\lambda t+\nu x}$ for localized initial conditions, locally in space. Therefore, pointwise linear stability is equivalent to requiring that $\Re\lambda < 0$ for all pinched double roots. Transforming to a frame of reference moving with speed s, the location of these pinched double roots varies with s and marginal stability is achieved when the pinched double root satisfies $\lambda^* \in \mathbf{i}\mathbb{R}$ for some value of $s = s_{lin}$; see for instance [120] for a recent and general account of the linear theory.

Definition 4.1.1 (Linear spreading speed)**.** *Consider the dispersion relation in a co-moving frame with speed s, $D_s^{\text{co}}(\lambda,\nu) = D(\lambda,\nu) + s\nu - \lambda$. A double root (λ^*,ν^*) of the dispersion relation,*

$$D_s^{\text{co}}(\lambda^*,\nu^*) = 0, \quad \partial_\nu D_s^{\text{co}}(\lambda^*,\nu^*) = 0,$$

is pinched if solving $D_s^{\text{co}}(\lambda,\nu)$ for $\nu_\pm = \nu(\lambda)$ with $\lim_{\lambda\to\lambda^*}\nu_\pm(\lambda) = \nu^*$, we have that $\Re(\nu_+(\lambda)) \to +\infty$ as $\Re(\lambda) \to +\infty$ and $\Re(\nu_-(\lambda)) \to -\infty$ as $\Re(\lambda) \to +\infty$. The linear spreading speed is defined as

$$s_{lin} = \sup_{s\in\mathbb{R}} \left\{ D_s^{\text{co}}(\lambda,\nu) \textit{ has a pinched double root with } \Re(\lambda) > 0 \right\}.$$

A more subtle analysis of the singularity of the Green's function predicts a slow convergence to the front, with relaxation of the speed $s \sim s_\infty - \frac{3}{2\Re\nu_* t}$; see [29] for a first proof of expansions for the speed in the case of the scalar KPP equation using probabilistic methods, [62] for an analysis in a more general context based on the Green's function, and the more recent [108] that partially recovers Bramson's result using PDE comparison techniques. Our emphasis here goes in a different direction, aimed at "zeroth order" speed selection in more complicated equations rather than higher order approximations in simple systems.

The possibly simplest example where complications arise is when the linearization has a skew-product structure that is, a subset of variables decouples from the others. It is then possible for multiple linear spreading speeds to exist within different components. To give an example, consider the system of equations studied in [116, 117]

$$\begin{cases} \partial_t u = d\partial_x^2 u + \alpha u - u^2 + \beta v, \\ \partial_t v = \partial_x^2 v + v - v^2. \end{cases}$$

Linearizing the system about the unstable state, the v component decouples and feeds into the u component as a source term. The dispersion relation for the full system is the product of the dispersion relations for the sub-systems, i.e. $D(\lambda,\nu) = D_u(\lambda,\nu)D_v(\lambda,\nu)$. The linear spreading speed for the v component is the Fisher-KPP speed of two and the solution converges to a traveling front with decay rate xe^{-x}. However, ahead of the front steeper decay rates are observed. These steep modes feed into the u component as a source term and, depending on the values of d and α, lead to faster spreading speeds. To determine the selected mode ν^*, one imposes that $D_u(\lambda^*,\nu^*) = D_v(\lambda^*,\nu^*)$. This *resonance condition*, together with a pinching requirement, implies that (λ^*,ν^*) is a pinched double root of the full dispersion relation $D(\lambda,\nu)$ and determines the associated linear spreading speed.

This study is based upon the observation that the pinched double root criterion may be insufficient in cases where there exist multiple bands of unstable modes. The previous example illustrates that spreading can be thought of as being enabled by "1 : 1–resonant coupling" between modes. A crucial factor is the presence of the term βv which enables the resonant coupling — spreading speeds are slower when $\beta = 0$; see also [94] for bidirectional coupling and associated discontinuity of spreading speeds. This point of view suggests that higher resonances may induce associated spreading speeds provided that nonlinear coupling terms are present in the system. The present work can be viewed as a case study for spreading speeds induced by *"2:1–resonant coupling"*. In other words, we suggest that spreading of localized disturbances into an unstable medium can be studied in a similar fashion to instabilities in bounded domains, that is, deriving amplitude equations that take into account the crucial effect of nonlinear interaction. A key difference is that such considerations here appear to be relevant even far from onset of instability, since speeds are determined in the leading edge of the front, at small amplitude, even when final states in the system have large finite amplitude.

4.1.2 Linear speeds from pointwise stability — min-max characterizations

The formulation of the linear spreading speed given in Definition 4.1.1 is difficult to generalize to encompass resonant interaction of modes. Here, we discuss an equivalent min-max formulation of the linear spreading speed.

To motivate the following definition of a resonant spreading speed, recall the criterion for linear spreading speeds in scalar equations. With $\lambda = \lambda(\nu)$ from the dispersion relation in a steady frame[1], we can define an envelope velocity $s_{\mathrm{env}}(\nu) = -\frac{\Re(\lambda(\nu))}{\Re\nu}$. In the simplest case of order preserving systems, assuming that perturbations travel at most as fast as linear perturbations, the linear (or 1 : 1-resonant) spreading speed can be obtained as a minimum of the envelope velocity,

$$s_{\mathrm{lin}} = \min_{\nu\in\mathbb{R}}(s_{\mathrm{env}}(\nu)).$$

We find the extremality condition by differentiating,

$$0 = -\Re(\lambda'(\nu*)) + \frac{\Re(\lambda(\nu_*))}{\Re(\nu_*)} =: \Re(s_{\mathrm{g}}(\nu_*)) - s_{\mathrm{env}}(\nu_*),$$

where we wrote $s_{\mathrm{g}}(\nu) := -\lambda'(\nu)$ for the group velocity, generalized to complex ν. Passing to a frame moving with speed s_{env}, we find the dispersion relation

$$D_s^{\mathrm{co}}(\lambda, \nu) := D(\lambda - s\nu, \nu), \tag{4.1.3}$$

with $s = s_{\mathrm{env}}$. Roots (λ, ν) in the steady frame translate to roots $(\lambda + s\nu, \nu)$ in the comoving frame. In particular, group velocity follows Galilean transformation laws and $\Re s_{\mathrm{g}}^{\mathrm{co}} = s_{\mathrm{g}} - s_{\mathrm{env}}$ in the frame moving with the linear spreading speed.

Beyond order preserving systems, we may allow modulations of the envelope and would then require that this minimum is taken over the maximal (with respect to modulations, that is, variations in $\Im(\nu)$) envelope speed

$$s_{\mathrm{lin}} = \min_{\Re\nu} \max_{\Im\nu} (s_{\mathrm{env}}(\nu)). \tag{4.1.4}$$

As a consequence, $s_{\mathrm{g}} \in \mathbb{R}$ and $\lambda'(\nu) = 0$ in a comoving frame, which is the classical double root criterion. We note that the min-max criterion can be obtained more systematically from a contour analysis of the Green's function in the complex plane.

[1] For systems, we take $\lambda(\nu)$ to be the root with largest real part

The real part of the group velocity is often interpreted as the speed at which a "localized patch" of the mode $e^{\nu x}$ spreads. In dispersive media, the group velocity provides the speed at which wave packets propagate, see for example [196]. It plays a similar role in the Fisher-KPP equation where the group velocity gives the slope of the ray in space-time for which solutions have a particular exponential decay rate, see [27, 117]. From this viewpoint, if the group velocity of a mode exceeds the envelope velocity then perturbations overtake the solution and marginal stability is not attained. On the other hand, if the group velocity is slower than the envelope velocity then the solution spreads faster than the perturbation and marginal stability is again not achieved. With this interpretation, the linear spreading speed is the speed at which the group velocity equals the envelope velocity. These conditions: that the group velocity is real and equal to the envelope velocity, in turn imply that the mode leading to these conditions is a critical point of the envelope velocity; see [189]. The group velocity of the mode ν can also be interpreted as giving the speed of the region in space-time for which the solution resembles the mode ν[2].

In order to justify this min-max characterization somewhat more explicitly, we start with the Fourier representation of solutions to the linear constant-coefficient equation $\partial_t u = \mathcal{L}u$,

$$u(t,x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{\ell x+\lambda(\mathbf{i}\ell)t}\hat{u}(0,\mathbf{i}\ell)\mathrm{d}\ell,$$

where the Fourier transform of the initial condition $\hat{u}(0,\mathbf{i}\ell)$ is analytic in ℓ. Under suitable assumptions on $\lambda(\nu)$, we can deform the contour in the complex plane,

$$u(t,x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{(\mathbf{i}\ell+\eta)x+\lambda(\mathbf{i}\ell+\eta)t}\hat{u}(0,\mathbf{i}\ell+\eta)\mathrm{d}\ell,$$

which can in turn be estimated as

$$|u(t,x)| \le C\sup_{\ell} e^{\Re\lambda(\mathbf{i}\ell+\eta)t},$$

again using mild assumptions on $\lambda(\nu)$. We can now optimize over η and obtain the optimal exponential growth estimates

$$|u(t,x)| \le C\inf_{\eta}\sup_{\ell} e^{\Re\lambda(\mathbf{i}\ell+\eta)t}.$$

The spreading speed is obtained by replacing $\lambda(\nu) \mapsto \lambda(\nu) + s\nu$ and finding the largest speed for which growth vanishes,

$$s_* = \sup\left\{ s|\inf_{\eta}\sup_{\ell}(\Re(\lambda(\mathbf{i}\ell+\eta)) + s\eta) = 0\right\}.$$

Introducing $\lambda_{\max}(\eta) := \sup_\ell \Re(\lambda(\mathbf{i}\ell+\eta))$, this simplifies to

$$s_* = \sup\left\{ s|\inf_{\eta}(\lambda_{\max}(\eta) + s\eta) = 0\right\}.$$

[2]This interpretation appears to be valid when the group velocity is real, but no longer so for complex group velocities. The interpretation of complex group velocities is less well understood, although headway has been made in several articles [103, 160, 186]. When the group velocity is complex, the ray in space time for which the mode is conserved is no longer a straight line.

Geometrically, $-s$ is the slope of the least steep line through the origin that touches the graph of $\lambda_{\max}(\eta)$. On the other hand, this slope can also be obtained as the minimum of $\lambda_{\max}(\eta)/(-\eta)$, which is of course the min-max criterion that we introduced in (4.1.4).

We remark that such min-max characterizations of spreading speeds go back to at least [107], for particular scalar examples, providing however also nonlinear characterizations of spreading speeds in these cases.

4.1.3 Linear speeds based on quadratic mode interaction — definition of s_{quad}

Going back to the possibility of quadratic interaction of modes, consider two modes $\nu_{2,3} \in \mathbb{C}$. Quadratic terms in the partial differential equation will couple these two modes and this interaction will potentially lead to amplification of the mode $\nu_1 = \nu_2 + \nu_3$ and faster spreading speeds. The temporal behavior of ν_1 will depend on the temporal behavior of the modes $\nu_{2,3}$ and the temporal behavior of ν_1 by itself. We identify the following criterion to predict the spreading speeds induced by this quadratic interaction.

Scalar case. For simplicity, we first state the criterion in the scalar case and then generalize it to systems in the next paragraph.

Definition 4.1.2 (2 : 1-resonant spreading speeds)**.** *The spreading speed s_{quad} induced by quadratic interaction of modes is a critical point of the envelope velocity s_{env} associated with pinched, space-time resonant modes ν_2, ν_3,*

$$s_{\text{quad}} = \min_{\Re(\nu_2+\nu_3)} \max_{\Im(\nu_2+\nu_3)} \left\{ s_{\text{env}}(\nu_2+\nu_3) \,|\, \nu_2, \nu_3 \text{ space-time resonant and pinched} \right\},$$

where

$$s_{\text{env}}(\nu_2+\nu_3) = -\frac{\Re(\lambda(\nu_2+\nu_3))}{\Re(\nu_2+\nu_3)},$$

and space-time resonance and pinching constraints on ν_2, ν_3 are

1. *(space-time- resonance) $\nu_1 = \nu_2 + \nu_3$ and $\lambda(\nu_1) = \lambda(\nu_2) + \lambda(\nu_3)$;*
2. *(pinching) solving $D_s^{\text{co}}(\lambda, \nu)$ for $\nu_j = \nu(\lambda_j)$, $s = s_{\text{quad}}$, we require $\Re(\nu_1(\lambda)) \to +\infty$ as $\Re(\lambda) \to +\infty$ and $\Re(\nu_{2,3}(\lambda)) \to -\infty$ as $\Re(\lambda) \to +\infty$.*

The corresponding quadratic coupling condition is

$$e^{-\nu_1 x} N_2[e^{\nu_2 x}, e^{\nu_3 x}] \not\equiv 0.$$

Remarks 4.1.1. *For pointwise functions $N(u)(x) = f(u(x))$, quadratic coupling simply requires that quadratic terms do not vanish, $f''(0) \neq 0$. On the other hand, the quadratic coupling condition is presumably not strictly necessary, coupling of almost-resonant modes $\tilde{\nu_j} = \nu_j + \epsilon_j$, with ϵ_j arbitrarily small, appears to be sufficient; see the discussion for more details.*

In other words, we mimic the procedure for scalar equations, but rather than combining two modes ν_1 and ν_2 "linearly" via a double root, we combine ν_1 and $\nu_2 + \nu_3$, where the latter is obtained from the quadratic interaction of modes ν_2 and ν_3.

