Pattern formation in the visual cortex

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Abstract

These notes correspond to research lectures on *Partial Differential Equations for Neurosciences* given at CIRM, 04-08/07/2017, during the Summer School on *PDE & Probability for Life Sciences*. They give a self-content overview of pattern formation in the primary visual cortex allowing one to explain psychophysical experiments and recordings of what is referred to as geometric visual hallucinations in the neuroscience community. The lecture is divided into several parts including a rough presentation on the modeling of cortical areas via neural field equations. Other parts deal with notions of equivariant bifurcation theory together with center manifold results in infinite-dynamical systems which will be the cornerstone of our analysis. Finally, in the last part, we shall use all the theoretical results to provide a comprehensive explanation of the formation of geometric visual hallucinations through Turing patterns.

Turing originally considered the problem of how animal coat patterns develop, suggesting that chemical markers in the skin comprise a system of diffusion-coupled chemical reactions among substances called morphogens [13]. He showed that in a two-component reaction-diffusion system, a state of uniform chemical concentration can undergo a diffusion-driven instability leading to the formation of a spatially inhomogeneous state. Ever since the pioneering work of Turing on morphogenesis, there has been a great deal of interest in spontaneous pattern formation in physical and biological systems. In the neural context, Wilson and Cowan [17] proposed a non-local version of Turing's diffusion-driven mechanism, based on competition between short-range excitation and longer-range inhibition. Here interactions are mediated, not by molecular diffusion, but by long-range axonal connections. Since then, this neural version of the Turing instability has been applied to a number of problems concerning cortical dynamics. Examples in visual neuroscience include the ring model of orientation tuning, cortical models of geometric visual hallucinations (that will be studied here) and developmental models of cortical maps.presentreview theoretical approaches to studying spontaneous pattern formation in neural field models, always emphasizing the important role that symmetries play.

Most of the material on center manifold is taken from the book of Haragus & Iooss [7] and on equivariant bifurcations from the book of Chossat & Lauterbach [2]. One other complementary reference is the book of Golubitsky-Stewart-Schaeffer [6]. On pattern formation, we refer to the very interesting book of Hoyle [9].

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Contents

1 Modeling cortical visual areas

1.1 Some properties of the visual cortex

In this very first section, we roughly describe the visual pathway (see Figure 1 for a sketch) and identify the specific visual area that we will be modeling, namely the primary visual cortex (V1 in short) which is the very first visual area receiving information from the retina through the lateral Geniculate body (LGN in short).

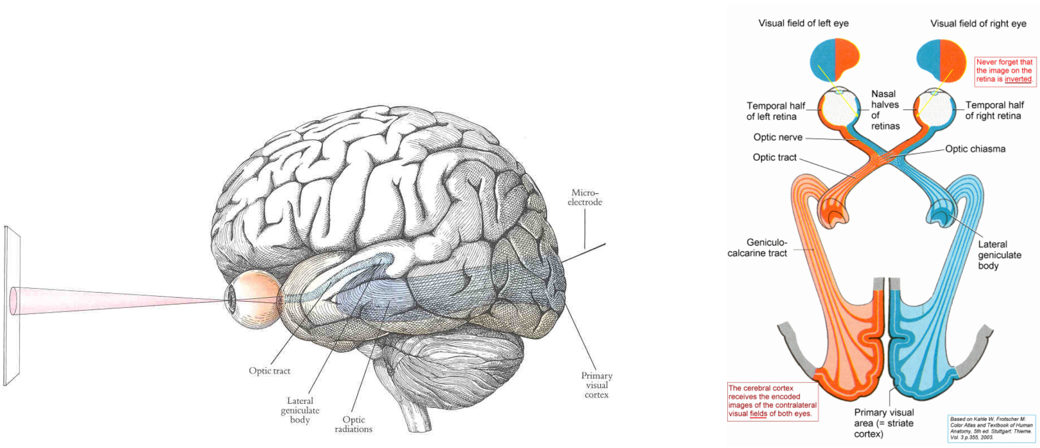


Figure 1: *Sketch of the human visual pathway.*

We first give a list of experimental observations that will be used in our modeling assumptions later on in this section.

- The cortex is a folded sheet of width 2cm.
- It has a layered structure (*i.e.* 6 identified layers) and is retinotopically organized: the mapping between the visual field and the cortical coordinates is approximatively log-polar (see Figure 2).
- From the LGN the information is transmitted to the visual cortex (back of the head) mostly to the area V1.
- Where does the information go after V1? Mainly: V2,V4, MT, MST... (there are 30 visual areas that are different by their architecture, connectivity or functional properties).
- V1 is spatially organized in columns that share the same preferred functional properties (orientation, ocular dominance, spatial frequency, direction of motion, color etc...).
- Local excitatory/inhibitory connections are homogeneous, whereas long-range connections (mainly excitatory neurons) are patchy, modulatory and anisotropic.

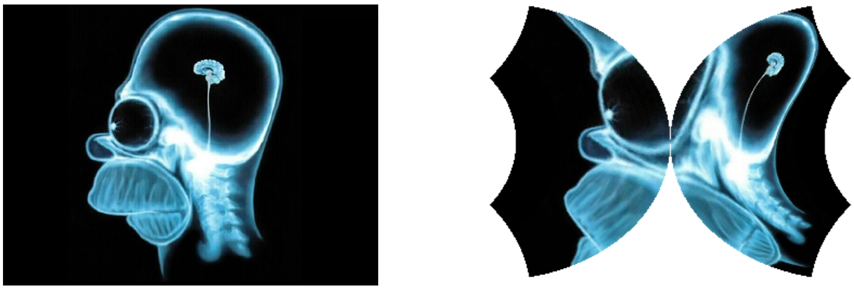


Figure 2: *Retinotopic organization of the primary visual cortex. To one point in the visual field (left image) corresponds one point in the primary visual cortex (right image). The associated map transformation is approximatively log-polar (roughly $f(z) = \log\left(\frac{z+0.33}{z+0.66}\right)$). In fact the left and right part of the visual field should be shifted in cortical space, but we did not intend to represent it on this cartoon.*

1.2 Neural fields models

In this section, we start by proposing a local models for n interacting neural masses that we will then generalize by taking a formal continuum limit. We suppose that each neural population i is described by its **average membrane potential** $V_i(t)$ or by its **average instantaneous firing rate** $\nu_i(t)$ with $\nu_i(t) = S_i(V_i(t))$, where S_i is of sigmoidal form (think of a tangent hyperbolic function). Then, a single action potential from neurons in population j, is seen as a **post-synaptic potential** $PSP_{ij}(t-s)$ by neurons in population i (s is the time of the spike hitting the synapse and t the time after the spike). The number of spikes arriving between t and $t+dt$ is $\nu_j(t)dt$, such that the average membrane potential of population i is:

$$V_i(t) = \sum_j \int_{t_0}^{t} PSP_{ij}(t-s)S_j(V_j(s))ds.$$

We further suppose that a post-synaptic potential has the same shape no matter which presynaptic population caused it, this leads to the relationship

$$PSP_{ij}(t) = w_{ij}PSP_i(t),$$

where w_{ij} is the average strength of the post-synaptic potential and if $w_{ij} > 0$ (resp. $w_{ij} < 0$) population j excites (resp. inhibts) population i. Now, if we assume that $PSP_i(t) = e^{-t/\tau_i}H(t)$ or equivalently

$$\tau_i \frac{\mathrm{d}PSP_i(t)}{\mathrm{d}t} + PSP_i(t) = \delta(t)$$

we end up with a system of ODEs:

$$\tau_i \frac{\mathrm{d}V_i(t)}{\mathrm{d}t} + V_i(t) = \sum_j w_{ij}S_j(V_j(t)) + I^i_{ext}(t),$$

which can be written in vector form:

$$\frac{\mathrm{d}\mathbf{V}}{\mathrm{d}t}(t) = -\mathbf{M}\mathbf{V}(t) + \mathbf{W}\cdot\mathbf{S}(\mathbf{V}(t)) + \mathbf{I}_{ext}(t).$$

Here the matrix $\mathbf{M}$ is set to the diagonal matrix $\mathbf{M} := \text{diag}\left[(1/\tau_i)_{i=1,\cdots,n}\right]$.

So far we have not made any assumptions about the topology of the underlying neural network, that is, the structure of the weight matrix $\mathbf{W}$ with components $w_{i,j}$. If one looks at a region of cortex such as primary visual cortex (V1), one finds that it has a characteristic spatial structure, in which a high density of neurons (10^5mm^{-3} in primates) are distributed according to an approximately two-dimensional (2D) architecture. That is, the physical location of a vertical column of neurons within the two-dimensional cortical sheet often reflects the specific information processing role of that population of neurons. In V1, we have already seen that there is an orderly retinotopic mapping of the visual field onto the cortical surface, with left and right halves of the visual field mapped onto right and left visual cortices respectively. This suggests labeling neurons according to their spatial location in cortex. This idea of labeling allows one to formally derive a continuum neural field model of cortex. Let $\Omega \subset \mathbb{R}^d$, $d = 1, 2, 3$ be a part of the cortex that is under consideration. If we note $\mathbf{V}(\mathbf{r}, t)$ the state vector at point $\mathbf{r}$ of Ω and if we introduce the $n \times n$ matrix function $\mathbf{W}(\mathbf{r}, \mathbf{r}', t)$, we obtain the following time evolution for $\mathbf{V}(\mathbf{r}, t)$

$$\frac{\partial \mathbf{V}(\mathbf{r}, t)}{\partial t} = -\mathbf{M}\mathbf{V}(\mathbf{r}, t) + \int_{\Omega} \mathbf{W}(\mathbf{r}, \mathbf{r}', t)\mathbf{S}(\mathbf{V}(\mathbf{r}', t))\mathrm{d}\mathbf{r}' + \mathbf{I}_{ext}(\mathbf{r}, t). \tag{1}$$

Here, $\mathbf{V}(\mathbf{r}, t)$ represents an average membrane potential at point $\mathbf{r} \in \Omega$ in the cortex and time t. We refer to the celebrated paper of Wilson-Cowan [16] for further discussion on the above derivation.

Remark 1.1. *Following the same basic procedure, it is straightforward to incorporate into the neural field equation* (1) *additional features such as synaptic depression, adaptive thresholds or axonal propagation delays.*

1.3 Geometric visual hallucinations

Geometric visual hallucinations are seen in many situations, for example, after being exposed to flickering lights, after the administration of certain anesthetics, on waking up or falling asleep, following deep binocular pressure on one's eyeballs, and shortly after the ingesting of drugs such as LSD and marijuana (this will be our modeling assumption). We refer to Figure 3 for various reproductions of experienced visual hallucinations. We would like to propose a cortical model which allows one to explain the formation of such geometric visual hallucinations. Our main assumption is that these hallucinations are solely produced in the primary visual cortex and should reflect the spontaneous emergence of spatial organisation of the cortical activity that we identify to the average membrane potential from the previous section. It is thus natural to apply the retinotopic map to see how such visual patterns look like in V1. For example, in the case of funnel and spiral (see Figure 3 (a)-(b)), we can deduce that the corresponding patterns in the visual cortex are stripes as shown in Figure 4. Applying the same procedure to other types of visual hallucination would lead to the conclusion that corresponding patterns in the visual cortex could spots organized on planar lattice (square or hexagonal).

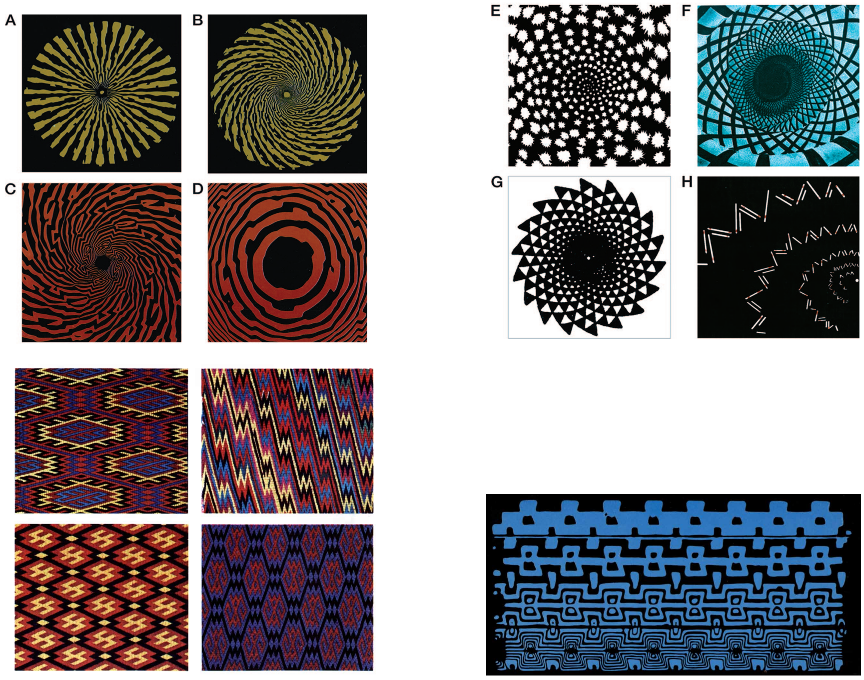


Figure 3: *Various reported visual hallucinations. Redrawn from Tyler (1978), Oster (1970) and Siegel (1977).*

With these conclusions, we can envision to propose a model which would produce Turing patterns in the sense of spatially periodic patterns on the visual area V1. We are going to see these patterns as spontaneous symmetry breaking bifurcated solutions of a neural field equation of the form of (1). Indeed, we assume that at rest, *i.e.* without any drug consumption and with closed eyes, the cortical activity is stationary and homogenous. Actually, to simplify the presentation, we will suppose that the activity is zero across V1. The ingestion of drug will be traduced by the increase of a parameter μ which will modify the nonlinear firing rate function $\mathbf{S}$. Hopefully, passing a critical value μ_c the rest sate will become to be unstable with respect to doubly periodic perturbations and a bifurcation will occur. Because of the symmetries that we will impose on our network, we will see emerging new branches of solutions (with less symmetry than the rest state which has always all the symmetries of the network). These new solutions will be interpreted as geometric visual hallucinations once seen in the visual field.

