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MATHEMATIQUES
Équations différentielles

NOTE ON THE GENERALIZED HÉNON-HEILES SYSTEM

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1. Introduction. In this note we study the topology and the algebraic structure of the Hamiltonian system corresponding to the following generalized Hénon-Heiles Hamiltonian [3,1]

$$H=\frac{1}{2}(p_1^2+p_2^2+Aq_1^2+Bq_2^2)-q_1^2q_2-\frac{\varepsilon}{3}q_2^3 \tag{1.1}$$

in the integrable case $\varepsilon=6$. This system reads

$$\begin{aligned}\dot q_1&=p_1 & \dot p_1&=2q_1q_2-Aq_1\\ \dot q_2&=p_2 & \dot p_2&=q_1^2+6q_2^2-Bq_2\end{aligned} \tag{1.2}$$

and the second integral of motion is [1]

$$F=q_1^4+4q_1^2q_2^2+4p_1(p_1q_2-p_2q_1)-4Aq_1^2q_2+(4A-B)(p_1^2+Aq_1^2). \tag{1.3}$$

Recently Newell, Tabor a. Zeng [2] found a Lax pair and integrated the system (1.2) under the additional assumption $A=B=0$. Adler a. van Moerbeke [4] also noted that it is an algebraically completely integrable system (i. e. it can be linearized on a family of Abelian surfaces). It turns out that the general case ($\varepsilon=6$, A and B — arbitrary) is algebraically integrable, too (Theorem 2). In Theorem 1 we solve the system (1.2) in terms of hyperelliptic genus-two theta functions of the complex time. The rich algebraic structure of the problem enables us to describe all generic bifurcations of invariant Liouville tori and cylinders. This result is formulated in Theorem 3.

2. Algebraic Structure of the System (1.2). Consider the hyperelliptic genus-two curve $S=\{w^2=z\,.\,P(z)\}$, where

$$P(z)=z(z-A)\,(4z-4A+B)^2+8hz-f. \tag{2.1}$$

In this section we suppose that the curve S is non-degenerated, i. e. A, B, f, h are such complex constants that $(A, B, f, h)\notin \mathrm{B}=\{(A, B, f, h)\in \mathbb{R}^4:\ \mathrm{disc}\ (z.P(z))=0\}$. We use the notation of [5]. Let γ_1, γ_2, δ_1, δ_2 be a canonical homology basis on S, $J(S)$ be the Jacobian variety of S, and $\theta(z_1, z_2)$ be the corresponding Riemann theta function. Let ζ be the Abel mapping with respect to the base point P_∞ (P_∞ is the 'infinite' point on S), $K=K(P_\infty)$ be the vector of Riemann constants. Denote by P_0 the Weierstrass point on S, corresponding to the root 0 of the polynomial $z\,.\,P(z)$.

Theorem 1. Every (complex) solution of the (complex) system (1.2) can be expressed in the following way

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$$q_1(t)=c_1\frac{\theta\left(-\frac{1}{2}at+t^0-\zeta(P_0)+K\right)}{\theta\left(-\frac{1}{2}at+t^0+K\right)}$$

$$q_2(t)=\frac{\partial^2}{\partial t^2}\ln\theta\left(-\frac{1}{2}at+t^0+K\right)+c_2.$$

Here $t^0=(t_1^0, t_2^0)$, where t_1^0 and t_2^0 are arbitrary constants, playing the role of initial conditions, c_1 and c_2 are suitable constants which depend upon $A, B, f, h, a=(a_1, a_2)$ where $a_1=\left.\frac{d\zeta_1}{d\tau}\right|_{P=P_\infty}$, $a_2=\left.\frac{d\zeta_2}{d\tau}\right|_{P=P_\infty}$ ($\tau=1/z^{1/2}$ is a local parameter around P_∞), and' $d\zeta_1$, $d\zeta_2$ form a basis of the space of holomorphic differentials on S.

The proof of Theorem 1 is based on the fact that the Hamilton-Jacobi equation separates in u, v variables, where

$$q_1^2=-4uv,\quad q_2=u+v+(B-4A)/4.$$

Remark. Denote by $\mathbf{A}_\mathbf{C}$ the complex invariant manifold $\mathbf{A}_\mathbf{C}=\{H=h,\ F=f\}$. Then the solutions described in Theorem 1 lie on $\mathbf{A}_\mathbf{C}$ for arbitrary constants A, B, f, h, t^0.

Theorem 2. The Hamiltonian system (1.2) is algebraically completely integrable. $\mathbf{A}_\mathbf{C}$ is a smooth complex manifold which is an affine part of an Abelian surface $\widetilde{\mathbf{A}}_\mathbf{C}$ of polarization (1,2). $\widetilde{\mathbf{A}}_\mathbf{C}=\mathbf{A}_\mathbf{C}\cup D_\infty$, where D_∞ is a smooth hyperelliptic genus-two curve. $\mathbf{A}_\mathbf{C}$ is a two-sheeted covering of the Jacobian variety $J(S)$, such that D_∞ becomes a two-sheeted covering of the curve D, the last being a translate of the Riemann's theta divisor. The Hamiltonian flows defined by H and F on $\mathbf{A}_\mathbf{C}$ extend holomorphically to flows on $\widetilde{\mathbf{A}}_\mathbf{C}$ which are straight-line motions.