Remarks 4.1.2. *Nonlinear resonant interaction is of course a well known phenomenon, in nonlinear dynamics as well as in the context of nonlinear waves [196]. There, one usually considers dispersive, Hamiltonian systems with dispersion relation $\omega(k) \in \mathbb{R}$, $\omega = \Im\lambda$, $k = \Im\nu$. Resonant triads correspond precisely to our space-time resonance condition, $\omega_1 = \omega_2 + \omega_3$, $k_1 = k_2 + k_3$. In this sense, our criterion could be viewed as an extension of the theory to complex wavenumbers.*

Generalization to systems of equations. Consider a system of equations

$$\partial_t u = \mathcal{L}u + N(u),$$

with $u \in \mathbb{R}^n$, $N(u) = N_2(u) + \mathcal{O}(|u|^3)$, and linear part $\mathcal{L}$ defined through its Fourier symbol $A(\mathbf{i}\ell)$. Applying the Fourier-Laplace transform to the linear equation $\partial_t u = \mathcal{L}u$, solutions are obtained for any triple ν, λ, v_ν for which

$$\left(A(\nu) - \lambda I\right) v_\nu = 0.$$

Note that such a solution exists precisely when (ν, λ) is a root of the dispersion relation,

$$D(\lambda, \nu) = \det\left(A(\nu) - \lambda I\right). \tag{4.1.5}$$

Assuming that a mode (λ, ν) is simple, that is, $\partial_\lambda D \neq 0$ at (λ, ν), we can solve

$$\left(A^*(\nu) - \bar{\lambda} I\right) w_\nu = 0, \quad (w_\nu, v_\nu) = 1,$$

where $(\cdot, \cdot)$ denotes the hermitian scalar product. Again using $\partial_\lambda D \neq 0$, we find a smooth family $\lambda(\nu)$ and expressions for envelope and group velocities of the mode ν,

$$s_{\mathrm{env}}(\nu) = -\frac{(A(\nu)v_\nu, w_\nu)}{\nu}, \qquad s_{\mathrm{g}}(\nu) = -\frac{d}{d\nu}(A(\nu)v_\nu, w_\nu).$$

Definition 4.1.3 (2 : 1-resonant spreading speeds — quadratic coupling in systems)**.** *The quadratic speed in systems is defined as for scalar systems, via the dispersion relation* (4.1.5)*. The quadratic coupling condition for extremal, space-time resonant, pinched modes (λ_j, ν_j), $j = 1, 2, 3$, is*[3]

$$\left(N_2[e^{\nu_2 x} v_{\nu_2}, e^{\nu_3 x} v_{\nu_3}], e^{-\bar{\nu}_1 x} w_{\nu_1}, \right) \not\equiv 0.$$

4.1.4 Application: unidirectionally coupled amplitude equations – Quadratic spreading speeds

Our goal here is to validate the criterion from Definition 4.1.3 in a simple case of unidirectionally coupled amplitude equations of the form

$$\begin{cases} \partial_t U = d\partial_x^2 U + (\alpha - 6A^2)U - U^3 + 2\gamma A^2, \\ \partial_t A = 4\partial_x^2 A + A - 3A^3. \end{cases} \tag{4.1.6}$$

[3] Again, for pointwise evaluation nonlinearities $f(u)$, the exponentials $e^{\nu_j x}$ can be omitted.

Such equations are typical amplitude equations near simultaneous onset of a Turing and a pitchfork bifurcation. Here, U stands for the amplitude of a homogeneous instability, and A represents the amplitude of a Turing mode, which we restricted to real values. It turns out that the spreading speeds observed in (4.1.6) are determined by the linearization about the unstable zero state. Depending on the parameter values (d, α), the spreading speed observed in (4.1.6) will be one of three speeds: the speed of the zero mode $s_U = 2\sqrt{d\alpha}$, the speed of the Turing mode $s_A = 4$, or the speed of the zero mode induced by the Turing mode through the quadratic interaction γA^2 as defined in Definition 4.1.3. We call this speed s_{AU}.

Lemma 4.1.1. *The speed s_{quad} induced by coupling modes $\nu_{2,3}$ from the equation for the Turing mode to modes ν_1 from the equation for the homogeneous mode U through the quadratic term γA^2, $\gamma \neq 0$, is faster than the single-mode speeds s_A and s_U in the region*

$$\mathrm{P} = \left\{(d, \alpha) \mid 4 - d < \alpha \ (0 < d \leq 1),\ 4 - d < \alpha < \frac{d}{d-1} \ (1 < d < 2)\right\}.$$

Proof. First, in a moving coordinate frame, the roots of the dispersion relation $D_s^{\text{co}}(\lambda, \nu)$ can be calculated explicitly,

$$\begin{cases} \nu_U^{\pm}(\lambda, s) = -\frac{s}{2d} \pm \frac{1}{2d}\sqrt{s^2 - 4d\alpha + 4d\lambda}, \\ \nu_A^{\pm}(\lambda, s) = -\frac{s}{8} \pm \frac{1}{8}\sqrt{s^2 - 16 + 16\lambda}. \end{cases} \tag{4.1.7}$$

The envelope velocities associated to modes $\nu \in \mathbb{R}^-$ are,

$$\begin{cases} s_U(\nu) = -d\nu - \frac{\alpha}{\nu}, \\ s_A(\nu) = -4\nu - \frac{1}{\nu}. \end{cases}$$

Group velocities in the second equation are simply $s_{\text{g}} = -\lambda'(\nu) = 8\nu$, so that "complex interaction" implies $\nu_2 = \nu_3$. Space-time resonance then implies that

$$2(4\nu_2^2 + 1) = d(2\nu_2)^2 + \alpha,$$

and therefore that

$$\nu_2 = -\frac{1}{2}\sqrt{\frac{\alpha - 2}{2 - d}}.$$

This implies that $\nu_1 = 2\nu_2$ and we calculate the envelope velocity $s_U(\nu_1)$ which yields the speed

$$s_{AU} = d\sqrt{\frac{\alpha - 2}{2 - d}} + \alpha\sqrt{\frac{2 - d}{\alpha - 2}}. \tag{4.1.8}$$

In order to find the restrictions on parameters $(\alpha, d) \in \mathrm{P}$ as stated in the theorem, we check the pinching condition which imposes restrictions on the parameter values. We first note that $\Re\nu = 0$ would give infinite envelope speed, certainly not a minimum in the definition of s_{quad}. Now $\Re\nu_2 < 0$ implies that, either (i), $\alpha > 2$ and $d < 2$, or (ii), $\alpha < 2$ and $d > 2$. Since $\nu_1 = 2\nu_2$, we see that ν_1 is a root of the dispersion relation in the comoving frame $D_{s_{AU}}(\lambda, \nu)$ with $\lambda = 0$. In order to verify the pinching condition, we need to track this root as $\mathrm{Re}(\lambda) \to +\infty$ and verify that this root tends to $+\infty$ as well.

Since we have explicit representations of roots, we only need to show that $\nu_1 = \nu_U^+(0, s_{AU})$. This is true if $\nu_1 + \frac{s_{AU}}{2d} > 0$, or, equivalently, if

$$\nu_1 + \frac{s_U(\nu_1)}{2d} = \frac{\nu_1}{2} - \frac{\alpha}{2d\nu_1} > 0.$$

Since $\nu_1 < 0$, this is equivalent to $\nu_1^2 < \frac{\alpha}{d}$. Expand this condition,

$$\frac{\alpha - 2}{2 - d} < \frac{\alpha}{d},$$

and solve for α to find,

$$\alpha(d-1) < d \text{ and } d < 2, \quad \text{or,} \quad \alpha(d-1) > d \text{ and } d > 2.$$

This condition holds automatically if $d \leq 1$ and gives a condition on α if $d > 1$.

In a similar fashion, we require $\nu_2 = \nu_A^-(0, s_A(\nu_2))$. Once again referencing (4.1.7), this is equivalent to the requirement that $\nu_2 + \frac{s_A}{8} < 0$. Expanding we find that

$$\alpha > 4 - d \text{ and } d < 2, \quad \text{or,} \quad \alpha < 4 - d \text{ and } d > 2.$$

Finally, note that requirements that $\frac{d}{d-1} < \alpha$ and $\alpha < 4 - d$ are not compatible for $d > 2$. As a consequence, we are left with the restrictions $d < 2$, $\alpha < d/(d-1)$, $\alpha > 4 - d$, which delimits precisely the region P.

One can check that the min-max criterion actually gives a finite value for $s, \nu_{2/3}$, which then necessarily coincides with the value of the unique critical point that we computed here. ∎

Now that we have identified s_{AU} and the region P, we can state the following theorem.

Theorem 4.1.1. *Choose $(\alpha, d) \in \mathrm{P}$, such that $s_{AU} > \max\{s_A, s_U\}$. Define the invasion point,*

$$\kappa(t) = \sup_{x \in \mathbb{R}} \{x \mid u(t,x) > \sqrt{\alpha - 2}\},$$

and the selected speed

$$s_{\text{sel}} = \lim_{t \to \infty} \frac{\kappa(t)}{t}.$$

For $(d, \alpha) \in \mathrm{P}$, and $\gamma \neq 0$, the solution of (4.1.6) with initial data consisting of compactly supported perturbations of Heaviside step functions will spread with speed

$$s_{AU} = d\sqrt{\frac{\alpha - 2}{2 - d}} + \alpha\sqrt{\frac{2 - d}{\alpha - 2}},$$

i.e. $s_{\text{sel}} = s_{AU}$. In the complement of the region P, one has

$$s_{\text{sel}} = \max\{s_A, s_U\}.$$

We refer to Figure 4.1 for an illustration.

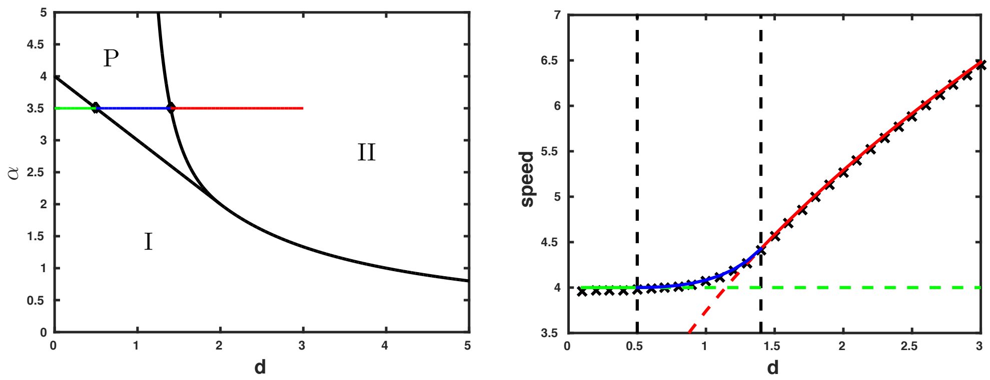


Figure 4.1: Comparison of numerically observed spreading speeds (crosses) with theoretical predictions (colored lines). Theory predicts transitions from speeds s_A (green) to s_{AU} (blue) to s_U (red) as d is increased (left).

The proof of Theorem 4.1.1 relies on the the construction of sub and super solutions and is based on ideas in [116, 117]. As usual, it is the construction of sub-solutions that consists in the most delicate part of the analysis. We note that for the particular case of amplitude equations in (4.1.6), the dynamics of the Turing mode is independent of the dynamics of the zero mode. This fact, that is, the absence of a back-coupling as seen in (4.1.1), is essential to the proof of Theorem 4.1.6, it is, however not generic for amplitude equations near Turing/pitchfork instabilities. On the other hand, this skew-product nature of the equation does not appear to be relevant to the underlying phenomenon and should be thought of as a technical assumption that allows the use of comparison principles.

We briefly comment on the mechanism leading to faster speeds in (4.1.6), which is similar to the one identified in [116, 117]. Starting from compactly supported initial data, the A component forms a traveling front propagating with speed 4. Ahead of the front interface, the solution decays to zero faster than any exponential. Through the quadratic coupling term, the A component acts as a source in the U equation with decay rates twice those of the original solution. Under the evolution of the equation governing U, these "steep" decay rates may actually be "weak" and lead to faster invasion speeds and the front profile will converge to a super-critical traveling front with weak exponential decay.

4.1.5 Extensions

The above theory can be naturally extended to treat more general form of resonances. To illustrate the robustness of our method, let us consider the following system of coupled reaction diffusion equations,

$$\begin{cases} \partial_t u = d\partial_x^2 u + \alpha u(1u) + \beta v^p(1-u) & , t > 0,\, x \in \mathbb{R}, \\ \partial_t v = \partial_x^2 v + v(1-v) & , t > 0,\, x \in \mathbb{R}, \\ u(0,x) = u_0(x), \quad v(0,x) = v_0(x) & , x \in \mathbb{R}, \end{cases} \tag{4.1.9}$$

with $d, \alpha, \beta, p > 0$ and given initial conditions $0 \le u_0, v_0 \le 1$ being compactly supported perturbations of the Heaviside step function $\mathbf{1}_{x\le 0}$. As usual, we define the invasion point as

$$\kappa(t) = \sup_{x\in\mathbb{R}} \left\{ x \,|\, u(t,x) \ge \frac{1}{2} \right\},$$

and we want to know the expression of the selected speed or spreading speed,

$$s_{\text{sel}} = \lim_{t\to\infty} \frac{\kappa(t)}{t}.$$

We have the following result.

Theorem 4.1.2. *Consider (4.1.9) with with $d, \alpha, \beta, p > 0$. Fix initial data $0 \le u(0,x) \le 1$ and $0 \le v(0,x) \le 1$, each consisting of a compactly supported perturbation of the Heaviside step function $\mathbf{1}_{x\le 0}$. Then, there exist domains $\mathrm{I}, \mathrm{II}, \mathrm{III}$, depending on p, so that the selected speed $s_{\text{sel}}(p)$ is given by*

$$s_{\text{sel}}(p) = \begin{cases} 2 & , (d,\alpha) \in \mathrm{I}, \\ 2\sqrt{d\alpha} & , (d,\alpha) \in \mathrm{II}, \\ s_{\text{anom}}(d,\alpha,p) & , (d,\alpha) \in \mathrm{III}, \end{cases}$$

with

$$s_{\text{anom}}(d,\alpha,p) = \sqrt{\frac{\alpha - p}{p - dp^2}} + \sqrt{\frac{p - dp^2}{\alpha - p}}, \tag{4.1.10}$$

and

$$\begin{aligned} \mathrm{I} &= \left\{ \alpha \le p(2-dp) \mid d \le \frac{1}{p} \right\} \cup \left\{ \alpha \le \frac{1}{d} \mid d > \frac{1}{p} \right\}, \\ \mathrm{II} &= \left\{ \alpha \ge \frac{dp^2}{2dp-1} \mid \frac{1}{2p} < d \le \frac{1}{p} \right\} \cup \left\{ \alpha \ge \frac{1}{d} \mid d > \frac{1}{p} \right\}, \\ \mathrm{III} &= \left\{ \alpha > p(2-dp) \mid d < \frac{1}{2p} \right\} \cup \left\{ p(2-dp) < \alpha < \frac{dp^2}{2dp-1} \mid \frac{1}{2p} < d \le \frac{1}{p} \right\}. \end{aligned}$$

Let first note that the system (4.1.9) has already been studied in the case $p = 1$ [116, 117]. It is important to note that when $p = 2$, we recover the "2 : 1- resonant spreading speed" from Definition 4.1.3. Actually, for any $p \ge 1$ being an integer, the spreading speed can be interpreted as a "p : 1- resonant spreading speed". However, for general $p > 0$, it is not possible to use directly Definition 4.1.3, and this is why we propose the natural generalization (4.1.14); see below. It is also important to remark that for $(d,\alpha) \in \mathrm{III}$ we have $s_{\text{anom}}(d,\alpha,p) > \max(2, 2\sqrt{d\alpha})$ and in that respect s_{anom} is referred to as an anomalous spreading speed. We will see that it is the coupling βv^p into the u component of system (4.1.9) that induces a resonance in the dynamics leading to this anomalous spreading speed. It is interesting to note that as $p \to +\infty$, the domain III of existence of the anomalous speed shrinks as it is shifted close to axis $d = 0$ where it imposes large values for α as we have $\alpha \ge p(2-dp)$ in that region. On the other hand, when $p \to 0^+$, the domain of existence of the anomalous speed becomes larger and eventually covers the whole quadrant $\alpha > 0$ and $d > 0$. In that respect, small values of p enhance anomalous spreading. The proof of Theorem 4.1.2 relies on the fact that each component of (4.1.9) satisfies the comparison principle, allowing us to apply the theory of sub- and super-solutions. In fact, for each domain, we explicitly construct sub- and super-solutions from which we will deduce Theorem 4.1.2.