Further modeling assumptions. One important remark is that the visual hallucinations that we consider here are static and thus we have to suppose that the topology of our network does not change in time, *i.e.* $\mathbf{W}(\mathbf{r}, \mathbf{r}', t) = \mathbf{W}(\mathbf{r}, \mathbf{r}')$. We also assume that the primary visual cortex does not receive any input from other cortical areas and so the external input $\mathbf{I}_{ext}$ is set to zero: $\mathbf{I}_{ext}(\mathbf{r}, t) = 0$. We will suppose that the cortical activity $\mathbf{V}(\mathbf{r}, t)$ is one-dimensional and we will denote it $u(\mathbf{r}, t)$ from now to emphasize its a scalar function. Regarding the assumption on the dependance of the function $\mathbf{S}$ with respect to the parameter μ, we will simply use $\mathbf{S}(u) := S(u, \mu)$ with $S(0, \mu) = 0$ for all μ and $D_u S(0, \mu) = \mu s_1$ for some $s_1 > 0$. Finally, we idealize the visual cortex Ω to the Euclidean plane $\mathbb{R}^2$, this is motivated by the essential two-dimensional structure of the visual cortex where we neglect the width. This hypothesis also allows us to impose some symmetry assumptions on the network topology, *i.e.* the connectivity kernel $\mathbf{W}$. Namely, we want Euclidean invariance for the connectivity kernel and one natural way to achieve it is to suppose that $\mathbf{W}(\mathbf{r}, \mathbf{r}') = w(\|\mathbf{r} - \mathbf{r}'\|)$, where $\|\cdot\|$ is the usual Euclidean norm. Finally, we will suppose that local excitatory (*i.e.* $w(0) > 0$) and laterally inhibitory (*i.e.* $w(r) < 0$ for r large enough).

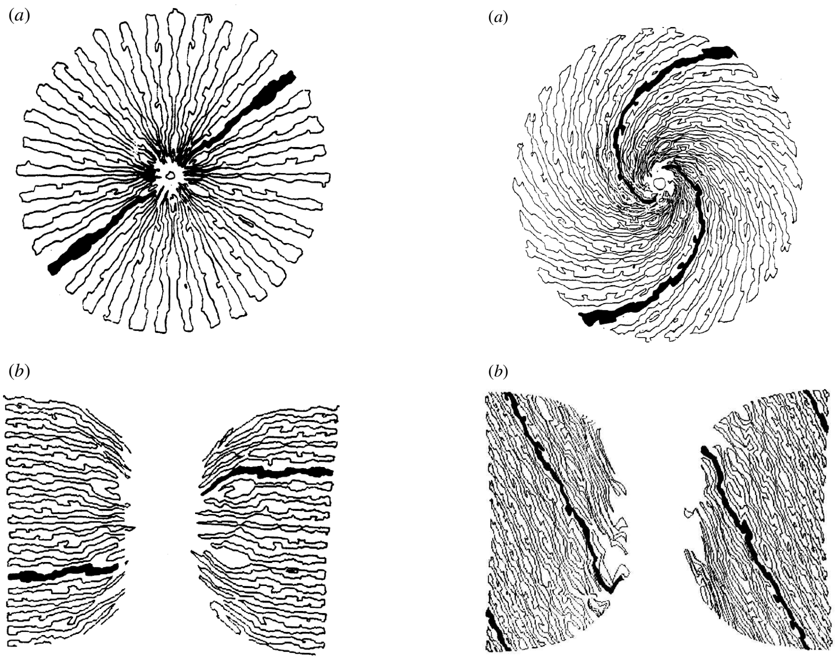


Figure 4: *Action of the retinocortical map on the funnel and spiral form constant. (a) Image in the visual field; (b) V1 map of the image.*

We are thus let to study the following neural field equation

$$\frac{\partial u(\mathbf{r}, t)}{\partial t} = -u(\mathbf{r}, t) + \int_{\mathbb{R}^2} w(\|\mathbf{r} - \mathbf{r}'\|) S(u(\mathbf{r}', t), \mu) \mathrm{d}\mathbf{r}', \tag{2}$$

where we have rescaled time to suppose that $\mathbf{M}$ can be taken to be equal to identity matrix (*i.e.* 1 in our scalar case). We can check that $u(\mathbf{r}, t) = 0$ is always a solution because of our hypothesis on the nonlinearity S (recall that we suppose that $S(0, \mu) = 0$ for all μ). Let us linearize the above equation (2) around this rest state:

$$\frac{\partial u(\mathbf{r}, t)}{\partial t} = -u(\mathbf{r}, t) + \mu s_1 \int_{\mathbb{R}^2} w(\|\mathbf{r} - \mathbf{r}'\|) u(\mathbf{r}', t) \mathrm{d}\mathbf{r}'. \tag{3}$$

In order to get the continuous part of the spectrum, we look for special solutions of the form $u(\mathbf{r}, t) = e^{\lambda t} e^{\mathbf{i}\mathbf{k}\cdot\mathbf{r}}$ for some given vector $\mathbf{k} \in \mathbb{R}^2$, and we obtain the dispersion relation

$$\lambda(\|\mathbf{k}\|, \mu) = -1 + \mu s_1 \widehat{w}(\|\mathbf{k}\|), \tag{4}$$

where $\widehat{w}$ is the Fourier transform of w. Here, we made a slight abuse of notation by explicitly writing $\widehat{w}$ as a function of $\|\cdot\|$. We show in Figure 5 how such a dispersion relation is modified as

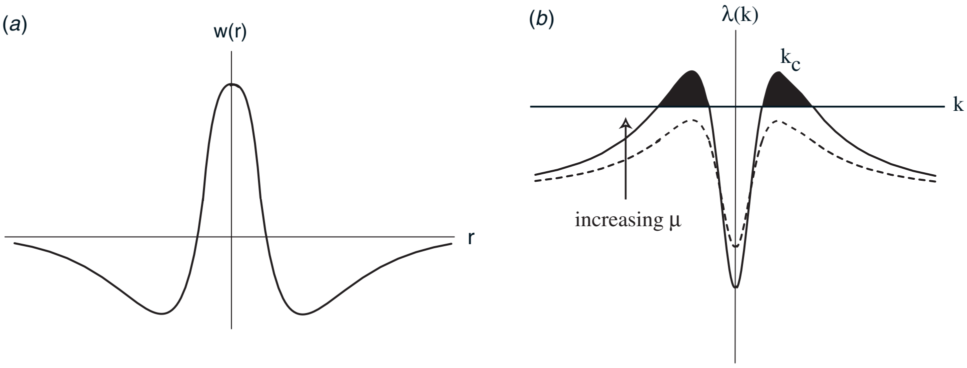


Figure 5: *Schematic visualization of the connectivity kernel w satisfying our assumptions (locally excitatory and laterally inhibitory) together the corresponding dispersion relation given in equation (4).*

μ is increased. As a consequence, in what will follow, we suppose that there exists a unique couple $(\mu_c, k_c) \in (0, \infty)^2$ such that the following conditions hold.

Hypothesis 1.1 (Dispersion relation)**.** *The dispersion relation (4) of the linearized equation (3) satisfies:*

(i) $\lambda(k_c, \mu_c) = 0$ *and* $\lambda(\|\mathbf{k}\|, \mu_c) \neq 0$ *for all* $\|\mathbf{k}\| \neq k_c$*;*

(ii) for all $\mu < \mu_c$*, we have* $\lambda(\|\mathbf{k}\|, \mu) < 0$ *for all* $\mathbf{k} \in \mathbb{R}^2$*;*

(iii) $k \to \lambda(k, \mu_c)$ *has a maximum at* $k = k_c$*.*

We clearly see that the above hypotheses imply that for $\mu > \mu_c$, there will be an annulus of unstable eigenmodes while for $\mu < \mu_c$ the rest state is linearly stable. We can already see what will be the main difficulties to overcome:

- there is continuous spectrum due to the Euclidean symmetry of the problem;
- a whole circle of eigenmodes becomes neutrally unstable at $\mu = \mu_c$ so that the center part of the spectrum is infinite-dimensional;
- passed $\mu > \mu_c$, the rest state is unstable to an annulus of unstable eigenmodes so that the dynamics nearby should be intricate.

Main idea. We restrict ourselves to the function space of doubly periodic functions such that the spectrum of the linearized operator is discrete with finitely many eigenvalues on the center part. We also only study the dynamics of (2) in a neighborhood of $u \simeq 0$ and $\mu \simeq \mu_c$ where one can rely on various techniques such as the construction of center manifolds and equivariant bifurcations, as our reduced problem will still have some symmetries reminiscent of the Euclidean ones.

2 Center manifolds in infinite-dimensional dynamical systems

Center manifolds are fundamental for the study of dynamical systems near critical situations and in particular in bifurcation theory. Starting with an infinite-dimensional problem, the center manifold theorem will reduce the study of small solutions, staying sufficiently close to 0, to that of small solutions of a reduced system with finite dimension. The solutions on the center manifold are described by a finite-dimensional system of ordinary differential equations, also called the reduced system. The very first results on center manifolds go back to the pioneering works of Pliss [12] and Kelley [10] in the finite-dimensional setting. Regarding extensions to the infinite-dimensional setting we can refer to [8, 11, 15] and references therein together with the recent comprehensive book of Haragus & Iooss [7] from which these notes are partially taken from. Center manifold theorems have proved its full strength in studying local bifurcations in infinite-dimensional systems and led to significant progress in understanding of some nonlinear phenomena in partial differential equations, including applications in pattern formation, water wave problems or population dynamics. In this lecture, we will see how to apply such results in the context of geometric visual hallucinations that can be interpreted as pattern forming states on the visual cortex.

2.1 Notations and definitions

Consider two (complex or real) Banach spaces $\mathcal{X}$ and $\mathcal{Y}$. We shall use the following notations:

- $\mathscr{C}^k(\mathcal{Y},\mathcal{X})$ is the Banach space of k-times continuously differentiable functions $F:\mathcal{Y}\to\mathcal{X}$ equipped with the norm on all derivatives up to order k,

$$\|F\|_{\mathscr{C}^k}=\max_{j=0,\dots,k}\left(\sup_{y\in\mathcal{Y}}\left(\|D^jF(y)\|_{\mathscr{L}(\mathcal{Y}^j,\mathcal{X})}\right)\right).$$

- $\mathscr{L}(\mathcal{Y},\mathcal{X})$ is the Banach space of linear bounded operators $\mathbf{L}:\mathcal{Y}\to\mathcal{X}$, equipped with operator norm:

$$\|\mathbf{L}\|_{\mathscr{L}(\mathcal{Y},\mathcal{X})}=\sup_{\|u\|_{\mathcal{Y}}=1}\left(\|\mathbf{L}u\|_{\mathcal{X}}\right),$$

  if $\mathcal{Y}=\mathcal{X}$, we write $\mathscr{L}(\mathcal{Y})=\mathscr{L}(\mathcal{Y},\mathcal{X})$.

- For a linear operator $\mathbf{L}:\mathcal{Y}\to\mathcal{X}$, we denote its range by $\mathrm{im}\mathbf{L}$:

$$\mathrm{im}\mathbf{L}=\{\mathbf{L}u\in\mathcal{X}\mid u\in\mathcal{Y}\}\subset\mathcal{X},$$

  and its kernel by $\ker\mathbf{L}$:

$$\ker\mathbf{L}=\{u\in\mathcal{Y}\mid\mathbf{L}u=0\}\subset\mathcal{Y}.$$

- Assume that $\mathcal{Y}\hookrightarrow\mathcal{X}$ with continuous embedding. For a linear operator $\mathbf{L}\in\mathscr{L}(\mathcal{Y},\mathcal{X})$, we denote by $\rho(\mathbf{L})$, or simply ρ, the resolvent set of $\mathbf{L}$:

$$\rho=\{\lambda\in\mathbb{C}\mid\lambda\mathrm{id}-\mathbf{L}:\mathcal{Y}\to\mathcal{X}\text{ is bijective }\}.$$

The complement of the resolvent set is the spectrum $\sigma(\mathbf{L})$, or simply σ,

$$\sigma = \mathbb{C} \setminus \{\rho\}.$$

Remark 2.1. *When* $\mathbf{L}$ *is real, both the resolvent set and the spectrum of* $\mathbf{L}$ *are symmetric with respect to the real axis in the complex plane.*

2.2 Local center manifold

Let $\mathcal{X}, \mathcal{Y}$ and $\mathcal{Z}$ be Banach spaces such that:

$$\mathcal{Z} \hookrightarrow \mathcal{Y} \hookrightarrow \mathcal{X}$$

with continuous embeddings. We consider a differential equation in $\mathcal{X}$ of the form:

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \mathbf{L}u + \mathbf{R}(u) \tag{5}$$

in which we assume that the linear part $\mathbf{L}$ and the nonlinear part $\mathbf{R}$ are such that the following holds.

Hypothesis 2.1 (Regularity)**.** *We assume that* $\mathbf{L}$ *and* $\mathbf{R}$ *in* (5) *have the following properties:*

(i) $\mathbf{L} \in \mathscr{L}(\mathcal{Z}, \mathcal{X})$*;*

(ii) for some $k \geq 2$*, there exists a neighborhood* $\mathcal{V} \subset \mathcal{Z}$ *of* 0 *such that* $\mathbf{R} \in \mathscr{C}^k(\mathcal{V}, \mathcal{Y})$ *and*

$$\mathbf{R}(0) = 0, \quad D\mathbf{R}(0) = 0.$$

Hypothesis 2.2 (Spectral decomposition)**.** *Consider the spectrum* σ *of the linear operator* $\mathbf{L}$*, and write:*

$$\sigma = \sigma_+ \cup \sigma_0 \cup \sigma_-$$

in which

$$\sigma_+ = \{\lambda \in \sigma \mid \mathrm{Re}\lambda > 0\}, \quad \sigma_0 = \{\lambda \in \sigma \mid \mathrm{Re}\lambda = 0\}, \quad \sigma_- = \{\lambda \in \sigma \mid \mathrm{Re}\lambda < 0\}.$$

We assume that:

(i) there exists a positive constant γ *such that*

$$\inf_{\lambda \in \sigma_+} (\mathrm{Re}\lambda) > \gamma, \quad \sup_{\lambda \in \sigma_-} (\mathrm{Re}\lambda) < -\gamma;$$

(ii) the set σ_0 *consists of a finite number of eigenvalues with finite algebraic multiplicities.*

Hypothesis 2.3 (Resolvent estimates)**.** *Assume that there exist positive constants $\omega_0 > 0$, $c > 0$ and $\alpha \in [0,1)$ such that for all $\omega \in \mathbb{R}$ with $|\omega| \geq \omega_0$, we have that $\mathrm{i}\omega$ belongs to the resolvent set of $\mathbf{L}$ and*

$$\begin{aligned}\|(\mathrm{i}\omega - \mathbf{L})^{-1}\|_{\mathscr{L}(\mathcal{X})} &\leq \frac{c}{|\omega|},\\ \|(\mathrm{i}\omega - \mathbf{L})^{-1}\|_{\mathscr{L}(\mathcal{Y},\mathcal{Z})} &\leq \frac{c}{|\omega|^{1-\alpha}}.\end{aligned}$$

Remark 2.2. *It is important to notice that the above Hypotheses can only be satisfied in the semilinear case $\mathcal{Y} \subset \mathcal{X}$ with $\mathcal{Y} \neq \mathcal{X}$. Usually, a weaker assumption is required for the linear operator $\mathbf{L}$, but we rather prefer to give the above characterizations as they are easier to verify in practice. It is also interesting to note that when, $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$ are all Hilbert spaces, then one needs only to check that only the first inequality of Hypothesis 2.3 is satisfied. In Hilbert spaces, for operators $\mathbf{L}$ that are sectorial and generate an analytic semigroup, then Hypothesis 2.3 is automatically satisfied.*

As a consequence of Hypothesis 2.2 (ii), we can define the spectral projection $\mathbf{P}_0 \in \mathscr{L}(\mathcal{X})$, corresponding to σ_0, by the Dunford formula:

$$\mathbf{P}_0 = \frac{1}{2\pi i}\int_\Gamma (\lambda \mathrm{id} - \mathbf{L})^{-1} \mathrm{d}\lambda, \tag{6}$$

where Γ is a simple, oriented counterclockwise, Jordan curve surrounding σ_0 and lying entirely in $\{\lambda \in \mathbb{C} \mid |\mathrm{Re}\lambda| < \gamma\}$. Then

$$\mathbf{P}_0^2 = \mathbf{P}_0, \quad \mathbf{P}_0\mathbf{L}u = \mathbf{L}\mathbf{P}_0 u \quad \forall u \in \mathcal{Z},$$

and $\mathrm{im}\mathbf{P}_0$ is finite-dimensional (σ_0 consists of a finite number of eigenvalues with finite algebraic multiplicities). In Particular, it satisfies $\mathrm{im}\mathbf{P}_0 \subset \mathcal{Z}$ and $\mathbf{P}_0 \in \mathcal{L}(\mathcal{X},\mathcal{Z})$. We define a second projector $\mathbf{P}_h : \mathcal{X} \to \mathcal{X}$ by

$$\mathbf{P}_h = Id - \mathbf{P}_0$$

which also satisfies

$$\mathbf{P}_h^2 = \mathbf{P}_h, \quad \mathbf{P}_h\mathbf{L}u = \mathbf{L}\mathbf{P}_h u \quad \forall u \in \mathcal{Z},$$

and

$$\mathbf{P}_h \in \mathscr{L}(\mathcal{X}) \cap \mathscr{L}(\mathcal{Y}) \cap \mathscr{L}(\mathcal{Z}).$$

We consider the spectral subspaces associated with these two projections:

$$\mathcal{E}_0 = \mathrm{im}\mathbf{P}_0 = \ker \mathbf{P}_h \subset \mathcal{Z}, \quad \mathcal{X}_h = \mathrm{im}\mathbf{P}_h = \ker \mathbf{P}_0 \subset \mathcal{X}$$

which provide the decomposition:

$$\mathcal{X} = \mathcal{X}_h \oplus \mathcal{E}_0.$$

We also denote

$$\mathcal{Z}_h = \mathbf{P}_h\mathcal{Z} \subset \mathcal{Z}, \quad \mathcal{Y}_h = \mathbf{P}_h\mathcal{Y} \subset \mathcal{Y}$$

and denote by $\mathbf{L}_0 \in \mathscr{L}(\mathcal{E}_0)$ and $\mathbf{L}_h \in \mathscr{L}(\mathcal{Z}_h, \mathcal{X}_h)$ the restrictions of $\mathbf{L}$ to $\mathcal{E}_0$ and $\mathcal{Z}_h$. The spectrum of $\mathbf{L}_0$ is σ_0 and the spectrum of $\mathbf{L}_h$ is $\sigma_+ \cup \sigma_-$.

Theorem 2.1 (Center manifold theorem)**.** *Assume that hypotheses 2.1, 2.2 and 2.3 hold. Then there exists a map $\Psi \in \mathscr{C}^k(\mathcal{E}_0, \mathcal{Z}_h)$, with*

$$\Psi(0) = 0, \quad D\Psi(0) = 0,$$

and a neighborhood $\mathcal{O}$ of 0 in $\mathcal{Z}$ such that the manifold:

$$\mathcal{M}_0 = \{u_0 + \Psi(u_0) \mid u_0 \in \mathcal{E}_0\} \subset \mathcal{Z}$$

has the following properties:

(i) $\mathcal{M}_0$ is locally invariant: if u is a solution of equation (5) satisfying $u(0) \in \mathcal{M}_0 \cap \mathcal{O}$ and $u(t) \in \mathcal{O}$ for all $t \in [0, T]$, then $u(t) \in \mathcal{M}_0$ for all $t \in [0, T]$.

(ii) $\mathcal{M}_0$ contains the set of bounded solutions of (5) staying in $\mathcal{O}$ for all $t \in \mathbb{R}$.

The manifold $\mathcal{M}_0$ is called a **local center manifold** *of (5) and the map Ψ is referred to as the* **reduction function***.*

Let u be a solution of (5) which belongs to $\mathcal{M}_0$, then $u = u_0 + \Psi(u_0)$ and u_0 satisfies:

$$\frac{\mathrm{d}u_0}{\mathrm{d}t} = \mathbf{L}_0 u_0 + \mathbf{P}_0 \mathbf{R}(u_0 + \Psi(u_0)). \tag{7}$$

The reduction function Ψ satisfies:

$$D\Psi(u_0)(\mathbf{L}_0 u_0 + \mathbf{P}_0 \mathbf{R}(u_0 + \Psi(u_0))) = \mathbf{L}_h \Psi(u_0) + \mathbf{P}_h \mathbf{R}(u_0 + \Psi(u_0)), \quad \forall u_0 \in \mathcal{E}_0. \tag{8}$$

Proof. The proof is in spirit very close to the one presented in the finite-dimensional case where one needs to work on the function space of exponentially growing functions and modify (truncate) the nonlinear part $\mathbf{R}(u)$ in order to obtain small Lipschitz constant via $\mathbf{R}^\epsilon(u) = \chi(u_0/\epsilon)\mathbf{R}(u)$ where χ is a smooth bounded cut-off function taking values in $[0, 1]$. If we write any solution of (5) $u = u_0 + u_h$, where $u_0 = \mathbf{P}_0 u \in \mathcal{E}_0$ and $u_h = \mathbf{P} u \in \mathcal{Z}_h$, we obtain a system

$$\frac{\mathrm{d}u_0}{\mathrm{d}t} = \mathbf{L}_0 u_0 + \mathbf{P}_0 \mathbf{R}^\epsilon(u), \tag{9a}$$

$$\frac{\mathrm{d}u_h}{\mathrm{d}t} = \mathbf{L}_h u_h + \mathbf{P}_h \mathbf{R}^\epsilon(u). \tag{9b}$$

Then the idea is to use a fixed-point argument for the above system (9). First, we notice that Hypothesis 2.3 allows us to solve the second equation on the hyperbolic part such that

$$u_h = \mathbf{K}_h \mathbf{P}_h \mathbf{R}^\epsilon(u),$$

for a linear map $\mathbf{K}_h \in \mathscr{L}\left(\mathscr{C}_\eta(\mathbb{R}, \mathcal{Y}_h), \mathscr{C}_\eta(\mathbb{R}, \mathcal{Z}_h)\right)$, and some $\eta > 0$, with

$$\|\mathbf{K}_h\|_{\mathscr{L}(\mathscr{C}_\eta(\mathbb{R}, \mathcal{Y}_h), \mathscr{C}_\eta(\mathbb{R}, \mathcal{Z}_h))} \leq C(\eta),$$

where $C : [0,\gamma] \to \mathbb{R}$ is continuous. We refer to [7, Appendix B.2] for a proof of the above statement. We can finally write system (9) as

$$u_0(t) = \mathbf{S}_{0,\epsilon}(u,t,u_0(0)) := e^{\mathbf{L}_0 t}u_0(0) + \int_0^t e^{\mathbf{L}_0(t-s)}\mathbf{P}_0\mathbf{R}^\epsilon(u(s))\mathrm{d}s, \tag{10a}$$

$$u_h = \mathbf{S}_{h,\epsilon}(u) = \mathbf{K}_h\mathbf{P}_h\mathbf{R}^\epsilon(u), \tag{10b}$$

where $u_0(0) \in \mathcal{E}_0$ is arbitrary. Note that $e^{\mathbf{L}_0 t}$ exists since $\mathcal{E}_0$ is finite-dimensional. We will look for solutions

$$u = (u_0, u_h) \in \mathcal{N}_{\eta,\epsilon} := \mathscr{C}_\eta(\mathbb{R}, \mathcal{E}_0) \times \mathscr{C}_0(\mathbb{R}, B_\epsilon(\mathcal{Z}_h)),$$

with $0 < \eta \leq \gamma$ and $\epsilon \in (0, \epsilon_0)$. More precisely, using a fixed point argument for the map

$$\mathbf{S}_\epsilon(u, u_0(0)) := \left(\mathbf{S}_{0,\epsilon}(u,\cdot,u_0(0)), \mathbf{S}_{h,\epsilon}(u)\right),$$

which enjoys the properties

- $\mathbf{S}_\epsilon(\cdot, u_0(0)) : \mathcal{N}_{\eta,\epsilon} \to \mathcal{N}_{\eta,\epsilon}$ is well defined,
- $\mathbf{S}_\epsilon(\cdot, u_0(0))$ is a contraction with respect to the norm of $\mathscr{C}_\eta(\mathbb{R}, \mathcal{X})$ for ϵ small enough and any $\eta \in [0, \gamma)$,

one can show that system (10) has a unique solution $u = (u_0, u_h) = \Lambda(u_0(0)) \in \mathcal{N}_{\eta,\epsilon}$ for any $u_0(0) \in \mathcal{E}_0$. We define the map Ψ of the theorem via

$$(u_0(0), \Psi(u_0(0))) := \Lambda(u_0(0))(0), \qquad \text{for all } u_0(0) \in \mathcal{E}_0.$$

The fixed point argument gives naturally the Lipschitz continuity of the map Ψ. In order to get the $\mathscr{C}^k$ regularity of Ψ one needs to use scale of Banach spaces to ensure the regularity of $\mathbf{R}^\epsilon$ on exponentially growing functions spaces. More precisely, it can be proved that $\mathbf{R}^\epsilon : \mathscr{C}_\eta(\mathbb{R}, \mathcal{Z}) \to \mathscr{C}_\zeta(\mathbb{R}, \mathcal{Y})$ is $\mathscr{C}^k$ for any $0 \leq \eta < \zeta/k$ and $\zeta > 0$ which in turn can be used to prove the desired regularity for Ψ (see [7, 14] for further details). ∎

2.3 Parameter-dependent center manifold

We consider a parameter-dependent differential equation in $\mathcal{X}$ of the form

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \mathbf{L}u + \mathbf{R}(u,\mu) \tag{11}$$

where $\mathbf{L}$ is a linear operator as in the previous section, and the nonlinear part $\mathbf{R}$ is defined for (u,μ) in a neighborhood of $(0,0) \in \mathcal{Z} \times \mathbb{R}^m$. Here $\mu \in \mathbb{R}^m$ is a paramter that we assume to be small. More precisely we keep hypotheses 2.2 and 2.3 and replace hypothesis 2.1 by the following:

Hypothesis 2.4 (Regularity)**.** *We assume that* $\mathbf{L}$ *and* $\mathbf{R}$ *in* (11) *have the following properties:*

(i) $\mathbf{L} \in \mathscr{L}(\mathcal{Z}, \mathcal{X})$,

(ii) for some $k \geq 2$, there exists a neighborhood $\mathcal{V}_u \subset \mathcal{Z}$ and $\mathcal{V}_\mu \subset \mathbb{R}^m$ of 0 such that $\mathbf{R} \in \mathscr{C}^k(\mathcal{V}_u \times \mathcal{V}_\mu, \mathcal{Y})$ and

$$\mathbf{R}(0,0) = 0, \quad D_u\mathbf{R}(0,0) = 0.$$

Theorem 2.2 (Parameter-dependent center manifold theorem)**.** *Assume that hypotheses 2.4, 2.2 and 2.3 hold. Then there exists a map $\Psi \in \mathscr{C}^k(\mathcal{E}_0 \times \mathbb{R}^m, \mathcal{Z}_h)$, with*

$$\Psi(0,0) = 0, \quad D_u\Psi(0,0) = 0,$$

and a neighborhood $\mathcal{O}_u \times \mathcal{O}_\mu$ of 0 in $\mathcal{Z} \times \mathbb{R}^m$ such that for $\mu \in \mathcal{O}_\mu$ the manifold:

$$\mathcal{M}_0(\mu) = \{u_0 + \Psi(u_0, \mu) \mid u_0 \in \mathcal{E}_0\} \subset \mathcal{Z}$$

has the following properties:

(i) $\mathcal{M}_0(\mu)$ is locally invariant: if u is a solution of equation (11) satisfying $u(0) \in \mathcal{M}_0(\mu) \cap \mathcal{O}_u$ and $u(t) \in \mathcal{O}_u$ for all $t \in [0,T]$, then $u(t) \in \mathcal{M}_0(\mu)$ for all $t \in [0,T]$;

(ii) $\mathcal{M}_0(\mu)$ contains the set of bounded solutions of (11) staying in $\mathcal{O}_u$ for all $t \in \mathbb{R}$.