Derivation of the spreading speed. For $p > 0$ and not an integer, we would like to proceed along similar lines as in the case $p = 1$, from [116, 117], where the strategy is to linearize the system around the unstable state $(0, 0)$. We consider the system

$$\begin{cases} \partial_t u = d\partial_x^2 u + \alpha u + \beta v^p & , t > 0,\ x \in \mathbb{R}, \\ \partial_t v = \partial_x^2 v + v & , t > 0,\ x \in \mathbb{R}, \end{cases} \tag{4.1.11}$$

which will serve as a natural super-system for (4.1.9). In a moving frame $y = x - st$, (4.1.11) writes

$$\begin{cases} \partial_t u = d\partial_y^2 u + s\partial_y u + \alpha u + \beta v^p & , t > 0,\ y \in \mathbb{R}, \\ \partial_t v = \partial_y^2 v + s\partial_y v + v & , t > 0,\ y \in \mathbb{R}. \end{cases} \tag{4.1.12}$$

The following heuristic approach on system (4.1.12) will give us an educated guess on the expressions of domains I, II, III and the expression of s_{anom} in the general case $p > 0$.

If one considers exponential solutions of the form

$$u(t, y) = e^{\Lambda t} e^{\nu_u(s,\Lambda) y}, \qquad v(t, y) = e^{\lambda t} e^{\nu_v(s,\lambda) y},$$

then in order for those functions to satisfy (4.1.12) we necessarily need

$$\Lambda = p\lambda, \text{ and } \nu_u(s, \Lambda) = p\nu_v(s, \lambda).$$

The heuristic is then the following : for fixed values of (d, α, p), we seek the couples (s, λ) solutions of any of the four equations

$$\begin{cases} \nu_u^+(s, \lambda) = \nu_u^-(s, \lambda), \\ \nu_v^+(s, \lambda) = \nu_v^-(s, \lambda), \\ \nu_u^\pm(s, p\lambda) = p\nu_v^\mp(s, \lambda), \end{cases} \tag{4.1.13}$$

and we want to find the value of the speed

$$s_{\text{lin}}(p) = \sup \{ s > 0 \,|\, \text{all couples } (s, \lambda) \text{ solutions of (4.1.13) satisfy } \mathrm{Re}(\lambda) > 0 \}. \tag{4.1.14}$$

We call that speed the linear speed despite (4.1.12) not being linear, as it serves as a predictor for the selected speed of the nonlinear system, just like the case $p = 1$. Obviously, $s_{\text{lin}}(1) = s_{\text{lin}}$ from [116, 117] and $s_{\text{lin}}(2) = s_{\text{quad}}$ from Theorem 4.1.1.

4.2 Locked fronts in reaction-diffusion equations

We study invasion fronts for general systems of reaction-diffusion equations,

$$\begin{cases} \partial_t u = \partial_x^2 u + F(u, v), \\ \partial_t v = \sigma\partial_x^2 v + G(u, v), \end{cases} \tag{4.2.1}$$

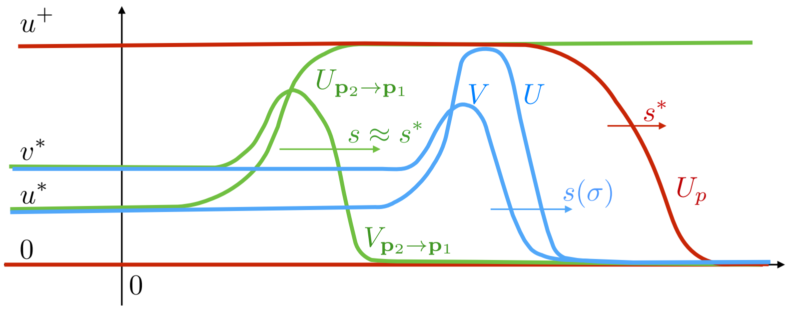


Figure 4.2: Illustration of our assumptions leading to the existence of locked traveling front solutions (in blue) of (4.2.1). In red, we have represented the pushed front $(U_p(x - s^*t), 0)$ connecting $\mathbf{p}_1 = (u^+, 0)$ to $\mathbf{p}_0 = (0, 0)$ that propagates to the right with speed s^* given by assumption **(H2)** below. In green, we have sketched one traveling front solution $(U_{\mathbf{p_2}\to\mathbf{p_1}}(x - st), V_{\mathbf{p_2}\to\mathbf{p_1}}(x - st))$ connecting $\mathbf{p}_2 = (u^*, v^*)$ to $\mathbf{p}_1 = (u^+, 0)$ that propagates to the right with some speed $s \approx s^*$ given by assumption **(H5)** below. Our main result demonstrates the existence of locked fronts $(U(x - s(\sigma)t), V(x - s(\sigma)t))$ connecting $\mathbf{p}_2 = (u^*, v^*)$ to $\mathbf{p}_0 = (0, 0)$ that propagates to the right with speed $s(\sigma)$ for $\sigma \approx \sigma_*$, see **(H3)** below for the definition of σ^*.

where $\sigma > 0$ and $x \in \mathbb{R}$. More specifically, we are interested in traveling wave solutions of the form $(u(x - st), v(x - st))$ which satisfy

$$\begin{cases} -su' = u'' + F(u, v), \\ -sv' = \sigma v'' + G(u, v), \end{cases}$$

where we have set $\xi = x - st$ and used the notation u' for $\dfrac{\mathrm{d}u}{\mathrm{d}\xi}$ and u'' for $\dfrac{\mathrm{d}^2u}{\mathrm{d}\xi^2}$. It will be more convenient to write this system as a first-order system

$$\begin{cases} u_1' = u_2, \\ u_2' = -su_2 - F(u_1, v_1), \\ v_1' = v_2, \\ \sigma v_2' = -sv_2 - G(u_1, v_1). \end{cases} \tag{4.2.2}$$

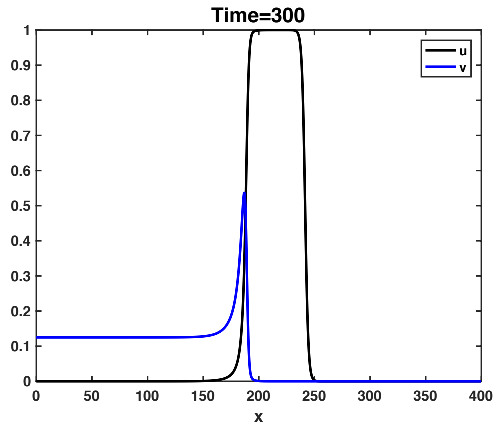
Throughout, the reaction terms are assumed to have the form,

$$F(u, v) = uf(u, v), \quad G(u, v) = vg(u, v), \quad \text{with } f(0, 0) > 0 \text{ and } g(0, 0) > 0. \tag{4.2.3}$$

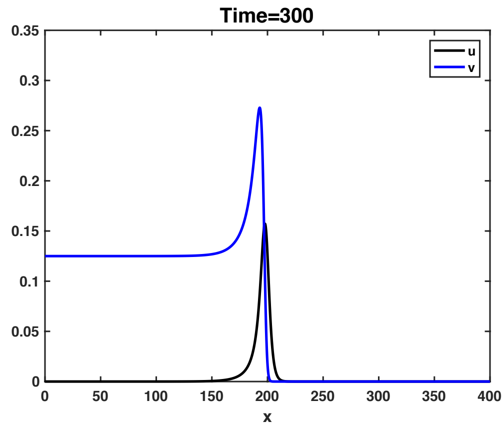
Precise assumptions regarding the functions $F(u, v)$ and $G(u, v)$ are listed below. We sketch those assumptions now to better set the stage and we refer to Figure 4.2 for an illustration.

(H1) System (4.2.1) has three nonnegative homogeneous steady states: $\mathbf{p}_0 = (0, 0)$, $\mathbf{p}_1 = (u^+, 0)$ and $\mathbf{p}_2 = (u^*, v^*)$ and the associated traveling wave equation (4.2.2) has three corresponding fixed points $\mathbf{P}_0 = (0, 0, 0, 0)$, $\mathbf{P}_1 = (u^+, 0, 0, 0)$ and $\mathbf{P}_2 = (u^*, 0, v^*, 0)$.

(H2) There exists a pushed front $(U_p(x - s^*t), 0)$ connecting $\mathbf{p}_1$ to $\mathbf{p}_0$ that propagates to the right with speed s^* and leaves the homogeneous state $\mathbf{p}_1$ in its wake.



(a) Staged invasion fronts.



(b) Locked fronts.

Figure 4.3: Profiles of the solutions of (4.2.1), evaluated at time $t = 300$, with nonlinear terms $f(u, v) = (1 - u)(u + 1/16) - v$ and $g(u, v) = 2u(1 - u) + 1/8 - v$ for different values of σ. (a) We observe a staged invasion process where the zero state is first invaded by the u component, then at some later time is subsequently invaded by the v component. Here we have set $\sigma = 0.25$. (b) We observe locked fronts with both components traveling at the same wave speed. Here we have set $\sigma = 0.3$. Note that $\mathbf{p}_0 = (0, 0)$, $\mathbf{p}_1 = (1, 0)$ and $\mathbf{p}_2 = (0, 1/8)$.

(H3) There exists a $\sigma^* > 0$ such that the linearization of the v component about the pushed front has marginally stable spectrum at $\sigma = \sigma^*$. If $\sigma < \sigma^*$, then small perturbations of the front $(U_p(x - s^*t), 0)$ in the v component propagate slower than s^* whereas for $\sigma > \sigma^*$ these perturbations spread faster than s^*.

(H4) We assume an ordering of the eigenvalues for the linearization of the traveling wave equation (4.2.2) near $\mathbf{P}_0$ and $\mathbf{P}_1$ together with a condition on the ratio of the eigenvalues.

(H5) There is a family of traveling front solutions connecting $\mathbf{p}_2$ to $\mathbf{p}_1$ for all wave speeds s near s^*. These fronts have weak exponential decay representing the fact that the invasion speed of $\mathbf{p}_2$ into $\mathbf{p}_1$ is slower than s^*.

One can think of u and v as representing independent species that diffuse through space and interact through the reaction terms $F(u, v)$ and $G(u, v)$. When σ is small, we expect the spreading speed of the u component to exceed that of the v component. The dynamics in this regime is that of a staged invasion process: the zero state is first invaded by the u component, then at some later time is subsequently invaded by the v component, see Figure 4.3(a). As σ is increased, the speed of this secondary front will increase until eventually the two fronts lock and form a coherent coexistence front where the unstable zero state $\mathbf{p}_0$ is invaded by the stable state $\mathbf{p}_2$, see Figures 4.2 and 4.3(b). Broadly speaking, this transition to locking is the phenomena that we are concerned with here. Our primary goal is to determine parameter values for which this onset to locking is to be expected and whether the speed of the combined front is faster or slower than the speed of the individual fronts. We refer to [42, 60, 105] for the study of stage invasion processes in different two-components reaction-diffusion systems.

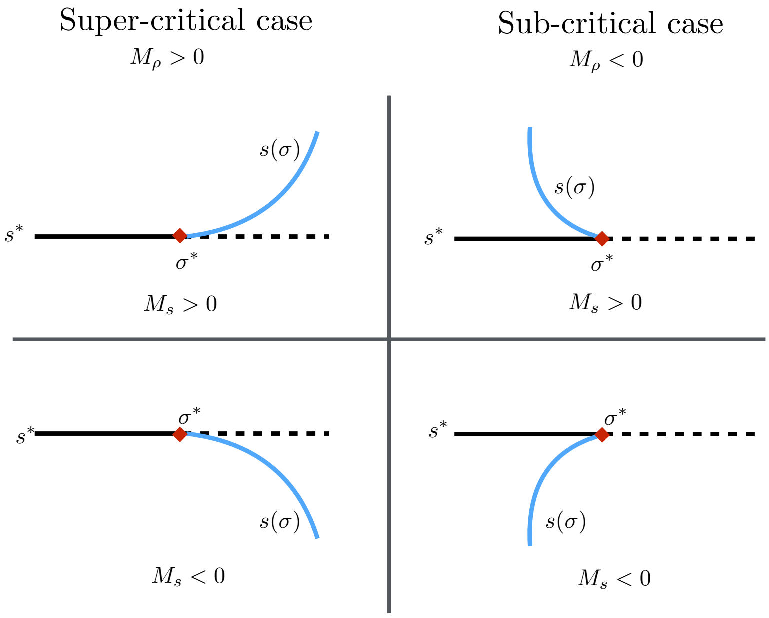


Figure 4.4: Sketch of the different bifurcation scenarios covered by our main result. In each panel, the horizontal black line $s = s^*$ illustrates the marginal stability assumption **(H3)** of the linearization of the v component about the pushed front. The red diamond indicates the critical value σ^* at which the pushed front has marginal stable spectrum. The solid part of the line indicates a negative principal eigenvalue of the corresponding linearized operator while the dashed part indicates a positive one. The bifurcating curve in blue illustrates the existence of locked front solutions with wave speed $s(\sigma)$ given by our main result. Two scenarios can happen: the bifurcation will occur either for $\sigma > \sigma^*$ (super-critical case) or for $\sigma < \sigma^*$ (sub-critical case), and in each case the direction of bifurcation can lead to larger wave speed (top panels) or slower wave speed (bottom panels). These different scenarios can be characterized by the signs of the constants M_ρ and M_s (see Theorem 4.2.1).

Our main result is the existence of a bifurcation leading to locked fronts occurring at the parameter values $(s, \sigma) = (s^*, \sigma^*)$. Depending on properties of the reaction terms the bifurcation will occur either for $\sigma > \sigma^*$ (super-critical) or for $\sigma < \sigma^*$ (sub-critical), see Figure 4.4 for a sketch. In the super-critical case, the coexistence front does not appear until after the bifurcation at σ^* and the speed of the locked front changes continuously following the bifurcation – varying quadratically in a neighborhood of the bifurcation point (see Figure 4.6 for an illustration on a specific example). The dynamics of the system in the sub-critical case are much different. In this scenario, the system transitions from a staged invasion process to locked fronts at a value of σ strictly less than the critical value σ^* and the spreading speed at this point is not continuous as a function of σ and we refer to Figure 4.7 for an illustration on a specific example.