Let u be a solution of (11) which belongs to $\mathcal{M}_0(\mu)$, then $u = u_0 + \Psi(u_0, \mu)$ and u_0 satisfies:

$$\frac{\mathrm{d}u_0}{\mathrm{d}t} = \mathbf{L}_0 u_0 + \mathbf{P}_0\mathbf{R}(u_0 + \Psi(u_0,\mu), \mu) \stackrel{def}{=} f(u_0, \mu) \tag{12}$$

where we observe that $f(0,0) = 0$ and $D_{u_0}f(0,0) = \mathbf{L}_0$ has spectrum σ_0. The reduction function Ψ satiafies:

$$D_{u_0}\Psi(u_0,\mu)f(u_0,\mu) = \mathbf{L}_h\Psi(u_0,\mu) + \mathbf{P}_h\mathbf{R}(u_0 + \Psi(u_0,\mu), \mu) \quad \forall u_0 \in \mathcal{E}_0.$$

Proof. The idea here is to consider the constant μ as an extra differential equation by saying that μ solves the equation

$$\frac{\mathrm{d}\mu}{\mathrm{d}t} = 0.$$

Then one augments equation (11) by

$$\frac{\mathrm{d}\tilde{u}}{\mathrm{d}t} = \widetilde{\mathbf{L}}\tilde{u} + \widetilde{\mathbf{R}}(\tilde{u}), \quad \tilde{u} = (u, \mu),$$

where $\widetilde{\mathbf{L}}\tilde{u} := (\mathbf{L}u + D_\mu\mathbf{R}(0,0)\mu, 0)$ and $\widetilde{\mathbf{R}}(\tilde{u}) = (\mathbf{R}(u,\mu) - D_\mu\mathbf{R}(0,0)\mu, 0)$. One then only need to check that Hypotheses 2.1, 2.2 and 2.3 hold for $\widetilde{\mathbf{L}}$ and $\widetilde{\mathbf{R}}$. ■

2.4 Equivariant systems

Hypothesis 2.5 (Equivariant equation)**.** *We assume that there exists a linear operator $\mathbf{T} \in \mathscr{L}(\mathcal{X}) \cap \mathscr{L}(\mathcal{Z})$, which communtes with vector field in equation (5):*

$$\mathbf{T}\mathbf{L}u = \mathbf{L}\mathbf{T}u, \quad \mathbf{T}\mathbf{R}(u) = \mathbf{R}(\mathbf{T}u)$$

We also assume that the restriction $\mathbf{T}_0$ of $\mathbf{T}$ to $\mathcal{E}_0$ is an isometry.

Theorem 2.3 (Equivariant center manifold)**.** *Under the assumption of theorem 2.1, we further assume that hypothesis 2.5 holds. Then one can find a reduction function* Ψ *which commutes with* $\mathbf{T}$*:*

$$\mathbf{T}\Psi u_0 = \Psi(\mathbf{T}_0 u_0), \quad \forall u_0 \in \mathcal{E}_0$$

and such that the vector field in the reduced equation (7) *commutes with* $\mathbf{T}_0$*.*

Proof. The uniqueness of the center manifold via the fixed point argument ensures that the manifold $\mathcal{M}_0$ is invariant under $\mathbf{T}$ provided that system (9) is equivariant under $\mathbf{T}$. This will be satisfied if the cut-off function χ satisfies

$$\chi(\mathbf{T}_0 u_0) = \chi(u_0) \text{ for all } u_0 \in \mathcal{E}_0,$$

which can always be achieved by choosing χ to be a smooth function of $\|u_0\|^2$ where $\|\cdot\|$ stands for the Euclidean norm on $\mathcal{E}_0$. Since $\mathbf{T}_0$ is an isometry on $\mathcal{E}_0$, the conclusion follows. ∎

Remark 2.3. *Analogous results hold for the parameter-dependent equation* (11)*.*

2.5 Empty unstable spectrum

Theorem 2.4 (Center manifold for empty unstable spectrum)**.** *Under the assumptions of theorem 2.1 and assume that* $\sigma_+ = \varnothing$*. Then in addition to propertries of theorem 2.1, the local center manifold* $\mathcal{M}_0$ *is locally attracting: any solution of equation* (5) *that stays in* $\mathcal{O}$ *for all* $t > 0$ *tends exponentially towards a solution of* (5) *on* $\mathcal{M}_0$*.*

3 Normal forms

The normal forms theory consists in finding a polynomial change of variable which improves locally a nonlinear system, in order to recognize more easily its dynamics. In applications, normal form transformation are performed after a center manifold reduction.

3.1 Main theorem

We consider a parameter-dependent differential equations in $\mathbb{R}^n$ of the form

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \mathbf{L}u + \mathbf{R}(u, \mu) \tag{13}$$

in which we assume that $\mathbf{L}$ and $\mathbf{R}$ satisfy the following hypothesis.

Hypothesis 3.1 (Regularity)**.** *Assume that* $\mathbf{L}$ *and* $\mathbf{R}$ *have the following properties:*

(i) $\mathbf{L}$ *is a linear map in* $\mathbb{R}^n$*;*

(ii) for some $k \geq 2$, there exist neighborhoods $\mathcal{V}_u \subset \mathbb{R}^n$ and $\mathcal{V}_\mu \subset \mathbb{R}^m$ of 0 such that $\mathbf{R} \in \mathscr{C}^k(\mathcal{V}_u \times \mathcal{V}_\mu, \mathbb{R}^n)$ and

$$\mathbf{R}(0,0) = 0, \quad D_u\mathbf{R}(0,0) = 0.$$

Theorem 3.1 (Normal form theorem)**.** *Assume that hypothesis 3.1 holds. Then for any positive integer p, $2 \leq p \leq k$, there exist neighborhoods $\mathcal{V}_1$ and $\mathcal{V}_2$ of 0 in $\mathbb{R}^n$ and $\mathbb{R}^m$ such that for $\mu \in \mathcal{V}_2$, there is a polynomial map $\Phi_\mu : \mathbb{R}^n \to \mathbb{R}^n$ of degree p with the following properties:*

(i) the coefficients of the monomials of degree q in Φ_μ are functions of μ of class $\mathscr{C}^{k-q}$ and

$$\Phi_0(0) = 0, \quad D_u\Phi_0(0) = 0$$

(ii) for $v \in \mathcal{V}_1$, the polynomial change of variable

$$u = v + \Phi_\mu(v)$$

transforms equation (13) into the normal form:

$$\frac{dv}{dt} = \mathbf{L}v + \mathbf{N}_\mu(v) + \rho(v,\mu)$$

and the following properties hold:

(a) for any $\mu \in \mathcal{V}_2$, $\mathbf{N}_\mu$ is a polynomial map $\mathbb{R}^n \to \mathbb{R}^n$ of degree p, with coefficients depending upon μ, such that the coefficients of the monomials of degree q are of class $\mathscr{C}^{k-q}$ and

$$\mathbf{N}_0(0) = 0, \quad D_v\mathbf{N}_0(0) = 0$$

(b) the equality $\mathbf{N}_\mu(e^{t\mathbf{L}^*}v) = e^{t\mathbf{L}^*}\mathbf{N}_\mu(v)$ holds for all $(t,v) \in \mathbb{R} \times \mathbb{R}^n$ and $\mu \in \mathcal{V}_2$

(c) the map ρ belongs to $\mathscr{C}^k(\mathcal{V}_1 \times \mathcal{V}_2, \mathbb{R}^n)$ and

$$\rho(v,\mu) = o(\|v\|^p) \quad \forall \mu \in \mathcal{V}_2$$

3.2 An example – The Hopf bifurcation

Consider an equation of the form (13) with a single parameter $\mu \in \mathbb{R}$ and satisfying the hypotheses of the center manifold theorem 2.2. Assume that the center part of the spectrum σ_0 of the linear operator $\mathbf{L}$ contains two purely imaginary eigenvalues $\pm \mathrm{i}\omega$, which are simple. Under these assumptions, we have $\sigma_0 = \{\pm \mathrm{i}\omega\}$ and $\mathcal{E}_0$ is two-dimensional spanned by the eigenvectors $\zeta, \bar{\zeta}$ associated with $\mathrm{i}\omega$ and $-\mathrm{i}\omega$ respectively. The center manifold theorem 2.2 gives

$$u = u_0 + \Psi(u_0, \mu), \quad u_0 \in \mathcal{E}_0,$$

and applying the normal form theorem 3.1 we find

$$u_0 = v_0 + \Phi_\mu(v_0),$$

which gives:

$$u = v_0 + \widetilde{\Psi}(v_0, \mu), \quad u_0 \in \mathcal{E}_0. \tag{14}$$

For $v_0(t) \in \mathcal{E}_0$, we write

$$v_0(t) = A(t)\zeta + \overline{A(t)\zeta}, \quad A(t) \in \mathbb{C}$$

Lemma 3.1. *The polynomial $\mathbf{N}_\mu$ in theorem 3.1 is of the form:*

$$\mathbf{N}_\mu(A, \overline{A}) = (AQ(|A|^2, \mu), \overline{AQ}(|A|^2, \mu)),$$

where Q is a complex-valued polynomial in its argument, satisfying $Q(0,0) = 0$ and of the form:

$$Q(|A|^2, \mu) = a\mu + b|A|^2 + O((|\mu| + |A|^2)^2).$$

In applications, one is interested in computing the values of a and b. We explain below a procedure which allows to obtain explicit formula for these coefficients. First, we write the Taylor expansion of $\mathbf{R}$ and $\widetilde{\Psi}$:

$$\mathbf{R}(u, \mu) = \sum_{1 \le q+l \le p} \mathbf{R}_{ql}(u^{(q)}, \mu^{(l)}) + o((|\mu| + \|u\|)^p)$$

$$\widetilde{\Psi}(v_0, \mu) = \sum_{1 \le q+l \le p} \widetilde{\Psi}_{ql}(v_0^{(q)}, \mu^{(l)}) + o((|\mu| + \|v_0\|)^p)$$

$$\widetilde{\Psi}_{ql}(v_0^{(q)}, \mu^{(l)}) = \mu^l \sum_{q_1+q_2=q} A^{q_1}\overline{A}^{q_2}\Psi_{q_1q_2l}$$

We differentiate equation (14) and obtain:

$$D_{v_0}\widetilde{\Psi}(v_0, \mu)\mathbf{L}_0 v_0 - \mathbf{L}\widetilde{\Psi}(v_0, \mu) + \mathbf{N}_\mu(v_0) = \mathbf{Q}(v_0, \mu)$$

where

$$\mathbf{Q}(v_0, \mu) = \Pi_p \Big(\mathbf{R}(v_0 + \widetilde{\Psi}(v_0, \mu), \mu) - D_{v_0}\widetilde{\Psi}(v_0, \mu)\mathbf{N}_\mu(v_0)\Big)$$

Here Π_p represents the linear map that associates to map of class $\mathscr{C}^p$ the polynomial of degree p in its Taylor expansion. We then replace the Taylor expansions of $\mathbf{R}$ and $\widetilde{\Psi}$ and by identifying the terms of order $O(\mu), O(A^2)$ and $O(|A|^2)$ we obtain:

$$\begin{aligned} -\mathbf{L}\Psi_{001} &= \mathbf{R}_{01} \\ (2\mathrm{i}\omega - \mathbf{L})\Psi_{200} &= \mathbf{R}_{20}(\zeta, \zeta) \\ -\mathbf{L}\Psi_{110} &= 2\mathbf{R}_{20}(\zeta, \bar{\zeta}) \end{aligned}$$

Here the operators $\mathbf{L}$ and $(2\mathrm{i}\omega - \mathbf{L})$ are invertible so that Ψ_{001}, Ψ_{200} and Ψ_{110} are uniquely determined. Next we identify the terms of order $O(\mu A)$ and $O(A|A|^2)$

$$\begin{aligned} (\mathrm{i}\omega - \mathbf{L})\Psi_{101} &= -a\zeta + \mathbf{R}_{11}(\zeta) + 2\mathbf{R}_{20}(\zeta, \Psi_{001}) \\ (\mathrm{i}\omega - \mathbf{L})\Psi_{210} &= -b\zeta + 2\mathbf{R}_{20}(\zeta, \Psi_{110}) + 2\mathbf{R}_{20}(\bar{\zeta}, \Psi_{200}) + 3\mathbf{R}_{30}(\zeta, \zeta, \bar{\zeta}) \end{aligned}$$

Since $\mathrm{i}\omega$ is a simple isolated eigenvalue of $\mathbf{L}$, the range of $(\mathrm{i}\omega - \mathbf{L})$ is of codimension one so we can solve these equations and determine Ψ_{101} and Ψ_{210}, provided the right hand sides satisfy one solvability condition. This solvability condition allows to compute coefficients a and b.

- If $\mathbf{L}$ has an adjoint $\mathbf{L}^*$ acting on the dual space $\mathcal{X}^*$, the solvability condition is that the right hand sides be orthogonal to the kernel of the adjoint $(-\mathrm{i}\omega - \mathbf{L}^*)$ of $(\mathrm{i}\omega - \mathbf{L})$. The kernel of $(-\mathrm{i}\omega - \mathbf{L}^*)$ is just one-dimensional, spanned by $\zeta^* \in \mathcal{X}^*$ with $\langle \zeta, \zeta^* \rangle = 1$. Here $\langle \cdot, \cdot \rangle$ denotes the duality product between $\mathcal{X}$ and $\mathcal{X}^*$. We find:

$$\begin{aligned} a &= \langle \mathbf{R}_{11}(\zeta) + 2\mathbf{R}_{20}(\zeta, \Psi_{001}), \zeta^* \rangle \\ b &= \langle 2\mathbf{R}_{20}\left(\bar{\zeta}, \Psi_{200}\right) + 2\mathbf{R}_{20}\left(\zeta, \Psi_{110}\right) + 3\mathbf{R}_{30}(\zeta, \zeta, \bar{\zeta}), \zeta^* \rangle \end{aligned}$$

- If the adjoint $\mathbf{L}^*$ does not exist, we use a Fredholm alternative since both equations have the form:

$$(\mathrm{i}\omega - \mathbf{L})\Psi = \mathbf{R}, \text{ with } \mathbf{R} \in \mathcal{X}$$

  We project with $\mathbf{P}_0$ and $\mathbf{P}_h$ on the subspaces $\mathcal{E}_0$ and $\mathcal{X}_h$ and we obtain

$$\begin{aligned} (\mathrm{i}\omega - \mathbf{L}_0)\mathbf{P}_0\Psi &= \mathbf{P}_0\mathbf{R} \\ (\mathrm{i}\omega - \mathbf{L}_h)\mathbf{P}_h\Psi &= \mathbf{P}_h\mathbf{R} \end{aligned}$$

  The operator $(i\omega - \mathbf{L}_h)$ is invertible, then the second equation has a unique solution. The first equation is two-dimensional, there is a solution Ψ_0 provided the solvability condition holds

$$\langle \mathbf{R}_0, \zeta_0^* \rangle = 0$$

  where $\zeta_0^* \in \mathcal{E}_0$ is the eigenvector in the kernel of the adjoint $(-\mathrm{i}\omega - \mathbf{L}_0^*)$ in $\mathcal{E}_0$ chosen such that $\langle \zeta, \zeta_0^* \rangle = 1$. If $\mathbf{P}_0^*$ is the adjoint of $\mathbf{P}_0$ and setting $\zeta^* = \mathbf{P}_0^*\zeta_0^*$ the solvability condition becomes $\langle \mathbf{R}, \zeta^* \rangle = 0$ which leads to the same formula for a and b as above.

4 Steady-state bifurcation with symmetry – General results

We first start this section by stating some basic definitions and results on groups and their representations. In most of the following, G will be a finite group or a closed subgroup of $\mathbf{O}(n)$, the group of $n \times n$ orthogonal matrices with real entries, acting isometrically in $\mathbb{R}^n$. Such a subgroup is also a submanifold of the Lie group $\mathbf{O}(n)$ and is therefore itself a Lie group.
Some Examples:

- $\mathbb{Z}_k = \mathbb{Z}/k\mathbb{Z}$, the k-cyclic group, is isomorphic to $\mathbf{C}_k$, the group generated by the 2 by 2 matrix

$$\rho_k := \begin{pmatrix} \cos\frac{2\pi}{k} & \sin\frac{2\pi}{k} \\ -\sin\frac{2\pi}{k} & \cos\frac{2\pi}{k} \end{pmatrix}.$$

  The group $\mathbf{C}_k$ acts isometrically in $\mathbb{R}^2$.

- We note $\mathbf{D}_k$ the group generated by ρ_k and by the reflection

$$\kappa := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

This is the symmetry group of a regular polygon with k vertices (k-th dihedral group). Its order is $|\mathbf{D}_k| = 2k$.

- $\mathbf{S}^1 = \mathbb{R}/2\pi\mathbb{Z}$ is isomorphic to $\mathbf{SO}(2)$, the group of 2 by 2 rotation matrices. The group generated by $\mathbf{SO}(2)$ and the reflection matrix σ is $\mathbf{O}(2)$, the symmetry group of the circle.

We shall also be led to consider the non compact Euclidean group $\mathbf{E}(n) = \mathbf{O}(n) \ltimes \mathbb{R}^n$ (at least in the case $n = 2$) and we refer to subsection 4.3 for further definitions.

4.1 Definitions

4.1.1 Irreducible representations

Definition 4.1. *A* **representation** *of a group* Γ *in a finite-dimensional or a Banach space* $\mathcal{X}$ *is a continuous homomorphism* $\tau : \Gamma \to \mathbf{GL}(\mathcal{X})$ *from* Γ *to the group of invertible linear maps in* $\mathcal{X}$*.*

Therefore a representation τ verifies that $\tau(\gamma_1\gamma_2) = \tau(\gamma_1)\tau(\gamma_2)$, in particular $\tau(\gamma^{-1}) = \tau^{-1}(\gamma)$ and $\tau(\mathrm{e}) = \mathrm{id}_{\mathcal{X}}$. Note also that if $\ker(\tau) = \{0\}$, the image of Γ under τ is a group isomorphic to Γ and we call it the transformation group associated with Γ. We denote by Γ this group.

Example: Let $\mathscr{C}(\mathbb{R}^n)$ be a space of functions $\mathbb{R}^n \to \mathbb{R}$ (e.g. continuous functions) and let Γ be a subgroup of $\mathbf{O}(n)$. Then the relation $\tau(\gamma) \cdot u(x) = u\left(\gamma^{-1}x\right)$ defines a representation of Γ in $\mathscr{C}(\mathbb{R}^n)$.

Definition 4.2. *A representation is* **irreducible** *if the only subspaces of* $\mathcal{X}$ *which are invariant by* $\tau(\gamma)$ *for all* $\gamma \in \Gamma$ *are* $\{0\}$ *and* $\mathcal{X}$ *itself.*

Examples: $\mathbf{C}_k$, $\mathbf{D}_k$, $\mathbf{SO}(2)$, $\mathbf{0}(2)$ act irreducibly in $\mathbb{R}^2$.

Definition 4.3. *Two representations* τ *and* τ' *of the same group* Γ *are called* **equivalent** *if there exists a matrix* $M \in \mathbf{GL}(\mathbb{R}^n)$ *(or an endomorphism* $M \in \mathbf{GL}(\mathcal{X})$*) such that* $\tau' = M \circ \tau \circ M^{-1}$*.*

It is important to remark that representations of finite or compact groups can always be decomposed into direct sums of irreducible ones. This decomposition might not be unique because equivalent representations can occur several times, allowing for many choices of the corresponding representation spaces. This problem of non-uniqueness can be overcome by grouping irreducible representations in equivalence classes. This leads to a block decomposition of a representation which is *unique* and called the *isotypic decomposition* of representation.

Lemma 4.1 (Schur's lemma)**.** *Let* τ*,* ρ *be two complex irreducible representations of a compact group* Γ *in* $\mathcal{X}$*,* $\mathcal{Y}$ *respectively. Let* $\mathscr{A} : \mathcal{X} \to \mathcal{Y}$ *be a linear map such that* $\mathscr{A}\tau(\gamma) = \rho(\gamma)\mathscr{A}$ *for all* $\gamma \in \Gamma$*. Then:*

(i) if τ *and* ρ *are equivalent, then* $\mathcal{X} \sim \mathcal{Y}$ *and* $\mathscr{A} = c \cdot \mathrm{id}$ *for some* $c \in \mathbb{C}$*;*

(ii) if τ and ρ are not equivalent, then $\mathscr{A} = 0$.

Proof. Note first that the kernel as well as the range of a linear map which commutes with a group representation are invariant under this representation. To prove the first part, note that $\mathscr{A}$ has at least one eigenvalue c. Hence $\ker(\mathscr{A} - c \cdot \mathrm{id}) \neq \{0\}$. This kernel is Γ invariant, therefore by irreducibility assumption, it is equal to $\mathcal{X}$. Now $\dim \mathcal{X} = \dim \mathcal{Y}$ by the equivalence of τ and ρ. It follows that $\mathscr{A} = c \cdot \mathrm{id}$. Suppose now $\mathscr{A}$ is invertible, then τ and ρ are clearly equivalent. Therefore assume $\mathscr{A}$ is not invertible. By the argument above, $\ker \mathscr{A} = \mathcal{X}$ and $\mathrm{im}\mathscr{A} = \mathcal{Y}$, which imply $\mathscr{A} = 0$. ∎

Definition 4.4. *A representation τ of a compact group Γ in a (finite dimensional) space $\mathcal{X}$ is* **absolutely irreducible** *if all linear maps $\mathscr{A}$ which commute with τ are scalar multiples of the identity.*

By Schur's lemma, any irreducible representation in a complex space is absolutely irreducible, but this is not true in general for representations in a real space.

4.1.2 Equivariant vector fields

Definition 4.5. *Let $\mathcal{X}$, $\mathcal{Y}$ be two vector spaces with representations τ and ρ resp. of a group Γ. A continuous map $f: \mathcal{X} \to \mathcal{Y}$ is Γ-equivariant if $f(\tau(\gamma)x) = \rho(\gamma)f(x)$ for all $\gamma \in \Gamma$ and $x \in \mathcal{X}$.*

Later on in this section, we will be dealing with differential equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x) \tag{15}$$

where f is Γ-equivariant. Obviously, a first consequence of the Γ-equivariance is that if $x(t)$ is a solution of (15), then $\tau(\gamma)x(t)$ is also a solution for all t. From one solution we therefore obtain a Γ-*orbit* of identical solutions up to symmetry, which are obtained by applying the transformations $\tau(\tau)$ to it. One can say more.

Definition 4.6. *We give the following definitions:*

(i) Let $x \in \mathcal{X}$, we define $\Sigma = \Gamma^x = \{\gamma \in \Gamma \mid \tau(\gamma)x = x\}$. Σ is the **isotropy subgroup** *of x. Note that the isotropy group of $\tau(\gamma)x$ is $\gamma\Sigma\gamma^{-1}$, and when one talks about classification of isotropy subgroups (for a given action), it means "classification of conjugacy classes".*

(ii) Given an isotropy subgroup Σ, let $\mathrm{Fix}(\Sigma) := \{x \in \mathcal{X} \mid \tau(\sigma)x = x$ for all $\sigma \in \Sigma\}$. This is a linear subspace of $\mathcal{X}$.

(iii) Let $x \in \mathcal{X}$, the set $\Gamma \cdot x = \{\tau(\gamma)x,\ \gamma \in \Gamma\}$ is called the Γ-orbit of x.

It is easily seen that any two points in the Γ-orbit of a point x have conjugated isotropy subgroups. The conjugacy classes of the isotropy subgroups of G for its action (representation) in X are called the **isotropy types** of this action.

Example. Let $\mathbf{D}_3$ act in $\mathbb{R}^2$ by its natural action. The isotropy subgroups are $\{\mathrm{id}\}$, which fixes all points in $\mathbb{R}^2$, the two-element groups of reflection across the axes of symmetry of an equilateral triangle (these axes are the subspaces $\mathrm{Fix}(\Sigma)$ for these subgroups), and Γ itself which fixes only the origin. Given a point away from the axes of symmetry, its $\mathbf{D}_3$-orbit consists of 6 points. If however we consider a point on one of the axes of symmetry, then its $\mathbf{D}_3$-orbit consists only of 3 points. The orbit of the origin is the origin itself. This is a general fact:

Lemma 4.2. *If Γ is finite, the number of elements in $\Gamma \cdot x$ is equal to $|\Gamma|/|\Gamma^x|$. If Γ is a (compact) Lie group, then $\Gamma \cdot x$ is a submanifold of $\mathcal{X}$ with dimension equal to $\dim(\Gamma) - \dim(\Gamma^x)$.*

The following lemma is fundamental.

Lemma 4.3. *Let f be Γ-equivariant. Then for any isotropy subgroup σ, $f : \mathrm{Fix}(\Sigma) \to \mathrm{Fix}(\Sigma)$.*

Proof. Observe that given $x \in \mathrm{Fix}(\Sigma)$ and $\sigma \in \Sigma$, we have that $\tau(\sigma)f(x) = f(\tau(\sigma)x) = f(x)$. ■

Therefore given an initial condition in $\mathrm{Fix}(\Sigma)$, the full trajectory belongs to the subspace $\mathrm{Fix}(\Sigma)$. We write $N(\Sigma) := \{\gamma \in \Gamma \mid \gamma\Sigma\gamma^{-1} = \Sigma\}$ the **normalizer** of a subgroup Σ of Γ.

Lemma 4.4. *The maximal subgroup of Γ acting faithfully (with no other fixed-point than 0) in $\mathrm{Fix}(\Sigma)$ is $N(\Sigma)/\Sigma$.*

Proof. We write the action of Γ as $(\gamma, x) \mapsto \gamma x$ to simplify notation. Let $x \in \mathrm{Fix}(\Sigma)$, then $\gamma x \in \mathrm{Fix}(\Sigma) \Rightarrow \sigma\gamma x = \gamma x$ for all $\sigma \in \Sigma$. Hence $\gamma^{-1}\sigma\gamma x = x$ and $\gamma^{-1}\sigma\gamma \in \Sigma$. The result follows immediately. ■

Therefore the group orbit of a point $x \in \mathrm{Fix}(\Sigma)$ is obtained by letting $\mathrm{Fix}(\Sigma)$ act on x.

Let x be an equilibrium point for equation (15). If Γ is a Lie group, $\Gamma \cdot x$ is a manifold with dimension equal to $\dim(\Gamma) - \dim(\Gamma^x)$. It may also happen that the vector field $f(x)$ be tangent to $\Gamma \cdot x$. If this happens, then $f(y)$ is tangent to $\Gamma \cdot x$ at any point $y \in \Gamma \cdot x$. In that case $\Gamma \cdot x$ is a flow-invariant manifold. This motivates the following definition.

Definition 4.7. *A trajectory of an equivariant dynamical system which lies in the group orbit of a point is called a* **relative equilibrium**.

Equilibria are particular cases of relative equilibria. What is in general the dynamics of a relative equilibrium? Let $x(t) = \Phi_t(x_0)$ be the solution with initial condition x_0. Here Φ_t denotes the Γ-equivariant 1-parameter group of transformations associated with the vector field f. For a relative equilibrium, at each t, there exists a group element γ_t such that $\Phi_t(x_0) = \tau(\gamma_t)x_0$. Moreover

$\gamma_{t+t'} = \gamma_t \gamma_{t'}$ by the group property of Φ_t. It follows that the set $\{\gamma_t,\ t \in \mathbb{R}\}$ is a one-parameter, abelian subgroup of Γ. The closure of an abelian subgroup in a compact Lie group is a torus. We conclude that the trajectories of a relative equilibrium fill tori in the Γ-orbit. The dimension of a torus group in a (compact) group Γ cannot exceed a value which defines the "maximal torus" in Γ. For example if $\Gamma = \mathbf{O}(2)$ then obviously the maximal torus is $\mathbf{S}^1$ (a circle). But if $\Gamma = \mathbf{SO}(3)$ or $\mathbf{O}(3)$ then the maximal torus has also dimension 1, despite the fact that $\dim \mathbf{SO}(3) = 3$. Therefore in these cases, the trajectories are closed (circles) and the relative equilibria are (at most) *periodic orbits*. One can even be more restrictive: since the trajectory of relative equilibrium lies inside a subspace $\mathrm{Fix}(\Sigma)$, the dimension of its closure cannot exceed the dimension of the maximal torus in the group $N(\Sigma)/\Sigma$.