We employ a dynamical systems approach and construct these traveling fronts as heteroclinic orbits of the corresponding traveling wave equation (4.2.2), see Figure 4.5. The traveling front solutions that we are interested in lie near a concatenation of traveling front solutions: the first being the pushed front connecting $\mathbf{P}_1$ to $\mathbf{P}_0$ (see **(H2)**) and the second connecting the stable coexistence state $\mathbf{P}_2$ to this intermediate state $\mathbf{P}_1$ (see **(H5)**). A powerful technique for constructing solutions near heteroclinic chains is Lin's method [152, 174, 175]. In this approach, perturbed solutions are obtained by variation of constants and these perturbed solutions are matched via Liapunov-Schmidt reduction leading to a system of bifurcation equations. Two common assumptions when using these techniques are a) that the dimensions of the stable and unstable manifolds of each fixed point in the chain are equal and b) the sum of tangent spaces of the intersecting unstable and stable manifolds have co-dimension one. Neither of these assumptions hold in our case. As fixed points of the traveling wave equation the stable coexistence state $\mathbf{P}_2$ has two unstable eigenvalues and two stable eigenvalues, the intermediate saddle state $\mathbf{P}_1$ has three stable eigenvalues and one unstable eigenvalue and the unstable zero state $\mathbf{P}_0$ has four stable eigenvalues. Restricting to fronts with strong exponential decay, the zero state can be thought of as having a two-two splitting of eigenvalues, but no such reduction is possible for the intermediate state.

One interesting phenomena that we observe is a discontinuity of the spreading speed as a function of σ in the sub-critical regime. The discontinuous nature of spreading speeds with respect to system parameters has been observed previously, see for example [81, 94, 116, 120]. However, the discontinuity in those cases is typically observed as a parameter is altered from zero to some non-zero value representing the onset of coupling of some previously uncoupled modes. The mechanism here appears to be different.

There is a large literature pertaining to traveling fronts in systems of reaction-diffusion equations. Directly related to the work here is [119], where system (4.2.1) is studied under the assumption that the second component is decoupled from the first, i.e. that $g(u, v) = g(v)$. Further assuming that the system obeys a comparison principle, precise statements regarding the evolution of compactly supported initial data can be made; see also [23]. Here, we do not assume monotonicity and therefore a dynamical system approach is required. A similar approach is used in [119], however, the decoupling of the v component reduces the traveling wave equation to a three dimensional system.

The present work is also partially motivated by recent studies of bacterial invasion fronts similar to [144]. In this context, the u component can be thought of as a bacterial population of cooperators while the v component are defectors. In a well mixed population the defectors out compete the cooperators. However, in a spatially extended system the cooperators may persist via spatial movement by outrunning the defectors. This depends on the relative diffusivities, where for σ small the cooperators are able to escape. However, for σ sufficiently large the defector front is sufficiently fast to lock with the cooperator front and slow its invasion. Our result characterizes how this locking may take place. See also [194, 195] for similar systems of equations.

4.2.1 Discussion of methods: a dynamical systems viewpoint

We have thus far focused primarily on properties of the PDE (4.2.1). Mathematically, our main result regards the construction of traveling fronts in the associated traveling wave ODE, (4.2.2). We include a short discussion now to connect these two perspectives; see also [190] for a longer discussion. To keep this discussion as straightforward as possible we restrict ourselves only to the simplest case of constant coefficient reaction-diffusion systems giving rise to fixed form traveling front solutions connecting homogeneous steady states and ignore complications that can arise for pattern forming systems, inhomogeneous problems, or systems including advective terms to name a few.

The notion of spreading speeds for a PDE typically refers to the asymptotic speed of invasion of compactly supported perturbations of an unstable state; see for example [8]. For scalar equations having a comparison principle or for monotone systems of equations, it is often possible to rigorously establish spreading speeds. In doing so, it is often the case that the compactly supported initial conditions eventually converge to a traveling front. Thus, the system identifies a unique *selected* front propagating at the selected spreading speed and the proof implies stability (in an appropriate sense) of this front with respect to a large class of initial conditions.

Many systems, including the ones considered here, lack a comparison structure and consequently it becomes extremely difficult to rigorously establish PDE spreading speeds in the traditional sense. In these cases, one approach is to consider the speed selection problem as a front selection problem and identify fronts which are consistent with selection from compactly supported initial data. In doing this, one weakens the "global" stability requirement of the selected front to a local stability criterion. This local stability criterion is referred to as *marginal stability*; see [57, 190].

Marginal stability requires that the selected front be pointwise marginally stable with respect to compactly supported perturbations. As fronts propagating into unstable states, the essential spectrum of any invasion front is unstable (in $L^2(\mathbb{R})$ for example). A common technique to stabilize the essential spectrum is to work in exponentially weighted spaces. Weights shift the essential spectrum and there is typically an optimal weight that pushes the essential spectrum as far to the left as possible; see [177] for an introduction to the absolute spectrum and its role in this regard. Marginal stability can then be defined in terms of stability properties in this optimally weighted space. Generally speaking, there are two possibilities. For a pushed front, the essential spectrum is stabilized while the point spectrum is stable with the exception of a translational eigenvalue on the imaginary axis. For a pulled front, the essential spectrum is itself marginally stable and there are no unstable eigenvalues.

Invasion fronts typically come in families parameterized by their speed of propagation. With the previous discussion in mind, given this family of fronts we seek to identify the unique marginally stable front. The speed of this marginally stable front then provides a prediction for the spreading speed of compactly supported initial conditions for the original PDE (4.2.1).

We are interested in constructing candidate pushed fronts for (4.2.1) by constructing heteroclinic orbits for (4.2.2). The fronts of interest must possess two qualitative features that are indicative of the existence of a pushed front. First, it must be possible to stabilize the essential spectrum using exponential weights. Secondly, the decay of the front must be sufficiently steep so that the derivative of the front profile remains as an eigenvalue in the weighted space.

For the problem considered in this paper, the second property is key and we focus on constructing traveling front solutions with sufficiently steep exponential decay rates. These are candidate solutions for the selected front and their speed then gives a prediction for the spreading speeds of the original PDE system (4.2.1). We do not pursue a full stability analysis of the fronts that we construct, although such an analysis is conceivably possible through similar means as those used in the existence proof. In fact, we do not necessarily believe these fronts to always be marginally stable. For example, in the sub-critical regime depicted in Figure 4.4 we expect the bifurcating fronts to be pointwise unstable and this feature is essential to the jump in spreading speed observed numerically in this regime.

In the next two subsections, we outline our assumptions in more detail and state our main result.

4.2.2 Set up and main assumptions

In this subsection, we specify the precise assumptions required of (4.2.1) in order to state our main result in the following one. We first make some assumptions on the reaction terms $F(u,v)$ and $G(u,v)$ that have the specific form defined in (4.2.3).

Hypothesis (H1) *Assume that that homogeneous system*

$$\left\{\begin{array}{l} \frac{\mathrm{d}u}{\mathrm{d}t} = F(u,v), \\ \frac{\mathrm{d}v}{\mathrm{d}t} = G(u,v), \end{array}\right.$$

with $F(u,v) = uf(u,v)$ and $G(u,v) = vg(u,v)$, has three non-negative equilibrium points which we denote by $\mathbf{p}_0 = (0,0)$, $\mathbf{p}_1 = (u^+,0)$ and $\mathbf{p}_2 = (u^*,v^*)$ for some $u^* \geq 0$ and $v^* > 0$. We assume that $f(\mathbf{p}_0) > 0$ and $g(\mathbf{p}_0) > 0$ so that $\mathbf{p}_0$ is an unstable node for the homogeneous system. We assume that $F_u(\mathbf{p}_1) < 0$ and $g(\mathbf{p}_1) > 0$ so that $\mathbf{p}_1$ is a saddle with one stable direction in the $v = 0$ coordinate axis and an unstable direction transverse to this axis. Finally, we assume that $\mathbf{p}_2$ is a stable node.

The traveling wave equation (4.2.2) naturally inherits equilibrium points from the homogeneous equation which we denote as $\mathbf{P}_0 = (0,0,0,0)$, $\mathbf{P}_1 = (u^+,0,0,0)$ and $\mathbf{P}_2 = (u^*,0,v^*,0)$. At either the fixed point $\mathbf{P}_0$ or $\mathbf{P}_1$, the linearization is block triangular and eigenvalues can be computed explicitly. At $\mathbf{P}_0$, the four eigenvalues are

$$\left\{\begin{array}{l} \mu_u^{\pm}(s) = -\frac{s}{2} \pm \frac{1}{2}\sqrt{s^2 - 4f(\mathbf{p}_0)}, \\ \mu_v^{\pm}(s,\sigma) = -\frac{s}{2\sigma} \pm \frac{1}{2\sigma}\sqrt{s^2 - 4\sigma g(\mathbf{p}_0)}, \end{array}\right.$$

where we used the fact that $F_u(\mathbf{p}_0) = f(\mathbf{p}_0)$ and $G_v(\mathbf{p}_0) = g(\mathbf{p}_0)$. Similarly, at $\mathbf{P}_1$, the linearization has eigenvalues

$$\left\{\begin{array}{l} \nu_u^{\pm}(s) = -\frac{s}{2} \pm \frac{1}{2}\sqrt{s^2 - 4F_u(\mathbf{p}_1)}, \\ \nu_v^{\pm}(s,\sigma) = -\frac{s}{2\sigma} \pm \frac{1}{2\sigma}\sqrt{s^2 - 4\sigma g(\mathbf{p}_1)}, \end{array}\right.$$

where once again we used the fact that $G_v(\mathbf{p}_1) = g(\mathbf{p}_1)$.

When the v component is identically zero, system (4.2.1) reduces to a scalar reaction-diffusion equation

$$\partial_t u = \partial_x^2 u + F(u,0), \tag{4.2.4}$$

and the traveling wave equation (4.2.2) reduces to the planar system

$$\left\{\begin{array}{l} u_1' = u_2, \\ u_2' = -su_2 - F(u_1,0). \end{array}\right.$$

We now list assumptions related to traveling front solutions of (4.2.4).

Hypothesis (H2) *We assume that there exists $s^* > 2\sqrt{f(\mathbf{p}_0)}$ for which (4.2.4) has a pushed front solution $U_p(x - s^*t)$ moving to the right with speed s^*. By pushed front, we mean that the solution has steep exponential decay $U_p(\xi) \sim Ce^{\mu_{\bar{u}}^{-}(s^*)\xi}$ as $\xi \to \infty$ and has stable spectrum in the weighted space $L^2_\alpha(\mathbb{R})$, for some $\alpha > 0$, with the exception of an eigenvalue at zero due to translational invariance. There is, in fact, a one parameter family of translates of these fronts and we therefore impose that $U_p''(0) = 0$ and restrict to one element of the family.*

To reiterate the connection to the PDE (4.2.1), we are interested in reaction terms for which non-negative and compactly supported initial data for (4.2.1) of the form $(u_0(x), 0)$ would spread with speed $s^* > 2\sqrt{f(\mathbf{p}_0)}$. Note that the quantity $2\sqrt{f(\mathbf{p}_0)}$ is the linear spreading speed of the u component near $\mathbf{p}_0$ and so we require faster than linear invasion speeds. For the traveling wave ODE, this translates to the existence of a marginally stable pushed front – which is exactly what is laid out by assumption **(H2)**.

Now consider the linearization of the v component of (4.2.1) around the traveling front solution $(U_p(x - s^*t), 0)$,

$$\mathcal{L}_v := \sigma\partial_\xi^2 + s^*\partial_\xi + g(U_p(\xi), 0).$$

The spectrum of this operator posed on $L^2(\mathbb{R})$ is unstable due to the instability of the asymptotic rest states. However, this spectrum may be stable when $\mathcal{L}_v$ is viewed as an operator on the exponentially weighted space

$$L^2_d(\mathbb{R}) = \left\{\phi(\xi) \in L^2(\mathbb{R}) \mid \phi(\xi)e^{d\xi} \in L^2(\mathbb{R})\right\}.$$

Let $d = \frac{s^*}{2\sigma}$. Then the operator $\mathcal{L}_v = \sigma\partial_\xi^2 + s^*\partial_\xi + g(U_p(\xi), 0)$ restricted to L^2_d is isomorphic to the operator $H_\sigma : L^2(\mathbb{R}) \to L^2(\mathbb{R})$, where

$$H_\sigma := \sigma\partial_\xi^2 + \left(-\frac{(s^*)^2}{4\sigma} + g(U_p(\xi), 0)\right).$$

We now state our assumptions on the spectrum of H_σ.

Hypothesis (H3) *We suppose that the most unstable spectra of H_σ is point spectra and define*

$$\lambda(\sigma) = \sup_{\omega \in \mathrm{spec}(H_\sigma)} \omega.$$

Let σ^* be defined such that $\lambda(\sigma^*) = 0$. Associated to this eigenvalue is a bounded eigenfunction which we denote $\tilde{\phi}(\xi)$. In the unweighted space, this eigenfunction becomes $\phi(\xi) = e^{-\frac{s^*}{2\sigma^*}\xi}\tilde{\phi}(\xi)$ which is unbounded as $\xi \to -\infty$. We further assume that $G_v(u, 0) = g(u, 0) > 0$ for all $u \in [0, u^+]$ such that $\phi'(\xi) < 0$ for all ξ.

We will require some properties of the eigenvalues of the linearization of $\mathbf{P}_0$ and $\mathbf{P}_1$ in a neighborhood of the critical parameter values (s^*, σ^*). These are outlined next.

Hypothesis (H4) *The eigenvalues of the linearization of (4.2.2) at* $\mathbf{P}_0$ *has four unstable eigenvalues. We assume for some open neighborhood of parameter space including* (s^*, σ^*) *that there exists an* $\alpha > 0$ *such that*

$$\mu_u^-(s) < -\alpha < \mu_u^+(s), \quad \mu_v^-(s,\sigma) < -\alpha < \mu_v^+(s,\sigma). \tag{4.2.5}$$

The fixed point $\mathbf{P}_1$ *is a saddle point of (4.2.2) with a* $3:1$ *splitting of the eigenvalues. We assume that the eigenvalues of the linearization at* $\mathbf{P}_1$ *can be ordered*

$$\nu_v^-(s,\sigma) < \nu_u^-(s) < \nu_v^+(s,\sigma) < 0 < \nu_u^+(s), \tag{4.2.6}$$

again for for some open set of parameters including (s^*, σ^*)*. In addition, we assume the following condition on the ratio of the eigenvalues:*

$$\nu_u^-(s) < 2\nu_v^+(s,\sigma). \tag{4.2.7}$$

The eigenvalue splitting (4.2.5) in Hypothesis **(H4)** guarantees the existence of a two dimensional strong stable manifold which we denote $W^{ss}(\mathbf{P}_0)$. Initial conditions in $W^{ss}(\mathbf{P}_0)$ correspond to solutions of (4.2.2) that decay to $\mathbf{P}_0$ with exponential rate greater than $e^{-\alpha\xi}$ at $\xi = +\infty$.

The final set of assumptions pertain to the existence and character of traveling front solutions connecting $\mathbf{P}_2$ to $\mathbf{P}_1$.