4.2 Equivariant Branching Lemma

Recall, that in the previous section we have considered parameter-dependent differential equations in $\mathcal{X}$ of the form

$$\frac{du}{dt} = \mathbf{L}u + \mathbf{R}(u,\mu) = \mathcal{F}(u,\mu) \tag{16}$$

where $\mathbf{L}$ is a linear operator, and the nonlinear part $\mathbf{R}$ is defined for (u,μ) in a neighborhood of $(0,0) \in \mathcal{Z} \times \mathbb{R}^m$. Here $\mu \in \mathbb{R}^m$ is a parameter that we assume to be small. We suppose that $\mathcal{F}$ is Γ-equivariant with respect to a representation τ of the group Γ. If we apply the parameter-dependent center manifold 2.3 theorem for the equivariant differential equation (16), the reduced equation on $\mathcal{E}_0$ has the general form:

$$\frac{\mathrm{d}u_0}{\mathrm{d}t} = f(u_0,\mu),$$

with

$$\tau(\gamma) f(u_0,\mu) = f(\tau(\gamma)u_0,\mu), \quad \forall u_0 \in \mathcal{E}_0 \text{ and } \forall \gamma \in \Gamma.$$

Since $\mathcal{E}_0$ is a real space of dimension n, we may regard f as a map $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$. Moreover, Γ acts on $\mathbb{R}^n$ and f is equivariant for this action.

Suppose now that the action of Γ on $\mathbb{R}^n$ possesses an isotropy subgroup Σ with a one-dimensonal fixed point space $\mathrm{Fix}(\Sigma)$. If we look for solutions in $\mathrm{Fix}(\Sigma)$, the reduced equation on the center manifold restricts to a scalar equation. Recall that if Γ acts absolutely irreducibly on $\mathcal{E}_0$ then the linearization of f at the origin is a multiple of the identity and we have $D_u f(0,\mu) = c(\mu) I_n$ where I_n is the identity map of $\mathbb{R}^n$.

Theorem 4.1 (Steady-state Equivariant Branching Lemma)**.** *We suppose that the assumptions of theorem 2.2 hold. Assume that the compact group Γ acts linearly and that $\mathcal{F}$ is Γ-equivariant. We suppose that Γ acts absolutely irreducibly on $\mathcal{E}_0$. We also suppose that $\mathbf{L}$ has 0 as an isolated eigenvalue with finite multiplicity. If Σ is an isotropy subgroup of Γ with $\dim \mathrm{Fix}(\Sigma) = 1$ and if $c'(0) \neq 0$, then it exists a unique branche of solutions with symmetry Σ.*

As a consequence, if the Hypotheses in Theorem (4.1) are satisfied we have the following characterization for each isotropy subgroup Σ of Γ such that $\dim \mathrm{Fix}(\Sigma) = 1$ in $\mathcal{E}_0$, where either one of the following situations occurs (where $f(u_0, \mu)$ is the reduced vector field in $\mathrm{Fix}(\Sigma)$):

(i) Suppose $\Sigma = \Gamma$. If $D_\mu f(0,0) \neq 0$, there exists one branch of solution $u_0(\mu)$. If in addition $D^2_{uu} f(0,0) \neq 0$, then $u^2 = \mathcal{O}(\|\mu\|)$ (**saddle-node bifurcation**).

(ii) Suppose $\Sigma < \Gamma$ and the normalizer $N(\Sigma)$ acts trivially in $\mathrm{Fix}(\Sigma)$. Then $f(u_0, \mu) = u_0 h(u_0, \mu)$ and if $D^2_{u\mu} f(0,0) \neq 0$ there exists a branch of solution $u_0(\mu)$. If in addition $D^2_{uu} f(0,0) \neq 0$, then $u_0 = \mathcal{O}(\|\mu\|)$ (**transcritical bifurcation**).

(iii) Suppose $\Sigma < \Gamma$ and the normalizer $N(\Sigma)$ acts as -1 in $\mathrm{Fix}(\Sigma)$ (*i.e.* $N(H)/H \simeq \mathbb{Z}_2$) . Then $f(u_0, \mu) = u_0 h(u_0, \mu)$ with h an even function of u_0. If $D^2_{u\mu} f(0,0) \neq 0$ there exists a branch of solution $\pm u_0(\mu)$ such that if $D^3_{uuu} f(0,0) \neq 0$, then $u_0^2 = \mathcal{O}(\|\mu\|)$ (**pitchfork bifurcation**).

Usually, we use the following terminology:

- If $\dim \mathrm{Fix}(\Sigma) = 1$, then Σ is a **maximal** isotropy subgroup.
- When $\Sigma < \Gamma$, the bifurcating solutions in $\mathrm{Fix}(\Sigma)$ have lower symmetry than the basic solution $u = 0$. This effect is called **spontaneous symmetry breaking**.

4.3 The Euclidean group & Planar lattices

4.3.1 Defintion

In real n-dimensional affine space R_n we chose an origin O and a coordinate frame so that any point P is determined by its coordinates $(x_1, \dots, x_n)$. The distance between P and Q is given by $d(P,Q)^2 = \sum_{i=1}^n (x_i - y_i)^2$. This gives R_n a Euclidean structure. The Euclidean Group $\mathbf{E}(n)$ is the group of all linear or affine linear isometries acting on R_n: all linear transformations which preserve the distances. It can be shown that any such transformation is a composition of an orthogonal transformation $\mathcal{O}$, i.e. an isometry which keeps the origin O fixed, and a translation by a vector ℓ where ℓ is a vector of $\mathbb{R}^n$. The group of isometries which keeps the origin O fixed is isomorphic to the real orthogonal group $\mathbf{O}(n)$ of $n \times n$ orthogonal matrices with real entries. Given any $\gamma \in \mathbf{E}(n)$ we write $\gamma = (\mathcal{O}, \mathbf{e}) \in \mathbf{O}(n) \times \mathbb{R}^n$. The composition of law is then:

$$\gamma \circ \gamma' = (\mathcal{O}\mathcal{O}', \mathcal{O}\ell + \ell')$$

This shows that the non compact Euclidean group $\mathbf{E}(n)$ is the semi-product $\mathbf{O}(n) \ltimes \mathbb{R}^n$. From now on, we will only focus on the two-dimensional case $n = 2$.

4.3.2 Group action

The group $\mathbf{E}(2)$ acts on $\mathbb{R}^2$ in the following way,

$$\gamma \cdot \mathbf{r} = \mathscr{O}\mathbf{r} + \ell, \text{ for any } \gamma = (\mathscr{O}, \ell) \text{ and } \mathbf{r} \in \mathbb{R}^2.$$

For future references, we will use the following notations

$$\begin{cases} \theta \cdot \mathbf{r} &=& \mathcal{R}_\theta \mathbf{r} \quad \text{(rotation)} \\ \kappa \cdot \mathbf{r} &=& \kappa \mathbf{r} \quad \text{(reflection)} \\ \ell \cdot \mathbf{r} &=& \mathbf{r} + \ell \quad \text{(translation)} \end{cases}$$

where

$$\mathcal{R}_\theta := \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \quad \kappa := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and ℓ is any vector in $\mathbb{R}^2$.

Finally, for any $\gamma \in \mathbf{E}(2)$, its action on a function $u(\mathbf{r})$ is given by

$$\gamma[u(\mathbf{r})] := u(\gamma^{-1} \cdot \mathbf{r}),$$

which defines the representation τ with $\tau(\gamma) \cdot u(\mathbf{r}) = \gamma[u(\mathbf{r})] = u(\gamma^{-1} \cdot \mathbf{r})$.

4.3.3 Planar lattices

Let ℓ_1, ℓ_2 be a basis of $\mathbb{R}^2$. The set

$$\mathcal{L} := \{m_1\ell_1 + m_2\ell_2 \mid (m_1, m_2) \in \mathbb{Z}^2\}$$

is a discrete subgroup of $\mathbb{R}^2$. It is called a **lattice group** because the orbit of a point in R_2, under the action of $\mathscr{L}$ forms a **periodic lattice** of points in R_2. We define the dual lattice of lattice $\mathscr{L}$ by

$$\mathcal{L}^* := \{m_1\mathbf{k}_1 + m_2\mathbf{k}_2 \mid (m_1, m_2) \in \mathbb{Z}^2\}$$

with $\ell_i \cdot \mathbf{k}_j = \delta_{i,j}$.

The largest subgroup of $\mathbf{O}(2)$ which keeps the lattice invariant is called the **holohedry** of the lattice. We summarize in Table 1 the different holohedries of the plane. As a consequence, the restriction of $\mathbf{E}(2)$ on a square lattice is the symmetry group $\Gamma = \mathbf{D}_4 \ltimes \mathbf{T}^2$ where $\mathbf{T}^2$ is the two-torus and $\mathbf{D}_4 = \langle \rho_4, \kappa \rangle$ where ρ_4 is the rotation center at 0 and of angle $\pi/2$.

4.3.4 Some further results on the square lattice

First, let us consider the vector space

$$V = \left\{ v = \sum_{j=1}^{2} z_j e^{2i\pi \mathbf{k}_j \cdot \mathbf{r}} + \text{c.c} \mid z_j \in \mathbb{C}, \|\mathbf{k}_j\| = 1 \right\} \cong \mathbb{C}^2,$$

where the isomorphism between V and $\mathbb{C}^2$ is given by $v \to \mathbf{z} = (z_1, z_2)$.

Name	Holohedry	Basis of $\mathcal{L}$	Basis of $\mathcal{L}^*$
Hexagonal	$\mathbf{D}_6$	$\ell_1 = (\frac{1}{\sqrt{3}}, 1)$, $\ell_2 = (\frac{2}{\sqrt{3}}, 0)$	$\mathbf{k}_1 = (0, 1)$, $\mathbf{k}_2 = (\frac{\sqrt{3}}{2}, -\frac{1}{2})$
Square	$\mathbf{D}_4$	$\ell_1 = (1, 0)$, $\ell_2 = (0, 1)$	$\mathbf{k}_1 = (1, 0)$, $\mathbf{k}_2 = (0, 1)$
Rhombic	$\mathbf{D}_2$	$\ell_1 = (1, -\cot\theta)$, $\ell_2 = (0, \cot\theta)$	$\mathbf{k}_1 = (1, 0)$, $\mathbf{k}_2 = (\cos\theta, \sin\theta)$

Table 1: *Lattices in two dimension.* $0 < \theta < \frac{\pi}{2}$ *and* $\theta \neq \frac{\pi}{3}$.

Lemma 4.5. *The action of* $\Gamma = \mathbf{D}_4 \ltimes \mathbf{T}^2$ *on* V *is given by:*

$$\begin{cases} \rho_4(\mathbf{z}) &= (\bar{z}_2, z_1), \\ \kappa(\mathbf{z}) &= (z_1, \bar{z}_2), \\ \Theta(\mathbf{z}) &= (e^{-2i\pi\theta_1} z_1, e^{-2i\pi\theta_2} z_2), \end{cases} \tag{17}$$

where $\Theta = \theta_1 \ell_1 + \theta_2 \ell_2$ *with* $\theta_1, \theta_2 \in [0, 1[$.

Proof. Let $v \in V$ and $(z_1, z_2) \in \mathbb{C}^2$ such that:

$$v(\mathbf{r}) = z_1 e^{2i\pi \mathbf{k}_1 \cdot \mathbf{r}} + z_2 e^{2i\pi \mathbf{k}_2 \cdot \mathbf{r}} + \text{c.c}$$

We have:

$$\begin{aligned} \rho_4[v(\mathbf{r})] &= v\left(\rho_4^{-1} \cdot \mathbf{r}\right) \\ &= z_1 e^{2i\pi \mathbf{k}_1 \cdot \left(\rho_4^{-1} \mathbf{r}\right)} + z_2 e^{2i\pi \mathbf{k}_2 \cdot \left(\rho_4^{-1} \mathbf{r}\right)} + \text{c.c} \\ &= z_1 e^{2i\pi (\rho_4 \mathbf{k}_1) \cdot \mathbf{r}} + z_2 e^{2i\pi (\rho_4 \mathbf{k}_2) \cdot \mathbf{r}} + \text{c.c} \\ &= z_1 e^{2i\pi \mathbf{k}_2 \cdot \mathbf{r}} + z_2 e^{-2i\pi \mathbf{k}_1 \cdot \mathbf{r}} + \text{c.c} \\ &= \bar{z}_2 e^{2i\pi \mathbf{k}_1 \cdot \mathbf{r}} + z_1 e^{2i\pi \mathbf{k}_2 \cdot \mathbf{r}} + \text{c.c} \end{aligned}$$

which implies that $\rho_4(\mathbf{z}) = (\bar{z}_2, z_1)$. We repeat the same procedure for κ and Θ. ∎

For the square lattice, we can also find all isotropy subgroups Σ (up to conjugation) with $\dim \text{Fix}(\Sigma) = 1$ and they are reported in Table 2.

Σ	Generators of Σ	$\text{Fix}(\Sigma)$	$\dim \text{Fix}(\Sigma)$	Name
$\mathbf{D}_4$	ρ_4, κ	$z_1 = z_2 \in \mathbb{R}$	1	Sport or Square
$\mathbf{O}(2) \times \mathbf{Z}_2$	$\rho_4^2, \kappa, [0, \theta_2]$	$z_1 \in \mathbb{R}$, $z_2 = 0$	1	Roll

Table 2: *Isotropy subgroups* Σ *(up to conjugation) with* $\dim \text{Fix}(\Sigma) = 1$.

Finally, we will conclude this section by computing a Taylor expansion of Γ-equivariant vector fields up to order three. Let suppose that we have a vector field of the form

$$f(\mathbf{z}) = (f_1(\mathbf{z}), f_2(\mathbf{z})),$$

that is equivarient with respect to the action (17). This means that for all $\gamma \in \Gamma$ we have

$$\gamma f(\mathbf{z}) = f(\gamma \mathbf{z}).$$

For example, for $\gamma = \rho_4$, this yields to the compatibility condition

$$\left(\overline{f_2(z_1, z_2)}, f_1(z_1, z_2)\right) = (f_1(\bar{z}_2, z_1), f_2(\bar{z}_2, z_1)).$$

Let consider first only first order terms

$$\begin{aligned} f_1(z_1, z_2) &= \mu_1 z_1 + c_1 \bar{z}_1 + c_2 z_2 + c_3 \bar{z}_2, \\ f_2(z_1, z_2) &= \mu_2 z_2 + d_1 \bar{z}_2 + d_2 z_1 + d_3 \bar{z}_1, \end{aligned}$$

where μ_j are bifurcation parameters, and the c_j and d_j are constants. Applying first, the compatibility condition for translation Θ, we find that all coefficients c_j and d_j must be zero for $j = 1, 2, 3$. Then applying the rotation tells us that $\mu_1 = \mu_2 = \mu$ where μ is real. As we should have had suspected find that to linear order

$$\begin{aligned} f_1(z_1, z_2) &= \mu z_1, \\ f_2(z_1, z_2) &= \mu z_2. \end{aligned}$$

Because of the translation equivariance, one can check that $f(\mathbf{z})$ cannot possess any quadratic terms and only cubic terms of the form

$$z_1|z_1|^2, \quad z_1|z_2|^2, \quad z_2|z_1|^2, \quad z_2|z_2|^2,$$

transform in the appropriate way. As a consequence, to cubic order, the vector field should have the form

$$\begin{aligned} f_1(z_1, z_2) &= \mu z_1 + a_1 z_1|z_1|^2 + a_2 z_1|z_2|^2, \\ f_2(z_1, z_2) &= \mu z_2 + b_1 z_2|z_1|^2 + b_2 z_2|z_2|^2, \end{aligned}$$

where a_j and b_j are constants. The reflection equivariance leads to a_1 and a_2 being real, while the rotation equivariance implies $b_1 = a_2$ and $b_2 = a_1$. So in the end, the $\mathbf{D}_4 \ltimes \mathbf{T}^2$-equivariant vector field truncated at cubic order is

$$\begin{aligned} f_1(z_1, z_2) &= \mu z_1 + a_1 z_1|z_1|^2 + a_2 z_1|z_2|^2, \\ f_2(z_1, z_2) &= \mu z_2 + a_1 z_2|z_2|^2 + a_2 z_2|z_1|^2, \end{aligned}$$

where μ, a_1 and a_2 are real.

5 Application – Pattern formation in the visual cortex

Let us recall that we study equation (2) which is of the form

$$\frac{\partial u(\mathbf{r},t)}{\partial t} = -u(\mathbf{r},t) + \int_{\mathbb{R}^2} w(\|\mathbf{r}-\mathbf{r}'\|)S(u(\mathbf{r}',t),\mu)\mathrm{d}\mathbf{r}',$$

with the following hypotheses.

Hypothesis 5.1 (Nonlinearity)**.** *We suppose that the nonlinear function S satisfies the following assumptions:*

(i) $(u,\mu) \mapsto S(u,\mu)$ *is analytic on* $\mathbb{R}^2$ *with* $|S(u,\mu)| \leq s_m$ *and* $0 \leq D_u S(u,\mu) \leq \mu s_m$ *for all* $(u,\mu) \in \mathbb{R} \times (0,+\infty)$ *for some* $s_m > 0$*;*

(ii) $S(0,\mu) = 0$ *for all* $\mu \in \mathbb{R}$ *and* $D_u S(0,\mu) = \mu s_1$ *for some* $s_1 > 0$*.*

Note that the first set of assumptions (analyticity of S with respect to both variables) is very strong and could be weakened to $S \in \mathscr{C}^k(\mathbb{R}^2,\mathbb{R})$ for some $k \geq 2$. But, in practice, the following sigmoidal function is used often

$$S(u,\mu) = \tanh(\mu u),$$

so that we decided to stick with such a strong assumption. The second one ensures that S is a bounded non decreasing function with uniform Lipschitz constant. The last set of hypotheses has already been discussed in the first section of these notes.

Hypothesis 5.2 (Kernel & Dispersion relation)**.** *We suppose that* $w \in H^2(\mathbb{R}^2) \cap L^1_\eta(\mathbb{R}^2)$ *is such that the dispersion relation* $\lambda(\|\mathbf{k}\|,\mu) = -1 + \mu s_1 \widehat{w}(\|\mathbf{k}\|)$ *satisfies:*

(i) $\lambda(k_c,\mu_c) = 0$ *and* $\lambda(\|\mathbf{k}\|,\mu_c) \neq 0$ *for all* $\|\mathbf{k}\| \neq k_c$*;*

(ii) for all $\mu < \mu_c$*, we have* $\lambda(\|\mathbf{k}\|,\mu) < 0$ *for all* $\mathbf{k} \in \mathbb{R}^2$*;*

(iii) $k \to \lambda(k,\mu_c)$ *has a maximum at* $k = k_c$*.*

The condition that $w \in H^2(\mathbb{R}^2)$ ensures by Sobolev embedding that $w \in L^\infty(\mathbb{R}^2)$ and the extra condition that $w \in L^1_\eta(\mathbb{R}^2) := \left\{u \in L^1(\mathbb{R}) \mid (\mathbf{r} \mapsto e^{\eta\|\mathbf{r}\|}u(\mathbf{r}) \in L^1(\mathbb{R}^2)\right\}$ is only there to ensures smoothness properties of the Fourier transform $\widehat{w}$. Finally, the set of assumptions $(i)-(iii)$ have been explained in length in the first section (see Figure 5). From now on, we assume that the hypotheses on the nonlinearity and the kernel are satisfied. It is possible to show that the Cauchy problem associated to the neural field equation (2) is well posed on various Banach spaces and that solutions are unique and global in time. Because our bifurcation problem is for the moment infinite-dimensional, we are going to restrict ourselves to solutions which are doubly periodic on a square lattice and in order to slightly simplify our notation we are going to suppose that $k_c = 1$ so that $\ell_1 = \mathbf{k}_1 = (1,0)$ and $\ell_2 = \mathbf{k}_2 = (0,1)$ are the generators of the square lattice $\mathcal{L}$ and its dual

$\mathcal{L}^*$. As our function is defined on $\mathbb{R}^2$ and as we wish to work on a commutative algebra for the function space, we will set our problem on $\mathcal{Z} = \{u \in H^2(\mathcal{D}) \mid u(\mathbf{r}+\ell) = u(\mathbf{r}), \forall \ell \in \mathcal{L}\}$, where $\mathcal{D}$ is the fundamental domain on the square lattice, from which we will have

$$\|uv\|_{\mathcal{Z}} \lesssim \|u\|_{\mathcal{Z}} \|v\|_{\mathcal{Z}}.$$

The above property is really important as it makes $\mathcal{Z}$ a commutative algebra with respect to pointwise multiplication. We denote $\mathcal{X} = L^2(\mathcal{D})$. It is worth mentioning that any function in $\mathcal{X}$ can be decomposed as a sum of Fourier modes that lie on the dual lattice:

$$u(\mathbf{r}, t) = \sum_{\mathbf{k} \in \mathcal{L}^*} z_{\mathbf{k}}(t) e^{2i\pi \mathbf{r}\cdot\mathbf{k}} + \text{c.c.} \quad .$$

Let us now write the neural field equation (2) into the following form

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \mathbf{L}u + \mathbf{R}(u, \epsilon), \quad \epsilon := \mu - \mu_c, \tag{18}$$

where

$$\mathbf{L}u(\mathbf{r}) := -u(\mathbf{r}) + \mu_c s_1 \int_{\mathbb{R}^2} w(\|\mathbf{r}-\mathbf{r}'\|) u(\mathbf{r}') \mathrm{d}\mathbf{r}', \tag{19a}$$

$$\mathbf{R}(u, \epsilon) := \int_{\mathbb{R}^2} w(\|\mathbf{r}-\mathbf{r}'\|) S(u(\mathbf{r}', t), \mu_c + \epsilon) \mathrm{d}\mathbf{r}' - \mu_c s_1 \int_{\mathbb{R}^2} w(\|\mathbf{r}-\mathbf{r}'\|) u(\mathbf{r}') \mathrm{d}\mathbf{r}'. \tag{19b}$$

It is straightforward to check that the following properties are satisfied.

Lemma 5.1. *Suppose that all the above hypotheses on w and S are satisfied, then we have:*

(i) $\mathbf{L} \in \mathscr{L}(\mathcal{Z}, \mathcal{X})$ *is compact and sectorial on* $\mathcal{Z}$ *and thus satisfies* $\|(\mathrm{i}\omega - \mathbf{L})^{-1}\|_{\mathscr{L}(\mathcal{X})} \leq \frac{c}{|\omega|}$ *for some constant c and $|\omega|$ large enough;*

(ii) for all $k \geq 0$, we have that $\mathbf{R} \in \mathscr{C}^k(\mathcal{Z} \times \mathbb{R}, \mathcal{Z})$*;*

(iii) the spectrum σ of $\mathbf{L}$ *is discrete and the set σ_0 consists of a finite number of eigenvalues with finite algebraic multiplicities;*

(iv) both $\mathbf{L}$ *and* $\mathbf{R}$ *are equivariant with respect to the group action of* $\Gamma = \mathbf{D}_4 \ltimes \mathbf{T}^2$ *via* $\gamma[u(\mathbf{r})] := u(\gamma^{-1} \cdot \mathbf{r})$ *for any $\gamma \in \Gamma$;*

(v) the representation $\tau : \mathcal{Z} \to \mathcal{Z}$ with $\tau(\gamma) \cdot u = \gamma[u]$ is absolutely irreducible.

The dimension of the bifurcation problem depends on the number of points $\mathbf{k} \in \mathcal{L}^*$ that lie on the critical circle of radius $k_c = 1$. Here, we work with the fundamental representation of $\mathbf{D}_4 \ltimes \mathbf{T}^2$ so that there exists two critical orthonormal vectors $\mathbf{k}_1 = (1, 0)$ and $\mathbf{k}_2 = (0, 1)$ that lie on the critical circle so that the corresponding center manifold if 4-dimensional.

Remark 5.1. *It is important to note that there exists another absolutely irreducible representation of* $\mathbf{D}_4 \ltimes \mathbf{T}^2$ *which is 8-dimensional, in that case we say that $\mathcal{L}$ is a* **superlattice** *(see [3, 5]).*

As a consequence, the kernel $\mathcal{E}_0$ of $\mathbf{L}$ is given by

$$\mathcal{E}_0 = \left\{ u \in \mathcal{Z} \mid u(\mathbf{r}) = \sum_{j=1}^{2} z_j e^{2i\pi \mathbf{k}_j \cdot \mathbf{r}} + \text{c.c} \quad \text{for} \quad (z_1, z_2) \in \mathbb{C}^2 \right\} \cong \mathbb{C}^2,$$

where the identification to $\mathbb{C}^2$ is done through the vector space V, defined in the previous section. We can apply the parameter center manifold theorem with symmetries and say that all small bounded solutions of (19) can be written as

$$u(\mathbf{r}, t) = u_0(\mathbf{r}, t) + \Psi(u_0(\mathbf{r}, t), \epsilon), \quad u_0(\mathbf{r}, t) = \sum_{j=1}^{2} z_j(t) e^{2i\pi \mathbf{k}_j \cdot \mathbf{r}} + \text{c.c} \ ,$$

where $(z_1(t), z_2(t))$ satisfy

$$\frac{\mathrm{d}z_1}{\mathrm{d}t} = z_1 \left(c(\epsilon) + a_1 |z_1|^2 + a_2 |z_2|^2 \right) + \text{h.o.t.}, \tag{20a}$$

$$\frac{\mathrm{d}z_2}{\mathrm{d}t} = z_2 \left(c(\epsilon) + a_1 |z_2|^2 + a_2 |z_1|^2 \right) + \text{h.o.t.}, \tag{20b}$$

where *h.o.t.* stands for higher order terms. Here, $c(\epsilon) I_4 = D_u f(0, \epsilon)$ where f is the associated reduced vector field. It is a direct computation to check that in our case

$$c(\epsilon) = \frac{\epsilon}{\mu_c} = \frac{\mu - \mu_c}{\mu_c},$$

such that the condition $c'(0) \neq 0$ of the Equivariant Branching Lemma is satisfied. As a consequence, for each isotropy subgroup $\Sigma \subset \Gamma$ with $\dim \text{Fix}(\Sigma) = 1$, there exists a bifurcating branch of solutions with symmetry Σ. All isotropy subgroups Σ with $\dim \text{Fix}(\Sigma) = 1$ are listed in Table 2. We have already seen that very close to the bifurcation $\mu \sim \mu_c$, the solutions should be well approximated, to leading order, by

$$u(\mathbf{r}) \cong z_1 e^{2i\pi \mathbf{k}_1 \cdot \mathbf{r}} + z_2 e^{2i\pi \mathbf{k}_2 \cdot \mathbf{r}} + \text{c.c.}$$

In the case of the symmetry branch $\Sigma = \mathbf{D}_4$, we have $z_1 = z_2 = z \in \mathbb{R}$ and

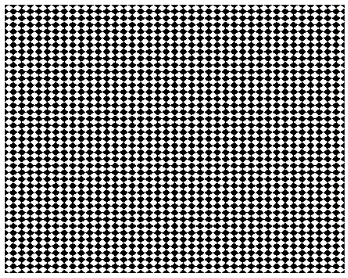
$$u(\mathbf{r}) \cong 2z \left(\cos(2\pi x) + \cos(2\pi y) \right)$$

and for $\Sigma = \mathbf{O}(2) \times \mathbf{Z}_2$, we have $z_1 = z \in \mathbb{R}$ and $z_2 = 0$, and we obtain

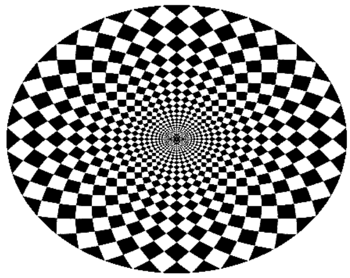
$$u(\mathbf{r}) \cong 2z \cos(2\pi x)$$

where $\mathbf{r} = (x, y) \in \mathbb{R}^2$.