Hypothesis (H5) *We assume a transverse intersection of the unstable manifold* $W^u(\mathbf{P}_2)$ *and stable manifold* $W^s(\mathbf{P}_1)$ *for all* (s,σ) *in a neighborhood of* (s^*, σ^*)*. For* (s^*, σ^*) *we assume the existence of a heteroclinic connection between* $\mathbf{P}_2$ *and* $\mathbf{P}_1$ *that approaches* $\mathbf{P}_1$ *tangent to the weak-stable eigenspace corresponding to the eigenvalue* $\nu_v^+(s^*, \sigma*)$*, see (4.2.6). Thus, the two dimensional tangent space of* $W^u(\mathbf{P}_2)$ *enters a neighborhood of* $\mathbf{P}_1$ *approximately tangent to the unstable/weak-stable manifold of* $\mathbf{P}_1$*.*

In terms of PDE assumptions, **(H5)** is consistent with a staged invasion process where compactly supported perturbations of the steady state $\mathbf{p}_1$ form a traveling front propagating with speed $s < s^*$ replacing the unstable state $\mathbf{p}_1$ with the stable state $\mathbf{p}_2$. Since the selected invasion speed of fronts propagating into the state $\mathbf{p}_1$ is slower than s^*, any traveling front solution with speed s^* should be *pointwise stable* which requires that they converge to $\mathbf{p}_1$ with weak exponential decay precluding the existence of a marginally stable translational eigenvalue.

Remarks on assumptions (H1)-(H5). We remark that **(H1)** and **(H4)** are straightforward to verify for a specific choice of $F(u,v)$ and $G(u,v)$. Assumption **(H2)** is more challenging, but due to the planar nature of the traveling wave equation it is plausible that such a condition could be checked in practice. We refer the reader to [153] for a general variational method suited to such problems. Assumption **(H3)** is yet more challenging to verify, however as a Sturm-Liouville operator there are many results in the literature pertaining to qualitative features of the spectrum of these operators. Finally, assumption **(H5)** is the most difficult to verify in practice, as it requires a rather complete analysis of a fully four dimensional system of differential equations (4.2.2). Nonetheless, our assumptions there simply state that the traveling front solutions have the most generic behavior possible as heteroclinic orbits between $\mathbf{P}_2$ and $\mathbf{P}_1$. In this sense, we argue that assumption **(H5)** is not so extreme, in spite of the challenge presented in actually verifying that it would hold in specific examples. We also remark that the precise ordering of the eigenvalues assumed in **(H4)** are technical assumptions and could likely be relaxed in some cases.

4.2.3 Main result and numerical illustrations

We can now state our main result.

Theorem 4.2.1. *Consider (4.2.1) and assume that Hypotheses **(H1)-(H5)** hold. Then there exists a constant M_ρ such that:*

- *(sub-critical) if $M_\rho < 0$ then there exists $\delta > 0$ such that there exists positive traveling front solutions $(U(x-s(\sigma)t), V(x-s(\sigma)t))$ for any $\sigma^* - \delta < \sigma < \sigma^*$ with speed*

$$s(\sigma) = s^* + M_s(\sigma - \sigma^*)^2 + \mathcal{O}(3);$$

- *(super-critical) if $M_\rho > 0$ then there exists $\delta > 0$ such that there exists positive traveling front solutions $(U(x-s(\sigma)t), V(x-s(\sigma)t))$ for any $\sigma^* < \sigma < \sigma^* + \delta$ with speed*

$$s(\sigma) = s^* + M_s(\sigma - \sigma^*)^2 + \mathcal{O}(3).$$

These traveling fronts belong to the intersection of the unstable manifold $W^u(\mathbf{P}_2)$ and the strong stable manifold $W^{ss}(\mathbf{P}_0)$.

First, we make several remarks.

Remark 1. As part of the proof of Theorem 4.2.1 we obtain expressions for M_ρ and M_s. In particular,

$$\begin{aligned}\operatorname{sign}(M_\rho) &= \operatorname{sign}\left(-r_2\int_{\xi_0}^{\infty} e^{\frac{s^*}{\sigma^*}\xi}\left(\frac{G_{uv}(U_p(\xi),0)}{\sigma^*}a_1(\xi)\phi(\xi)^2 + \frac{G_{vv}(U_p(\xi),0)}{2\sigma^*}\phi^3(\xi)\right)\mathrm{d}\xi\right.\\ &\quad - \left. r_1\left(\tilde{\phi}''(\xi_0)\tilde{\phi}(\xi_0) - (\tilde{\phi}'(\xi_0))^2\right) + \frac{1}{r_2}e^{\frac{s^*}{\sigma^*}\xi_0}\gamma^{(2)}(s^*,\sigma^*)\left(\nu_v^-(s^*,\sigma^*)\phi(\xi_0) - \phi'(\xi_0)\right)\right),\end{aligned}$$

where $r_{1,2}$, $a_1(\xi)$ and $\gamma^{(2)}(s^*,\sigma^*)$ are all explicitly characterized. A similar expression holds for M_s, but is quite complicated.

Remark 2. We comment on the sub-critical case. Our analysis holds only in a neighborhood of the bifurcation point. However, we expect that this curve could be followed in (s,σ) parameter space to a saddle-node bifurcation where the curve would subsequently reverse direction with respect to σ. This curve can be found numerically using numerical continuation methods, see Figure 4.7. These numerics reveal two branches of fronts that appear via a saddle node bifurcation. It is the lower branch of solutions that appear to be marginally stable and reflect the invasion speed of the system.

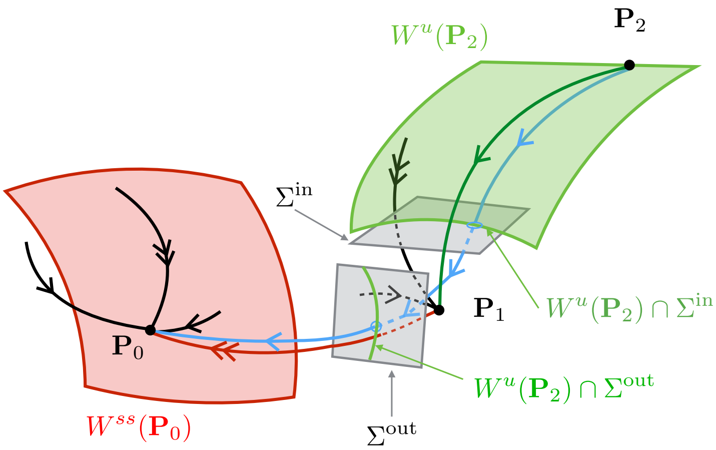


Figure 4.5: Geometrical illustration in $\mathbb{R}^4$ of the construction of locked fronts. Locked fronts are heteroclinic orbits connecting $\mathbf{P}_2$ to $\mathbf{P}_0$ that lie at the intersection of the unstable manifold $W^u(\mathbf{P}_2)$ and the strong stable manifold $W^{ss}(\mathbf{P}_0)$. We track $W^{ss}(\mathbf{P}_0)$ backwards along the pushed front heteroclinic $(U_p(\xi), U_p'(\xi), 0, 0)^T$, represented by the dark red heteroclinic orbit on the figure, to a neighborhood of $\mathbf{P}_1$ and track $W^u(\mathbf{P}_2)$ forwards past the fixed point $\mathbf{P}_1$ from Σ^{in} to Σ^{out} to compare the two manifolds near a common point on the heteroclinic $(U_p(\xi), U_p'(\xi), 0, 0)^T$ in Σ^{out}. In the figure, we represented in dark green one heteroclinic orbit connecting $\mathbf{P}_2$ to $\mathbf{P}_1$ within $W^u(\mathbf{P}_2)$. Schematically, the locked front, represented by the blue heteroclinic orbit on the figure, is found to be close to the concatenation of the two heteroclinic orbits connecting first $\mathbf{P}_2$ to $\mathbf{P}_1$ (dark green) and then $\mathbf{P}_1$ to $\mathbf{P}_0$ (dark red). In that respect, our strategy of proof is a variation of Lin's method.

For systems of equations without a comparison principle, the selected front is classified as the marginal stable front, see [57, 190]. It is interesting to note that in these examples there appear to be two marginally (spectrally) stable fronts – the original front $(U_p(x - s^*t), 0)$ and the coexistence front – and the full system selects the slower of these two fronts.

Sketch of the proof. We now comment on the strategy of the proof that employs a variation of Lin's method; see Figure 4.5 for a geometrical illustration of our dynamical systems approach. The traveling fronts that we seek are heteroclinic orbits in the traveling wave equations connecting $\mathbf{P}_2$ to $\mathbf{P}_0$. We further require that these fronts have strong exponential decay in a neighborhood of $\mathbf{P}_0$. As such, these traveling fronts belong to the intersection of the unstable manifold $W^u(\mathbf{P}_2)$ and the strong stable manifold $W^{ss}(\mathbf{P}_0)$. Therefore, the goal is to track $W^{ss}(\mathbf{P}_0)$ backwards along the pushed front heteroclinic $(U_p(\xi), U_p'(\xi), 0, 0)^T$ to a neighborhood of $\mathbf{P}_1$. The dependence of this manifold on the parameters s and σ can be characterized using Melnikov type integrals and the manifold can be expressed as a graph over the strong stable tangent space. To track $W^u(\mathbf{P}_2)$ forwards we use **(H5)** to get an expression for this manifold as it enters a neighborhood of $\mathbf{P}_1$. To track this manifold past the fixed point requires a Shilnikov type analysis near $\mathbf{P}_1$. Finally, we compare the two manifolds near a common point on the hete-

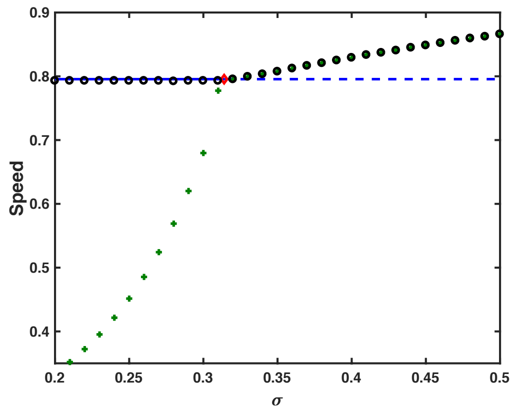


Figure 4.6: Numerically computed wave speeds of the u-component, black circles, and of the v-component, green plus sign for $\epsilon = 1$ in (4.2.8). The horizontal blue line $s = s^* = \sqrt{2}(a + 1/2)$ represents the sign of the associated principal eigenvalue of the operator H_σ and the red diamond indicates the critical value σ^* at which this principal eigenvalue vanishes. The solid part of the line indicates a negative principal eigenvalue while the dashed part indicates a positive one. Here, $\sigma^* \simeq 0.314$. For all numerical simulations we have set $a = 1/16$.

roclinic $(U_p(\xi), U_p'(\xi), 0, 0)^T$ and following a Liapunov-Schmidt reduction we obtain the required expansions of s as a function of σ.

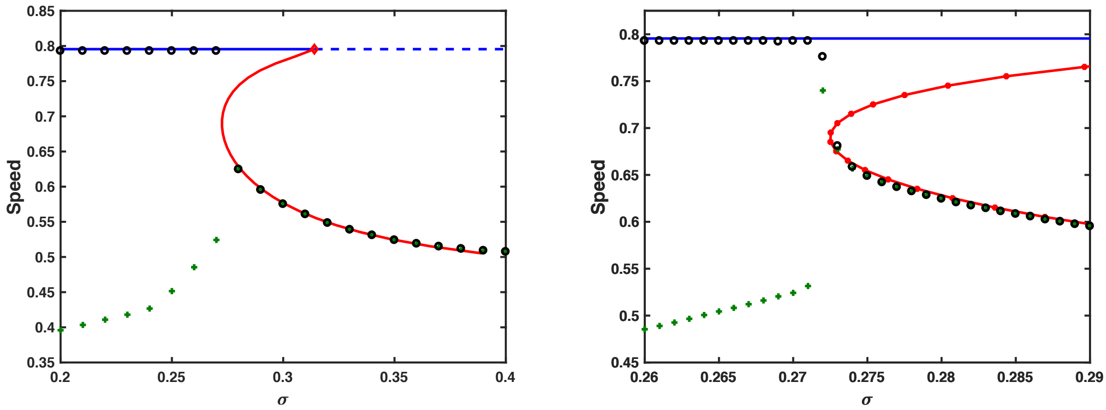
Numerical illustration of the main result. To conclude, we illustrate the main result on an example. We consider the following nonlinear functions $f_\epsilon(u, v)$ and $g(u, v)$ that lead to a supercritical bifurcation when $\epsilon = 1$ and exhibit a sub-critical bifurcation for $\epsilon = -1$:

$$f_\epsilon(u, v) = (1 - u)(u + a) + \epsilon v, \text{ and } \quad g(u, v) = 2u(1 - u) + 2a - v, \tag{4.2.8}$$

where $\epsilon \in \{\pm 1\}$. In both cases, when v is set to zero the system reduces to the scalar Nagumo's equation

$$\partial_t u = \partial_x^2 u + u(1 - u)(u + a). \tag{4.2.9}$$

The dynamics of (4.2.9) are well understood, see for example [91]. For $a < 1/2$, the system forms a pushed front propagating with speed $s^* = \sqrt{2}\left(\frac{1}{2} + a\right)$. For the numerical computations presented in both Figures 4.6 and 4.7, we have discretized (4.2.1) by the method of finite differences and used a semi-implicit scheme with time step $\delta t = 0.05$ and space discretization $\delta x = 0.05$ with $x \in [0, 400]$ and imposed Neumann boundary conditions. All simulations are done from compactly initial data and the speed of each component was calculated by computing how much time elapsed between the solution surpassing a threshold at two separate points in the spatial domain. In Figure 4.6, we present the case of a super-critical bifurcation where locked fronts are shown to exist past the bifurcation point $\sigma = \sigma^*$. In Figure 4.7, we illustrate the case of a sub-critical bifurcation where locked fronts are shown to exist before the bifurcation point $\sigma = \sigma^*$. We observe a discontinuity of the wave speed as σ is increased. We then implemented a numerical continuation scheme to continue the wave speed of these locked fronts back to the bifurcation point $\sigma = \sigma^*$. In the process, we see a turning point for some value of σ near 0.273.



(a) $\epsilon = -1$.

(b) Zoom of Figure (a) near $\sigma \sim 0.27$.

Figure 4.7: (a) Numerically computed wave speeds of the u-component, black circles, and of the v-component, green plus sign for $\epsilon = -1$ in (4.2.8). We observe a discontinuity in the value of the measured wave speed as σ is varied indicating a sub-critical bifurcation of the locked fronts. The horizontal blue line $s = s^* = \sqrt{2}(a + 1/2)$ represents the sign of the associated principal eigenvalue of the operator H_σ and the red diamond indicates the critical value σ^* at which this principal eigenvalue vanishes. The solid part of the line indicates a negative principal eigenvalue while the dashed part indicates a positive one. Here, $\sigma^* \simeq 0.314$. The red curve is a continuation of the wave speed of locked fronts up to the bifurcation point $\sigma = \sigma^*$. (b) Refinement of Figure (a) near the fold point. Here, the red dots are wave speeds obtained by numerical continuation. For all numerical simulations we have set $a = 1/16$.

We expect that locked fronts on this branch to be unstable as solutions of (4.2.1) which explains why one observes the lower branch of the bifurcation curve. It is interesting to note the relative good agreement between the wave speed obtained by numerical continuation and the wave speed obtained by direct numerical simulation of the system (4.2.1).

4.3 Some stability results of fronts in reaction-diffusion equations

In this section, we investigate two types of stability results of fronts in reaction-diffusion equations. The first result is about the asymptotic stability of the critical pulled front of the scalar Fisher-KPP equation, and the second result investigates the asymptotic stability of planar traveling fronts in nonlocal reaction-diffusion equations with bistable dynamics. By planar front, we refer to as a traveling front in one direction and constant along the transverse directions. Both results rely on pointwise semigroup estimates and in both cases the boundary of the essential spectrum touches the imaginary axis resulting in an algebraic decay of the perturbations in some well-chosen functional spaces.

4.3.1 Asymptotic stability of the critical Fisher-KPP front using pointwise estimates

We revisit the asymptotic stability analysis of Gallay [97] for the critical Fisher-KPP front of the following scalar parabolic equation

$$\partial_t u = \partial_x^2 u + c\partial_x u + f(u), \quad t > 0, \quad x \in \mathbb{R}, \tag{4.3.1}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a $\mathscr{C}^2$ map satisfying $f(0) = f(1) = 0$, $f'(0) > 0$, $f'(1) < 0$ and $f''(u) < 0$ for all $u \in (0,1)$ and $c > 0$. For such an example, it is well known that for any wavespeed $c \geq 2\sqrt{f'(0)} := c_*$, there exist monotone traveling front solutions $q(x)$ connecting $u = 1$ at $-\infty$ and $u = 0$ at $+\infty$ where the front profile q is solution of the second order ODE

$$0 = q'' + cq' + f(q). \tag{4.3.2}$$

The stability of traveling fronts for the Fisher-KPP equation has been studied by many authors. For the super-critical family of fronts propagating with speeds $c > c_*$, stability was established by Sattinger using exponential weights to stabilize the essential spectrum and yield exponential in time stability; see [181]. Stability of the critical front pulled q_* was established by [141], with extensions and refinements achieved in [35, 64, 97]. The sharpest of these results for the Fisher-KPP equation is [97], where perturbations of the critical front are shown to converge in an exponentially weighted L^∞ space with algebraic rate $t^{-3/2}$. Of course, we also mention that strong results concerning the convergence of compactly supported initial data to traveling fronts are possible for (4.3.1) using comparison principle techniques; see for example [9].

The primary challenge presented by the critical front is that it is not possible to stabilize the essential spectrum using exponential weights, see Figure 4.8 for an illustration. This is due to the presence of absolute spectrum at $\lambda = 0$ in the form of a branch point of the dispersion relation of the asymptotic system near $+\infty$. The presence of continuous spectrum near the origin suggests algebraic decay and one might further anticipate heat kernel type decay of perturbations. As we note above, perturbations of the critical front are known to converge slightly faster – in an exponentially weighted L^∞ space with algebraic rate $t^{-3/2}$; see [97].

Our approach is similar to that of [181] where the linear eigenvalue problem is studied in an exponentially weighted space and resolvent estimates are obtained via inverse Laplace transform. For the super-critical fronts studied in [181] the Laplace inversion contours can be placed in the stable half plane thereby simplifying the analysis. No such extension is possible here and we instead approach the problem using pointwise semigroup estimates. Pointwise semigroup methods were introduced by Zumbrun and Howard [204] and have been developed over the past several decades to address stability problems where the essential spectrum can not be separated from the imaginary axis. Applications include stability of viscous shock waves; see [122, 204], stability and instability of spatially periodic patterns; see [130], stability of defects in reaction-diffusion equations; see [19], and more recently stability of stationary reaction-diffusion fronts; see [149], to mention a few.

A rough outline of our approach is as follows.

- Write the solutions of (4.3.1) as $u(t,x) = q_*(x) + v(t,x)$, and obtain the following equation for the perturbation $v(t,x)$:

$$\partial_t v = \partial_x^2 v + c_* \partial_x v + f'(q_*)v + f(q_* + v) - f(q_*) - f'(q_*)v. \tag{4.3.3}$$

- For some well chosen smooth weight function $\omega(x) > 0$ (see (4.3.10)), perform a change of variable of the form $v(t,x) = \omega(x)p(t,x)$, where p now satisfies

$$\partial_t p = \partial_x^2 p + \left(c_* + 2\frac{\omega'}{\omega}\right)\partial_x p + \left(f'(q_*) + c_*\frac{\omega'}{\omega} + \frac{\omega''}{\omega}\right)p + \mathcal{N}(q_*, \omega p)p, \tag{4.3.4}$$

with nonlinear terms

$$\mathcal{N}(\mu,\nu) := \frac{1}{\nu}\left(f(\mu+\nu) - f(\mu) - f'(\mu)\nu\right),$$

and $\mathcal{L}$ denotes the linear operator

$$\mathcal{L}p := p_{xx} + \left(c_* + 2\frac{\omega'}{\omega}\right)p_x + \left(f'(q_*) + c_*\frac{\omega'}{\omega} + \frac{\omega''}{\omega}\right)p, \tag{4.3.5}$$

with dense domain $H^2(\mathbb{R})$ in $L^2(\mathbb{R})$.

- Construct bounded solutions $\varphi^\pm(x)$ for the eigenvalue problem $\mathcal{L}p = \lambda p$ on $\mathbb{R}^\pm$ where $\mathcal{L}$ given in (4.3.5).

- Find bounds for the pointwise Green's function,

$$\mathbf{G}_\lambda(x,y) = \begin{cases} \dfrac{\varphi^+(x)\varphi^-(y)}{\mathbb{W}_\lambda(y)}, & x \geq y, \\[2ex] \dfrac{\varphi^-(x)\varphi^+(y)}{\mathbb{W}_\lambda(y)}, & x \leq y, \end{cases}$$

where the Wronskian $\mathbb{W}_\lambda(y) := \varphi^+(y){\varphi^-}'(y) - {\varphi^+}'(y)\varphi^-(y)$, is often referred to as the Evans function; see [2].

- Apply the inverse Laplace transform, and by a suitable choice of inversion contour show that the Green's function

$$\mathbf{G}(t,x,y) = \frac{1}{2\pi\mathbf{i}}\int_\Gamma e^{\lambda t}\mathbf{G}_\lambda(x,y)\mathrm{d}\lambda, \tag{4.3.6}$$

decays pointwise with algebraic rate $t^{-3/2}$. Actually, we prove that for some constants $\kappa > 0$, $r > 0$ and $C > 0$, the Green's function $\mathbf{G}(t,x,y)$ satisfies the following estimates.

(i) For $|x-y| \geq Kt$ or $t < 1$, with K sufficiently large,

$$|\mathbf{G}(t,x,y)| \leq C\frac{1}{t^{1/2}}e^{-\frac{|x-y|^2}{\kappa t}}.$$

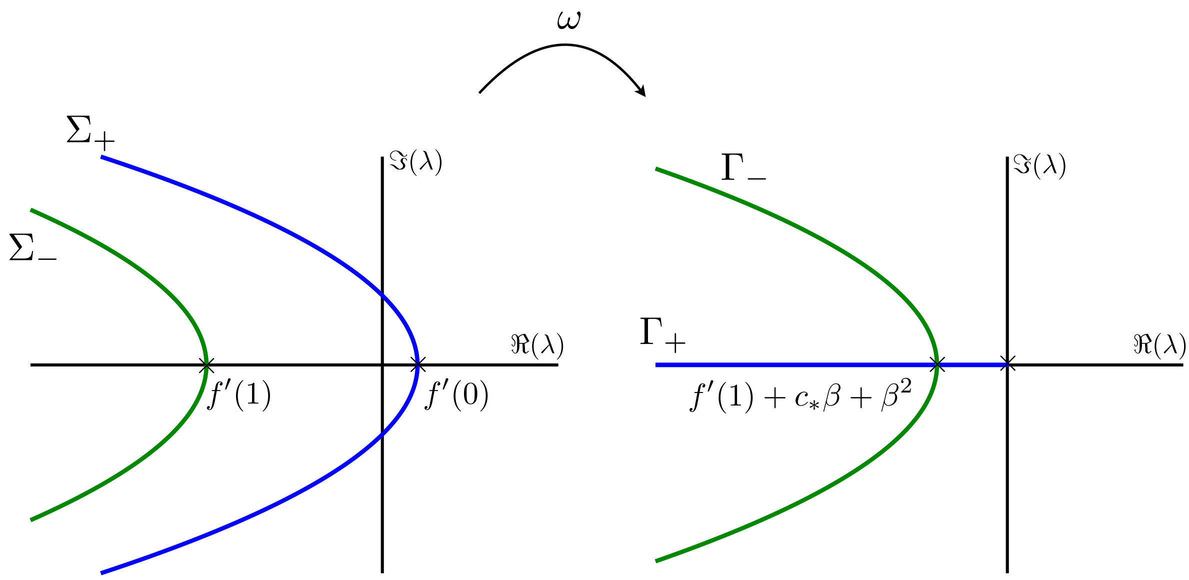


Figure 4.8: Illustration of the action of the weight function ω on the boundaries $\Sigma_\pm$ of the essential spectrum of the linearized equation around the critical traveling front solution q_*. Note that Σ_- is mapped to Γ_- while Σ_+ is mapped to the negative real axis Γ_+ inclusive of the point at 0.

(ii) For $|x-y| \leq Kt$ and $t \geq 1$, with K as above,

$$|\mathbf{G}(t,x,y)| \leq C\left(\frac{1+|x-y|}{t^{3/2}}\right) e^{-\frac{|x-y|^2}{\kappa t}} + Ce^{-rt}.$$

- Deduce from the above inequalities that for $t < 1$ the usual short time estimate holds,

$$\left|\int_{\mathbb{R}} \mathbf{G}(t,x,y)h(y)\mathrm{d}y\right| \leq C\|h\|_{L^\infty}, \tag{4.3.7}$$

and for $t \geq 1$, the large time estimate is

$$\left|\int_{\mathbb{R}} \mathbf{G}(t,x,y)h(y)\mathrm{d}y\right| \leq C\frac{1+|x|}{(1+t)^{3/2}} \int_{\mathbb{R}} (1+|y|)|h(y)|\mathrm{d}y. \tag{4.3.8}$$

- Apply estimates (4.3.7) and (4.3.8) to the nonlinear solution expressed using Duhamel's formula,

$$p(t,x) = \int_{\mathbb{R}} \mathbf{G}(t,x,y)p_0(y)\mathrm{d}y + \int_0^t \int_{\mathbb{R}} \mathbf{G}(t-\tau,x,y)\omega(y)\mathcal{N}(q_*(y),\omega(y)p(\tau,y))p(\tau,y)\mathrm{d}y\mathrm{d}\tau,$$

to show that the nonlinear system also exhibits the same algebraic decay rate.

This approach is motivated by the observation in [179] that the faster algebraic decay rate is a consequence of the lack of an embedded zero of the Evans function at $\lambda = 0$ (the analytic extension of the Evans function to the branch point is possible due to the Gap Lemma; see [100, 134]). Indeed, the critical front has weak exponential decay near $x = +\infty$,

$$q_*(x) \underset{+\infty}{\sim} bxe^{-\gamma_* x}, \tag{4.3.9}$$

where $\gamma_* := c_*/2$ and for some $b > 0$. This weak exponential decay implies that the derivative of the wave also has weak exponential decay and therefore does not lead to a zero of $\mathbb{W}_\lambda(y)$ at $\lambda = 0$. In the outline of our argument, this fact comes into play when we require bounds on the supremum of $\mathbf{G}(t, x, y)$. We find that this quantity is dominated by a region where $\mathbf{G}_\lambda(x, y)$ resembles $Ce^{-\sqrt{\lambda}(x-y)}$. Of course, this is exactly the Laplace transform of the derivative of the heat kernel; from which we naturally expect algebraic decay with rate $t^{-3/2}$.

We now state our main result. Let $\omega(x) > 0$ be a positive, bounded, smooth weight function of the form

$$\omega(x) = \begin{cases} e^{-\gamma_* x} & x \geq 1, \\ e^{\beta x} & x \leq -1, \end{cases} \tag{4.3.10}$$

for some $0 < \beta < -\frac{c_*}{2} + \sqrt{\frac{c_*^2}{4} - f'(1)}$.

Theorem 4.3.1. *Consider (4.3.1) with initial data $u(0, x) = q_*(x) + v_0(x)$ satisfying $0 \leq u(0, x) \leq 1$. There exist $C > 0$ and $\epsilon > 0$ such that if $v_0(x)$ satisfies,*

$$\left|\left|\frac{v_0(\cdot)}{\omega(\cdot)}\right|\right|_{L^\infty} + \left|\left|(1 + |\cdot|)\frac{v_0(\cdot)}{\omega(\cdot)}\right|\right|_{L^1} < \epsilon,$$

then the solution $u(t, x)$ is defined for all time and the critical front is nonlinearly stable in the sense that

$$\left|\left|\frac{1}{(1 + |\cdot|)}\frac{v(t, \cdot)}{\omega(\cdot)}\right|\right|_{L^\infty} \leq \frac{C\epsilon}{(1 + t)^{3/2}}, \quad t > 0,$$

where $v(t, x) := u(t, x) - q_*(x)$.

Theorem 4.3.1 recovers the sharp algebraic in time L^∞ decay rate of perturbations of the critical front that was obtained in [97]. The proof in [97] uses as a weight the derivative of the front profile. In this weighted space, the linearized operator as $x \to \infty$ is equivalent to the radial Laplacian in three dimensions; for which the fundamental solution possesses algebraic decay rate $t^{-3/2}$. The nonlinear argument relies on scaling variables and the application of renormalization group techniques. In comparing Theorem 4.3.1 to the main result in [97] we note small differences in the spatial decay rates of the allowable perturbations and note that the result in [97] is stronger than the one presented here in that the author is able to identify an asymptotic profile for the solution in addition to its decay rate.

The main novel contribution of our study is to present an alternative proof based upon pointwise semigroup methods and make rigorous the observation in [179] that the faster algebraic decay rate is a consequence of the lack of an embedded zero of the Evans function at $\lambda = 0$. We contend that the proof of Theorem 4.3.1 presented here is more elementary than that of [97] as it relies on (rather coarse) ODE estimates, contour integration and a standard nonlinear stability argument avoiding the technical PDE estimates and renormalization group theory of [97]. Furthermore, this alternative method paves the way to tackle a broader class of problems. For example, one could consider the extended Fisher-KPP equation

$$\partial_t u = -\gamma \partial_x^4 u + \partial_x^2 u + f(u), \quad t > 0, \quad x \in \mathbb{R}, \tag{4.3.11}$$

where $\gamma > 0$ is a small parameter and $f(u)$ is as in (4.3.1). For such an equation, there exists a family of fronts with wavespeed $c \geq c_*(\gamma)$, in the limit $\gamma \to 0$, which were shown to be stable in exponentially weighted spaces [172]. It could be possible to adapt the above ideas to prove that the critical front decays algebraically with rate $t^{-3/2}$ in a weighted L^∞ space. The calculations in that case are more involved (a four-dimensional system of ODEs), but the general key ingredients remain unchanged. Along similar lines, the approach developed in this paper could be used to establish precise stability results for pulled invasion fronts in systems of reaction-diffusion equations. For example, refinements of the stability results in [93, 170] may be achievable.