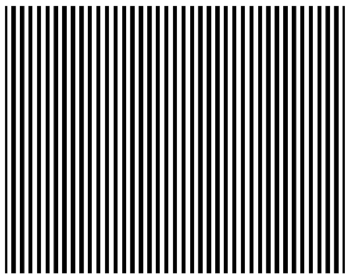
In Figure 6, we represented each geometric structures using the following strategy. When $u(\mathbf{r}) > 0$ we say that the cortical area is activated (black) and when $u(\mathbf{r}) < 0$ the area is inactive (white). As a consequence, Figures 6(b) and 6(d) are the first visual hallucinations that we recover from this mathematical analysis. Now, we would like to know which one of these two possible hallucinations is stable with respect to the dynamics. The very first task is to compute the constants a_1 and a_2 which appear in (20).



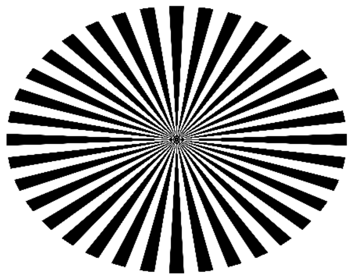
(a) $\mathbf{D}_4$



(b) $\mathbf{D}_4$



(c) $\mathbf{O}(2) \times \mathbf{Z}_2$



(d) $\mathbf{O}(2) \times \mathbf{Z}_2$

Figure 6: *Geometrical structures (planforms) corresponding to each isotropy subgroups from Table 2. To the left, the planforms are represented in V1 and to the right they are given in the retinal field and thus correspond to possible visual hallucinations.*

We have the following Lemma.

Lemma 5.2. *The coefficients a_1 and a_2 are given by:*

$$
\begin{aligned}
a_1 &= \widehat{w}_{\mathbf{k}_c}\left(s_2^2\left[\frac{\widehat{w}_0}{1-\widehat{w}_0/\widehat{w}_{\mathbf{k}_c}}+\frac{\widehat{w}_{2\mathbf{k}_c}}{2(1-\widehat{w}_{2\mathbf{k}_c}/\widehat{w}_{\mathbf{k}_c})}\right]+\frac{s_3}{2}\right)\\
a_2 &= \widehat{w}_{\mathbf{k}_c}\left(s_2^2\left[\frac{\widehat{w}_0}{1-\widehat{w}_0/\widehat{w}_{\mathbf{k}_c}}+2\frac{\widehat{w}_{\mathbf{k}_1+\mathbf{k}_2}}{1-\widehat{w}_{\mathbf{k}_1+\mathbf{k}_2}/\widehat{w}_{\mathbf{k}_c}}\right]+s_3\right),
\end{aligned}
$$

where $s_k := \partial_u^k S(0,\mu_c)$ and $\widehat{w}_{\mathbf{k}} := \int_{\mathbb{R}^2} w(\|\mathbf{r}\|)e^{-2i\pi\mathbf{k}\cdot\mathbf{r}}d\mathbf{r}$.

Proof. We first remark that:

$$S(u,\mu) = \mu s_1 u + \frac{s_2}{2}u^2 + \frac{s_3}{6}u^3 + \text{ h.o.t.}$$

Then we define a scalar product on $\mathcal{X}$:

$$\langle u, v\rangle = \int_{\mathcal{D}} u(\mathbf{r})\bar{v}(\mathbf{r})d\mathbf{r}$$

where $\bar{v}(\mathbf{r})$ is the complex conjugate of $v(\mathbf{r})$ and $\mathcal{D} = [0,1]\times[0,1]$ is the fundamental domain of the lattice. We denote

$$\begin{cases}\zeta_1 &= e^{2i\pi\mathbf{k}_1\cdot\mathbf{r}}\\ \zeta_2 &= e^{2i\pi\mathbf{k}_2\cdot\mathbf{r}}\end{cases}$$

We write

$$u(\mathbf{r},t) = z_1(t)\zeta_1 + \overline{z_1(t)\zeta_1} + z_2(t)\zeta_2 + \overline{z_2(t)\zeta_2} + \Psi(z_1,\bar{z}_1,z_2,\bar{z}_2,\mu),$$

with the Taylor expansion

$$\Psi(z_1,\bar{z}_1,z_2,\bar{z}_2,\mu) = \sum_{l_1,l_2,p_1,p_2,r>1} z_1^{l_1}\bar{z}_1^{l_2}z_2^{p_1}\bar{z}_2^{p_2}\mu^r\Psi_{l_1,l_2,p_1,p_2,r}.$$

We obtain after identification at each order the system

$$
\begin{cases}
0 &= -2\mathbf{L}\Psi_{2,1,0,0,0} + 2a_1\zeta_1 - 4\mathbf{R}_2(\Psi_{1,1,0,0,0},\zeta_1) - 4\mathbf{R}_2(\Psi_{2,0,0,0,0},\bar{\zeta}_1) - 6\mathbf{R}_3(\zeta_1,\zeta_1,\bar{\zeta}_1),\\
0 &= -\mathbf{L}\Psi_{1,1,1,0,0} + a_2\zeta_2 - 2\mathbf{R}_2(\Psi_{0,1,1,0,0},\zeta_1) - 2\mathbf{R}_2(\Psi_{1,0,1,0,0},\bar{\zeta}_1) - 2\mathbf{R}_2(\Psi_{1,1,0,0,0},\zeta_2)\\
& \quad -6\mathbf{R}_3(\zeta_2,\zeta_1,\bar{\zeta}_1).
\end{cases}
$$

So that we find that

$$
\begin{cases}
a_1 &= \langle 2\mathbf{R}_2(\Psi_{1,1,0,0,0},\zeta_1) + 2\mathbf{R}_2(\Psi_{2,0,0,0,0},\bar{\zeta}_1) + 3\mathbf{R}_3(\zeta_1,\zeta_1,\bar{\zeta}_1),\zeta_1\rangle,\\
a_2 &= \langle 2\mathbf{R}_2(\Psi_{0,1,1,0,0},\zeta_1) + 2\mathbf{R}_2(\Psi_{1,0,1,0,0},\bar{\zeta}_1) + 2\mathbf{R}_2(\Psi_{1,1,0,0,0},\zeta_2)\\
& \quad +6\mathbf{R}_3(\zeta_2,\zeta_1,\bar{\zeta}_1),\zeta_2\rangle.
\end{cases}
$$

Here, we have set

$$\mathbf{R}_2(u_1,u_2)(\mathbf{r}) := \frac{s_2}{2}\int_{\mathbb{R}^2} w(\|\mathbf{r}-\mathbf{r}'\|)u_1(\mathbf{r}')u_2(\mathbf{r}')d\mathbf{r}', \tag{21}$$

$$\mathbf{R}_3(u_1,u_2,u_3)(\mathbf{r}) := \frac{s_3}{6}\int_{\mathbb{R}^2} w(\|\mathbf{r}-\mathbf{r}'\|)u_1(\mathbf{r}')u_2(\mathbf{r}')u_3(\mathbf{r}')d\mathbf{r}'. \tag{22}$$

Finally, we have to solve the following set of equations

$$\begin{cases} 0 &=& 2\mathbf{L}\Psi_{2,0,0,0,0} + 2\mathbf{R}_2(\zeta_1,\zeta_1), \\ 0 &=& \mathbf{L}\Psi_{1,1,0,0,0} + 2\mathbf{R}_2(\zeta_1,\bar{\zeta}_1), \\ 0 &=& \mathbf{L}\Psi_{0,1,1,0,0} + 2\mathbf{R}_2(\zeta_2,\bar{\zeta}_1), \\ 0 &=& \mathbf{L}\Psi_{1,0,1,0,0} + 2\mathbf{R}_2(\zeta_1,\zeta_2), \end{cases}$$

which solutions are given by

$$\begin{cases} \Psi_{2,0,0,0,0} &=& \mathrm{Vect}(\zeta_1,\bar{\zeta}_1,\zeta_2,\bar{\zeta}_2) + \frac{s_2}{2}\frac{\widehat{w}_{2\mathbf{k}_c}}{1-\mu_c s_1 \widehat{w}_{2\mathbf{k}_c}}\zeta_1^2, \\ \Psi_{1,1,0,0,0} &=& \mathrm{Vect}(\zeta_1,\bar{\zeta}_1,\zeta_2,\bar{\zeta}_2) + 2\frac{s_2}{2}\frac{\widehat{w}_0}{1-\mu_c s_1 \widehat{w}_0}, \\ \Psi_{0,1,1,0,0} &=& \mathrm{Vect}(\zeta_1,\bar{\zeta}_1,\zeta_2,\bar{\zeta}_2) + 2\frac{s_2}{2}\frac{\widehat{w}_{\mathbf{k}_1+\mathbf{k}_2}}{1-\mu_c s_1 \widehat{w}_{\mathbf{k}_1+\mathbf{k}_2}}\zeta_2\bar{\zeta}_1, \\ \Psi_{1,0,1,0,0} &=& \mathrm{Vect}(\zeta_1,\bar{\zeta}_1,\zeta_2,\bar{\zeta}_2) + 2\frac{s_2}{2}\frac{\widehat{w}_{\mathbf{k}_1+\mathbf{k}_2}}{1-\mu_c s_1 \widehat{w}_{\mathbf{k}_1+\mathbf{k}_2}}\zeta_2\zeta_1, \end{cases}$$

where we used the fact that $\widehat{w}_{\mathbf{k}_1+\mathbf{k}_2} = \widehat{w}_{\mathbf{k}_2-\mathbf{k}_1}$ and $\widehat{w}_{j\mathbf{k}_c} = \widehat{w}_{j\mathbf{k}_1} = \widehat{w}_{j\mathbf{k}_2}$ for $j = 1, 2$. Then, it is enough to notice that for example,

$$\langle \mathbf{R}_2(\Psi_{1,1,0,0,0},\zeta_1),\zeta_1\rangle = \frac{s_2^2}{2}\frac{\widehat{w}_0\widehat{w}_{\mathbf{k}_c}}{1-\mu_c s_1 \widehat{w}_0},$$

and use the fact that $\mu_c s_1 \widehat{w}_{\mathbf{k}_c} = 1$ to obtain the desired formula for a_1 and a_2. ∎

Lemma 5.3. *For the reduced system* (20)*, we have the following dichotomy:*

- *The square solution $z_1 = z_2 = z \in \mathbb{R}$ with symmetry $\mathbf{D}_4$ is stable if and only if $a_1 < -|a_2| < 0$.*
- *The roll solution $z_1 = z \in \mathbb{R}$, $z_2 = 0$ with symmetry $\mathbf{O}(2) \times \mathbf{Z}_2$ is stable is and only if $a_2 < a_1 < 0$.*

These two branches of solutions are mutually exclusive for the stability, i.e. we cannot have at the same time both solutions stable.

As an extension, to this Lemma, these solutions will remain stable for the full dynamics of equation (19) within the class of perturbations having the same symmetries as they are normally hyperbolic for the reduced system (20). Finally, depending on the specific form of the nonlinearity S and the connectivity kernel w, we expect to see either spots or stripes close to the bifurcation $\mu \sim \mu_c$.

6 Conclusion & Perspectives

Take home message. Geometric visual hallucinations can be explained simply by symmetry-breaking bifurcation (Turing patterns) on the visual cortex abstracted by $\mathbb{R}^2$ by the action of discrete subgroups of the Euclidean group and the correspondance between visual field and visual cortex with a log-polar map.

What's missing? In our case study, we have only focused on the square lattice and totally ignored the hexagonal case. The analysis in that case is slightly more involved and the fundamental absolutely irreducible representation is now 6-dimensional. We let as an exercice to find all the axial isotropy subgroups (*i.e.* all isotropy subgroups with one-dimensional fix space) and conduct the same analysis as we did here. The results can be found in [1, 3, 5].

Extensions. This study can be extended to incorporate some kind of functional architecture of the visual cortex, see the beautiful paper [1], where this time the visual cortex is idealized to $\mathbb{R}^2 \times \mathbf{S}^1$. That is, to each point of the visual field we associate a point in the cortex $(\mathbf{r}, \theta)$ where θ retains the preferred local orientation. This model has been extended into several directions by adding more features (spatial frequency and texture), see [5] for a recent review on the subject.

Equivariant bifurcation. Steady-state equivariant bifurcations are now well documented but in the case of Hopf bifurcations with square or hexagonal symmetries, there are still some open problem left. One can ask the question to wether such symmetry-breaking bifurcations can be transposed into non-Euclidean geometry. This question has been partially treated in the case of hyperbolic geometry (Poincaré disk) in [5], see the references therein.

Disclaimer. One should not take for granted the neural fields formalism. Indeed, any other evolution equation equivariant with respect to the Euclidean symmetries and having a steady-state bifurcation would produce exactly the same type of geometric visual hallucinations. The equations are over simplified and our primary visual cortex does not reduce to a single neural field equation (see Figure 7). In fact, our analysis only tells us something about the geometry of our network: *i.e.* the invariance of connectivity kernel with resect to some action of the Euclidean group of transformations of the plane.

$$\partial_t u(x,t) = -u(x,t) + \int_{\mathbb{R}} W(x-y)S(u(y,t))dy$$

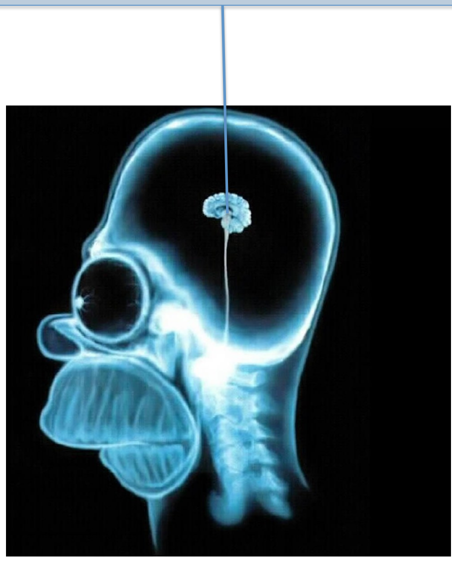


Figure 7

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