From the perspective of pointwise semigroup methods, the application here is fairly straightforward. To reinforce the discussion in the preceding paragraph, we regard this relative simplicity to be a strength of this study. One mathematical feature of interest is the presence of the branch point at the origin which prevents the continuation of any contour integrals into the left half of the complex plane. An important reference in this regard is Howard [122] where a marginally stable branch point also arises when considering the stability of degenerate viscous shock waves. The Fisher-KPP equation being studied here is quite different, but considerable similarities remain between the approach taken here and the one in [122].

4.3.2 Multidimensional stability of planar traveling fronts for bistable nonlocal reaction-diffusion equations

We consider the scalar nonlocal reaction-diffusion equation (1.0.3)

$$\begin{aligned} \partial_t u(t,\mathbf{x}) &= -u(t,\mathbf{x}) + \int_{\mathbb{R}^n} \mathcal{K}(\mathbf{x}-\mathbf{y})u(t,\mathbf{y})\mathrm{d}\mathbf{y} + f(u(t,\mathbf{x})) \\ &:= -u(t,\mathbf{x}) + \mathcal{K} *_{\mathbf{x}} u(t,\mathbf{x}) + f(u(t,\mathbf{x})) \end{aligned} \tag{4.3.12}$$

where $u \in \mathbb{R}$, $(t,\mathbf{x}) \in \mathbb{R}^+ \times \mathbb{R}^n$ and f is a smooth function of bistable type with three zeros, 0, 1 and $a \in (0,1)$. As usual, a prototypical example for f is the cubic nonlinearity of form $f(u) = u(1-u)(u-a)$. Here $\mathcal{K} \in L^1(\mathbb{R})$ is a nonnegative function with $\int_{\mathbb{R}^n} \mathcal{K}(\mathbf{x})\mathrm{d}\mathbf{x} = 1$ and that is even with respect to each variable. A planar traveling wave $\varphi(\xi)$ is a smooth function of the variable $\xi = \mathbf{e}\cdot\mathbf{x} - ct$, for $\mathbf{e} \in \mathbb{S}^{n-1}$ and some $c \in \mathbb{R}$, which is a solution of (4.3.12) satisfying the limits $\lim\limits_{\xi\to-\infty} \varphi(\xi) = 1$ and $\lim\limits_{\xi\to+\infty} \varphi(\xi) = 0$. Without loss of generality, we suppose that $\mathbf{e} = (1,0,\dots,0)$. In the moving frame $\mathbf{x} = (\xi,\mathbf{z}) \in \mathbb{R}\times\mathbb{R}^{n-1}$, equation (4.3.12) can be written as

$$\partial_t u(t,\mathbf{x}) - c\partial_\xi u(t,\mathbf{x}) = -u(t,\mathbf{x}) + \int_{\mathbb{R}^n} \mathcal{K}(\mathbf{x}-\mathbf{y})u(t,\mathbf{y})\mathrm{d}\mathbf{y} + f(u(t,\mathbf{x})) \tag{4.3.13}$$

such that the traveling wave $\varphi(\xi)$ is a stationary solution of (4.3.13). If we define $\mathcal{K}_0 : \mathbb{R} \to \mathbb{R}$ as

$$\mathcal{K}_0(\xi) = \int_{\mathbb{R}^{n-1}} \mathcal{K}(\xi,\mathbf{z})\mathrm{d}\mathbf{z} \tag{4.3.14}$$

then (φ, c) satisfies

$$-c\varphi'(\xi) = -\varphi(\xi) + \int_{\mathbb{R}} \mathcal{K}_0(\xi-\zeta)\varphi(\zeta)\mathrm{d}\zeta + f(\varphi(\xi)), \quad \lim_{\xi\to-\infty}\varphi(\xi) = 1 \text{ and } \lim_{\xi\to+\infty}\varphi(\xi) = 0, \tag{4.3.15}$$

where φ' stands for $\frac{\mathrm{d}\varphi}{\mathrm{d}\xi}$ and φ is decreasing.

Main assumptions. Throughout the paper, we will assume the following hypotheses for f and $\mathcal{K}$ which ensure the existence and uniqueness (modulo translation) of a solution (φ, c) to (4.3.15), see Theorem 1.2.5.

Hypothesis 4.3.1. *We suppose that the nonlinearity f satisfies the following properties:*

(i) $f \in \mathscr{C}^\infty(\mathbb{R})$;

(ii) $f(u) = 0$ *precisely when* $u \in \{0, a, 1\}$;

(iii) $f'(0) < 0$, $f'(1) < 0$ *and* $f'(a) > 0$.

Note that we only need $f \in \mathscr{C}^2(\mathbb{R})$ to obtain the existence result of [16] and here we require more regularity to obtain uniform bounds on the nonlinear terms in our stability analysis.

Hypothesis 4.3.2. *We suppose that the kernel $\mathcal{K}$ satisfies the following properties:*

(i) $\mathcal{K} \geq 0$, *is even with respect to each variable;*

(ii) $\mathcal{K} \in W^{1,1}(\mathbb{R}^n)$;

(iii) $\int_{\mathbb{R}^n} \mathcal{K}(\mathbf{x})\mathrm{d}\mathbf{x} = 1$, $\int_{\mathbb{R}^n} \|\mathbf{x}\|\mathcal{K}(\mathbf{x})\mathrm{d}\mathbf{x} < \infty$ *and* $\int_{\mathbb{R}^n} \|\mathbf{x}\|^2\mathcal{K}(\mathbf{x})\mathrm{d}\mathbf{x} < \infty$;

(iv) $\widehat{\mathcal{K}}(\mathbf{k}) = 1 - d_0\|\mathbf{k}\|^2 + o(\|\mathbf{k}\|^2)$ *as* $\mathbf{k} \to 0$ *with* $d_0 > 0$.

Here, $W^{k,p}(\mathbb{R}^n)$ denotes the Sobolev space with its usual norm and we use the notation $H^k(\mathbb{R}^n) := W^{k,2}(\mathbb{R}^n)$. The symbol $\widehat{\mathcal{K}}$ denotes the Fourier transform of $\mathcal{K}$ defined as

$$\widehat{\mathcal{K}}(\mathbf{k}) = \int_{\mathbb{R}^n} \mathcal{K}(\mathbf{x})e^{-\mathbf{i}\mathbf{k}\cdot\mathbf{x}}\mathrm{d}\mathbf{x}, \quad \mathbf{k} \in \mathbb{R}^n.$$

The first assumption is natural from a modeling point of view while the second and third assumptions are required to ensure the existence of traveling wave solution φ to equation (4.3.15). The third and forth assumptions also imply that

$$\forall j \in [\![1, n]\!] \quad \int_{\mathbb{R}^n} x_j\mathcal{K}(\mathbf{x})\mathrm{d}\mathbf{x} = 0 \text{ and } d_0 := \frac{1}{2n}\int_{\mathbb{R}^n} \|\mathbf{x}\|^2\mathcal{K}(\mathbf{x})\mathrm{d}\mathbf{x} > 0.$$

Furthermore, as $-1 + \widehat{\mathcal{K}}(\mathbf{k}) \sim -d_0\|\mathbf{k}\|^2$ for $\mathbf{k} \to 0$, in the long wavelength limit, the linear operator $u \mapsto -u + \mathcal{K} * u$ approaches the Laplacian $d_0\Delta_{\mathbb{R}^n}$ and we recover the classical Allen-Cahn equation. Remark that in the short wavelength limit we have $-1 + \widehat{\mathcal{K}}(\mathbf{k}) \sim -1$ for $\|\mathbf{k}\| \to \infty$ such that $u \mapsto -u + \mathcal{K} * u$ is a bounded operator which is a very different feature from the Laplacian. Note that with Hypothesis 4.3.2 for the kernel $\mathcal{K}$ we recover all the hypotheses of [16] for $\mathcal{K}_0$.

In this paper, we are concerned with determining the stability of the traveling wave φ. We are thus let to study the spectral properties of the linear operator $\mathcal{L}$

$$\begin{array}{rccl} \mathcal{L}: & H^1(\mathbb{R}^n) & \longrightarrow & L^2(\mathbb{R}^n) \\ & u & \longmapsto & -u + \mathcal{K} *_{\mathbf{x}} u + c\frac{\mathrm{d}}{\mathrm{d}\xi} + f'(\varphi)u. \end{array} \tag{4.3.16}$$

It is natural to assume that the wave φ is linearly stable in one space dimension to get stability in higher in space dimensions. In fact, it is consequence of Hypotheses 4.3.1 and 4.3.2 on f and $\mathcal{K}$. First, define the linear operator $\mathcal{L}_0$ associated to equation (4.3.15)

$$\begin{array}{rccl} \mathcal{L}_0: & H^1(\mathbb{R}) & \longrightarrow & L^2(\mathbb{R}) \\ & u & \longmapsto & -u+\mathcal{K}_0 *_\xi u + c\frac{\mathrm{d}}{\mathrm{d}\xi} + f'(\varphi)u. \end{array} \tag{4.3.17}$$

and its adjoint operator $\mathcal{L}_0^*$

$$\begin{array}{rccl} \mathcal{L}_0^*: & \mathcal{D}(\mathcal{L}_0^*) \subset L^2(\mathbb{R}) & \longrightarrow & L^2(\mathbb{R}) \\ & u & \longmapsto & -u+\mathcal{K}_0 *_\xi u - c\frac{\mathrm{d}}{\mathrm{d}\xi} + f'(\varphi)u. \end{array} \tag{4.3.18}$$

Lemma 4.3.1 ([12, 44]). *Suppose that Hypotheses 4.3.1 and 4.3.2 are satisfied, then*

(i) 0 is an algebraic simple eigenvalue of $\mathcal{L}_0$ with negative eigenfunction φ';

(ii) there exists $\gamma_0 > 0$ such that $\sigma_{ess}(\mathcal{L}_0) \subset \{\lambda \mid |\Re(\lambda)| < -\gamma_0\}$;

(iii) there exists a unique negative solution $\psi \in H^1(\mathbb{R})$ which solves $\mathcal{L}_0^*\psi = 0$ with

$$\int_{\mathbb{R}} \varphi'(\xi)\psi(\xi)\mathrm{d}\xi = 1.$$

Since the eigenvalue zero is isolated, there exists a spectral projection operator, $\mathcal{P}$, onto the null space of $\mathcal{L}_0$ given by

$$\mathcal{P}u = \frac{1}{2\pi\mathbf{i}}\int_\Gamma (\mathcal{L}_0 - \lambda)^{-1} u \mathrm{d}\lambda, \tag{4.3.19}$$

where Γ is a simple closed curve in the complex plane enclosing the zero eigenvalue. If $\langle\cdot,\cdot\rangle$ denotes the scalar product on $L^2(\mathbb{R})$ then we can write $\mathcal{P}$ as

$$\mathcal{P}u(\xi,\mathbf{z}) = \langle\psi,u\rangle(\mathbf{z})\varphi'(\xi) := \left(\int_{\mathbb{R}} \psi(\xi)u(\xi,\mathbf{z})\mathrm{d}\xi\right)\varphi'(\xi). \tag{4.3.20}$$

We define the operator $\mathcal{Q}$ as $\mathcal{Q}u := u - \mathcal{P}u$.

Main result. We can now state our main result. The perturbation of the wave will be written as

$$u(t,\mathbf{x}) := \varphi(\xi - \rho(t,\mathbf{z})) + v(t,\xi-\rho(t,\mathbf{z}),\mathbf{z}) \tag{4.3.21}$$

where $\rho : \mathbb{R}^{n-1} \to \mathbb{R} \in H^k(\mathbb{R}^{n-1})$ and $v : \mathbb{R}^n \to \mathbb{R} \in H^k(\mathbb{R}^n)$ is in the range of the operator $\mathcal{L}_0$ that is $\mathcal{P}v = 0$. And we set

$$\mathcal{E}_0 := \|v_0\|_{W^{1,1}(\mathbb{R}^n)} + \|v_0\|_{H^k(\mathbb{R}^n)} + \|\rho_0\|_{W^{1,1}(\mathbb{R}^{n-1})} + \|\rho_0\|_{H^{k+1}(\mathbb{R}^{n-1})}.$$

Theorem 4.3.2. *Let $n \geq 2$ and $k \geq \left[\frac{n+1}{2}\right]$. Suppose that Hypotheses 4.3.1 and 4.3.2 are satisfied. There exists $C > 0$ such that if $\mathcal{E}_0$ is sufficiently small, then the traveling wave solution*

φ of equation (4.3.13) is stable in the sense that the perturbation (ρ, v) given in (4.3.21) satisfies the decay estimates for all $t \geq 0$

$$\|v(t)\|_{H^k(\mathbb{R}^n)} \leq C(1+t)^{-\frac{n-1}{4}-1}\mathcal{E}_0, \tag{4.3.22a}$$

$$\|\rho(t)\|_{H^k(\mathbb{R}^{n-1})} \leq C(1+t)^{-\frac{n-1}{4}}\mathcal{E}_0, \tag{4.3.22b}$$

$$\|\nabla_{\mathbf{z}} \cdot \rho(t)\|_{H^k(\mathbb{R}^{n-1})} \leq C(1+t)^{-\frac{n+1}{4}}\mathcal{E}_0, \tag{4.3.22c}$$

where $\nabla_{\mathbf{z}} = (\partial_{x_2}, \cdots, \partial_{x_n})$.

Note that Theorem 4.3.2 is well known in the case of local diffusion, namely when the nonlocal term $-u + \mathcal{K} *_{\mathbf{x}} u$ in equation (4.3.12) is replaced by the standard Laplacian $\Delta = \sum_{i=1}^{n} \partial_{x_i}^2$ on $\mathbb{R}^n$. Xin [200] was the first to prove these results in dimension $n \geq 4$ in the local case. His results were then extended to the remaining dimensions $n = 2, 3$ in [148] and generalized to systems of bistable reaction-diffusion equations by Kapitula [133]. Our strategy of proof is similar to as [133, 200] where semigroup estimates for the associated linearized operator are used to prove the multidimensional stability of the traveling wave φ. It is important to remark that in dimension $n \geq 4$, these semigroup estimates are sufficient to prove Theorem 4.3.2 (see the last paragraph below). For the remaining dimensions $n = 2, 3$, the proof essentially relies on the decomposition of the perturbation as written in (4.3.21) which basically allows one to split the problem into two parts. One part controls the drift of the perturbations along the translates of the wave and another part which controls the remaining part of the perturbations and will decay faster in time. Although our proof will follow the strategy developed in [133, 200], we still have to deal with the nonlocal nature of our equations. In our case, we use point-wise Green's functions estimates to obtain sharp decay estimates of the linear part of our linearized operator. These types of estimates are reminiscent of the ones obtained by Hoffman and coworkers [115] in the study of multi-dimensional stability of planar traveling of lattice differential equations, which are discrete version of equation (4.3.12). In the nonlocal setting, using super- and sub- solution technique, Chen [43] has been able to prove the uniform multidimensional stability of the traveling wave φ of equation (4.3.12). As a direct consequence, our Theorem 4.3.2 generalizes Chen's result.

An application. This present work was initially motivated by the study of Bates and Chen [11] where they prove a multidimensional stability result for a slightly different multidimensional bistable nonlocal reaction-diffusion equation. Their idea was to consider a generalization of the Laplacian in n-dimension for which, each component $\partial_{x_i}^2$ of Δ is approximated by the convolution operator $-u + \mathcal{J} *_{x_i} u$. They obtain an equation of form

$$\partial_t u = \sum_{i=1}^{n} (-u + \mathcal{J} *_{x_i} u) + f(u), \tag{4.3.23}$$

with

$$\mathcal{J} *_{x_i} u(\mathbf{x}) := \int_{\mathbb{R}} \mathcal{J}(y) u(x_1, \cdots, x_i - y, \cdots, x_n) \mathrm{d}y.$$

The kernel $\mathcal{J}$ satisfies the following Hypothesis.

Hypothesis 4.3.3. *We suppose that the kernel $\mathcal{J}$ satisfies the following properties:*

(i) $\mathcal{J} \geq 0$, *is even;*

(ii) $\mathcal{J} \in W_\eta^{1,1}(\mathbb{R})$ *for* $\eta > 0$.

Here $W_\eta^{1,1}(\mathbb{R})$ denotes the exponentially weighted function space defined as

$$W_\eta^{1,1}(\mathbb{R}) := \left\{ u \in W^{1,1}(\mathbb{R}) \mid e^{\eta|\cdot|}u \in L^1(\mathbb{R}) \text{ and } e^{\eta|\cdot|}\partial_x u \in L^1(\mathbb{R}) \right\}.$$

A direct consequence of Hypothesis 4.3.3 is that $\widehat{\mathcal{J}}(k) = 1 - d_0 k^2 + o(k^2)$ as $k \to 0$ for $d_0 > 0$. In this setting, the traveling wave φ is solution of (4.3.15) with $\mathcal{K}_0 = \mathcal{J}$ and the linearized operator $\mathcal{L}_0$ has the same expression as in equation (4.3.17) and thus Lemma 4.3.1 is also verified provided that f satisfies Hypothesis 4.3.1.

To study the stability of the traveling wave φ, we work with the same decomposition

$$u(t, \mathbf{x}) = \varphi(\xi - \rho(t, \mathbf{z})) + v(t, \xi - \rho(t, \mathbf{z}), \mathbf{x}), \quad t \geq 0, \tag{4.3.24}$$

with $\rho : \mathbb{R}^{n-1} \to \mathbb{R} \in H^k(\mathbb{R}^{n-1})$ and $v : \mathbb{R}^n \to \mathbb{R} \in H^k(\mathbb{R}^n)$ is in the range of the operator $\mathcal{L}_0$ that is $\mathcal{P}v = 0$. We also set

$$\widetilde{\mathcal{E}}_0 := \|v_0\|_{L^1(\mathbb{R}^n)} + \|v_0\|_{H^k(\mathbb{R}^n)} + \|\rho_0\|_{W^{1,1}(\mathbb{R}^{n-1})} + \|\rho_0\|_{H^{k+1}(\mathbb{R}^{n-1})}.$$

As a bi-product of our proof, we obtain the following result.

Theorem 4.3.3. *Let* $n \geq 2$ *and* $k \geq \left[\frac{n+1}{2}\right]$. *Suppose that Hypotheses 4.3.1 and 4.3.3 are satisfied. There exists* $C > 0$ *such that if* $\widetilde{\mathcal{E}}_0$ *is sufficiently small, then the traveling wave solution* φ *of equation* (4.3.23) *is stable in the sense that the perturbation* (ρ, v) *given in* (4.3.24) *satisfies the decay estimates for all* $t \geq 0$

$$\|v(t)\|_{H^k(\mathbb{R}^n)} \leq C(1+t)^{-\frac{n+1}{2}}\widetilde{\mathcal{E}}_0, \tag{4.3.25a}$$

$$\|\rho(t)\|_{H^k(\mathbb{R}^{n-1})} \leq C(1+t)^{-\frac{n-1}{4}}\widetilde{\mathcal{E}}_0, \tag{4.3.25b}$$

$$\|\nabla_{\mathbf{z}} \cdot \rho(t)\|_{H^k(\mathbb{R}^{n-1})} \leq C(1+t)^{-\frac{n+1}{4}}\widetilde{\mathcal{E}}_0. \tag{4.3.25c}$$

Note that Bates and Chen [11] only proved Theorem 4.3.3 in dimension $n \geq 4$ and thus our result generalizes their analysis to the remaining dimensions $n = 2, 3$. Compared to Theorem 4.3.2, we obtain a sharper decay of v component of the perturbation. This is a consequence of the fact that the projection $\mathcal{P}$ commutes with each linear operator $-u + \mathcal{J} *_{x_i} u$ for $i = 2 \cdots n$.

Stability in dimension $n \geq 4$. In this paragraph, we give a simple proof of Theorem 4.3.2 in the high-dimensional case $n \geq 4$, following ideas that have been developed for the multidimensional local case [98, 133, 200]. The main ingredient of the proof is an estimate for the linearized evolution operator $\mathcal{L}$.

First, we consider a solution $u(t, \mathbf{x}) = \varphi(\xi) + v(t, \mathbf{x})$ of (4.3.12) which satisfies the equation

$$\partial_t v(t, \mathbf{x}) = \mathcal{L}v(t, \mathbf{x}) + \mathcal{N}(v(t, \mathbf{x})), \tag{4.3.26}$$

where

$$\mathcal{N}(v) := f(\varphi + v) - f(\varphi) - f'(\varphi)v. \tag{4.3.27}$$

The Cauchy problem associated to equation (4.3.26) with initial condition $v_0 \in H^k(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$, with $k \geq n+1$ and $n \geq 4$ is locally well-posed in $H^k(\mathbb{R}^n)$. This is equivalent to say that for any $v_0 \in H^k(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ there exists a time $T > 0$ such that (4.3.26) as a unique mild solution in $H^k(\mathbb{R}^n)$ defined on $[0, T]$ satisfying $v(0) = v_0$. The integral formulation of (4.3.26) is given by

$$v(t) = \mathcal{S}_{\mathcal{L}}(t)v_0 + \int_0^t \mathcal{S}_{\mathcal{L}}(t-s)\mathcal{N}(v(s))\mathrm{d}s, \tag{4.3.28}$$

where $\mathcal{S}_{\mathcal{L}}$ is the semigroup associated to the linear operator $\mathcal{L}$. From [72, Propositions 3.3 & 3.4], there exist positive constants C and $\theta > 0$ such that

$$\|\mathcal{S}_{\mathcal{L}}(t)v\|_{H^k(\mathbb{R}^n)} \leq C\left((1+t)^{-\frac{n-1}{4}}\|v\|_{L^1(\mathbb{R}^n)} + e^{-\theta t}\|v\|_{H^k(\mathbb{R}^n)}\right). \tag{4.3.29}$$

The nonlinear contribution $\mathcal{N}(v)$ is at least quadratic in v close to the origin. As a consequence, we can find a positive nondecreasing function $\kappa : \mathbb{R}_+ \to \mathbb{R}_+$ such that, for all $t \in [0, T]$,

$$|\mathcal{N}(v)| \leq \kappa(R)|v|^2, \quad \text{for } |v| \leq R.$$

Let $T_* > 0$ be the maximal time of existence of a solution $v \in H^k(\mathbb{R}^n)$ with initial condition $v_0 \in H^k(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$. For $t \in [0, T*)$ we define

$$\Phi(t) = \sup_{0 \leq s \leq t} (1+s)^{\frac{n-1}{4}}\|v(s)\|_{H^k(\mathbb{R}^n)}.$$

Using estimate (4.3.29) directly into the integral formulation (4.3.28) yields

$$\begin{aligned}
\|v(t)\|_{H^k(\mathbb{R}^n)} &\leq \|\mathcal{S}_{\mathcal{L}}(t)v_0\|_{H^k(\mathbb{R}^n)} + \int_0^t \|\mathcal{S}_{\mathcal{L}}(t-s)\mathcal{N}(v(s))\|_{H^k(\mathbb{R}^n)}\mathrm{d}s \\
&\lesssim (1+t)^{-\frac{n-1}{4}}\|v_0\|_{L^1(\mathbb{R}^n)} + e^{-\theta t}\|v_0\|_{H^k(\mathbb{R}^n)} + \kappa(\Phi(t))\int_0^t (1+t-s)^{-\frac{n-1}{4}}\|v(s)\|^2_{H^k(\mathbb{R}^n)}\mathrm{d}s \\
&\quad + \kappa(\Phi(t))\int_0^t e^{-\theta(t-s)}\|v(s)\|^2_{H^k(\mathbb{R}^n)}\mathrm{d}s.
\end{aligned}$$

Here, we use the notation $A \lesssim B$ whenever $A \leq cB$ for $c > 0$ a constant independent of time t. From [200], there exist constants $C_1 > 0$ and $C_2 > 0$ so that

$$\begin{aligned}
\int_0^t (1+t-s)^{-\frac{n-1}{4}}(1+s)^{-\frac{n-1}{2}}\mathrm{d}s &\leq C_1(1+t)^{-\frac{n-1}{4}}, \\
\int_0^t e^{-\theta(t-s)}(1+s)^{-\frac{n-1}{2}}\mathrm{d}s &\leq C_2(1+t)^{-\frac{n-1}{4}}.
\end{aligned}$$

Note that the first inequality is a consequence of our careful choice of n. Indeed, this inequality is only true for $\frac{n-1}{4} > \frac{1}{2}$ $(n \geq 4)$. Then, for all $t \in [0, T_*)$ we have

$$\Phi(t) \leq C_0\left(\|v_0\|_{L^1(\mathbb{R}^n)} + \|v_0\|_{H^k(\mathbb{R}^n)}\right) + \widetilde{C}_0\kappa(\Phi(t))\Phi(t)^2,$$

for some positive constants C_0 and $\widetilde{C}_0$. Suppose that the initial condition v_0 is small enough so that

$$2C_0\left(\|v_0\|_{L^1(\mathbb{R}^n)}+\|v_0\|_{H^k(\mathbb{R}^n)}\right)<1 \quad \text{and} \quad 4C_0\widetilde{C}_0\kappa(1)\left(\|v_0\|_{L^1(\mathbb{R}^n)}+\|v_0\|_{H^k(\mathbb{R}^n)}\right)<1,$$

then

$$\Phi(t)\leq 2C_0\left(\|v_0\|_{L^1(\mathbb{R}^n)}+\|v_0\|_{H^k(\mathbb{R}^n)}\right)<1,$$

for all $t\in[0,T_*)$ by continuity of Φ. This implies that the maximal time of existence is $T_*=+\infty$ and the solution v of (4.3.26) satisfies:

$$\sup_{t\geq 0}\,(1+t)^{\frac{n-1}{4}}\|v(t)\|_{H^k(\mathbb{R}^n)}\leq 2C_0\left(\|v_0\|_{L^1(\mathbb{R}^n)}+\|v_0\|_{H^k(\mathbb{R}^n)}\right).$$

Chapter 5

What is not presented in this memoir

The works [49, 70, 74–77, 84, 85] were completed during my PhD and will thus not be discussed here. The follow up paper [48] presents an overview of pattern formation on the hyperbolic plane for the Swift-Hohenberg equation which is a canonical model equation for pattern forming systems. Three different types of patterns are considered: spatially periodic stationary solutions (based on [74, 75]), radial solutions (based on [85]) and traveling waves (see [47]). Still directly related to my PhD is the collaboration [169], following up on [84, 85], which deals with numerical continuation of localized coherent structures in neural field equations. These two works [48, 169] could be almost considered as part of my doctoral thesis since they were completed shortly afterwards and are thus not presented.

The paper [89] written with J. Touboul investigates the role of delays on the stability of some coherent structures (localized stationary states and stationary interfaces) in neural field equations. Linear stability analysis allowed to reveal the existence of Hopf bifurcation curves induced by these delays, along different modes that may be symmetric or asymmetric. We showed that instabilities strongly depend on the dimension, and in particular may exhibit transversal instabilities along invariant directions. These instabilities yield pulsatile localized activity, and depending on the symmetry of the destabilized modes, either produce spatiotemporal *breathing* or *sloshing* patterns. I have decided to leave this work aside as it is not related to propagation phenomena.

In [138], we explored with Z. Kilpatrick the combined effects of additive noise and linear adaptation on localized unimodal symmetric stationary states of neural field equations in periodic domain. It is shown that for the deterministic system these stationary states undergo a pitchfork bifurcation by increasing the strength of adaptation and destabilize into propagating pulses. Near this criticality, we derived a stochastic amplitude equation describing the dynamics of these bifurcating pulses when the noise and the deterministic instability are of comparable magnitude. Away from this bifurcation, we investigated the effects of additive noise on the propagation of traveling pulses and demonstrated that noise induces wandering of traveling pulses. Although this work deals with traveling pulses, it does not relate well to the topic of this memoir as it primarily focuses about stochastic dynamics, and thus I decided not to include it here.

The paper [38] is the result of a Research Experience for Undergraduate (REU in short) program which I jointly mentored and sponsored with A. Scheel during 6 weeks in the Summer 2014 at the School of Mathematics, University of Minnesota. It is about the study of coherent structures in the form of invasion fronts in scalar feed-forward networks. It was published in SIAM Undergraduate Research Online (SIURO) and was entirely written by C. Browne & A.L. Dickerson, and as a consequence, it will not be further discussed.

Finally, the most important paper not included here is the rigorous derivation of the nonlocal FitzHugh-Nagumo system obtained in [55] with my PhD student J. Crevat and F. Filbet. It is proved that the hydrodynamic limit of a spatially extended transport kinetic FitzHugh-Nagumo model converges towards the classical nonlocal reaction-diffusion FitzHugh-Nagumo system. Our approach is based on a relative entropy method, where the macroscopic quantities of the kinetic model are compared with the solution to the nonlocal reaction-diffusion system. Even though this work opens very promising research perspectives, it is apart from my work on propagation phenomena in reaction-diffusion equations and more importantly I felt that it belongs to Joachim's forthcoming PhD thesis and not to this memoir.

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