To prove Theorem 2 we follow the procedure used in [6].

3. Topological Analysis. In this section we consider the system (1.2) as a system of real differential equations. The constants A, B, f, h will be real constants. Denote by $\mathbf{A}_\mathbf{R}$ the real invariant manifold $\mathbf{A}_\mathbf{R}=\{H=h,\ F=f\}$ of the system (1.2). According to Theorem 2, if $(A, B, f, h)\in\mathbf{R}^4\setminus\mathbf{B}$, then $\mathbf{A}_\mathbf{R}$ is a smooth real manifold, and hence $\mathbf{A}_\mathbf{R}$ may change the topological type only as the point (A, B, f, h) passes through a point $(A_0, B_0, f_0, h_0)\in\mathbf{B}$.

Definition. A point $(A_0, B_0, f_0, h_0)\in\mathbf{B}$ is said to be generic provided that in a neighbourhood of this point $\mathbf{B}_0$ is a smooth three-dimensional real manifold.

Definition. A bifurcation of the set $\mathbf{A}_\mathbf{R}$ is said to be generic provided that the point (A, B, f, h) passes, transversally to $\mathbf{B}$, through a generic point $(A_0, B_0, f_0, h_0)\in\mathbf{B}$.

Definition. Two intersections $\mathbf{B}\cap\{A=A_1,\ B=B_1\}$ and $\mathbf{B}\cap\{A=A_2,\ B=B_2\}$ are topologically equivalent, if there exist continuous functions $A=A(s)$, $B=B(s)$, $s\in[0,1]$, such that $A(0)=A_1$, $B(0)=B_1$, $A(1)=A_2$, $B(1)=B_2$ and all intersections $\mathbf{B}\cap\{A=A(s), B=B(s)\}$ $(s\in[0,1])$ are homeomorphic to each other.

Consider the sets $M_1, M_2, \ldots, M_n$. If $M_i\cap M_j$ consists of one point for $|i-j|=1$, and it is the empty set for $|i-j|\neq 1, 0$, then we denote $M_1\vee M_2\vee\ldots\vee M_n=\bigcup_{i=1}^{n}M_i$.

Denote also by $mT+nC$ a disjoint union of m tori and n cylinders and by P a set homeomorphic to a non-trivial bundle with base — the circle S^1 and a fibre $S^1\vee S^1$.

Consider the following bifurcations: $T\to S^1\to\varnothing$, $C\to\mathbf{R}^1\to\varnothing$, $T\to P\to T$, $C\to(S^1\vee S^1)\times\mathbf{R}^1\to 2C$, $C\to(S^1\vee S^1\vee S^1)\times\mathbf{R}^1\to 3C$, $2C\to S^1\times(\mathbf{R}^1\vee\mathbf{R}^1)\to 2C$,, $T+C\to S^1\times(S^1\vee\mathbf{R}^1)\to C$, $T+2C\to S^1\times(\mathbf{R}^1\vee S^1\vee\mathbf{R}^1)\to 2C$.

Theorem 4. The set $\mathbf{R}^4\setminus\mathbf{B}$ consists of 14 open, connected, non-intersecting each other domains. All topologically different intersections $\mathbf{B}\cap\{A=\text{const.},\ B=\text{const.}\}$ are given in Fig. 1. The topological type of $\mathbf{A}_\mathbf{R}$ does not change in each of the above 14 domains, and it is described in the Table. Any generic bifurcation of connected components of $\mathbf{A}_\mathbf{R}$ can be found among the bifurcations considered above. Their precise description is also given in the Table.

Remark. Here we explain the Table and Fig. 1. Each of the 20 intersections $\mathbf{B} \cap \{A=\text{const.}, B=\text{const.}\}$ shown in Fig. 1 corresponds to an arbitrary point (A, B) lying in one of the 20 subsets, shown in Fig. 2 (10 half-lines and 10 open 'triangular' domains). According to Theorem 4 the intersections $\mathbf{B} \cap \{A = A_1, B = B_1\}$, and $\mathbf{B} \cap \{A = A_2, B = B_2\}$ are topologically equivalent iff the points (A_1, B_1) and (A_2, B_2) lie in one and the same domain, shown in Fig. 2. It is seen from Fig. 1 that the set $\mathbf{R}^4 \setminus \mathbf{B}$ consists of 14 open domains. The notation $i \xrightarrow{k} j$ in the Table means a generic bifurcation between two subdomains of $\mathbf{R}^4 \setminus \mathbf{B}$ numbered by i and j in Fig. 1. If the point (A, B, f, h) passes transversally through a point $(A_0, B_0, f_0, h_0) \in \mathbf{B}$ then the integer k is equal to the number of the branch of the curve $\{\text{disc}\,(P(z))=0\} \cap \{A=A_0, B=B_0\} \subset \mathbf{R}^2\{f, h\}$ (see Fig. 3) in which the point (f_0, h_0) lies. For the sake of brevity we denote $2C \to C$ in the Table instead of $2C \to (S^1 \vee S^1) \times \mathbf{R}^1 \to C$, etc...

The proof of Theorem 4 is based on the following observation: Consider the antiholomorphic involution τ on S, $\tau: S \to S: (w, z) \to (\overline{w}, \overline{z})$ τ induces an antiholomorphic involution on $J(S)$, and hence on $\widetilde{\mathbf{A}}_\mathbf{C} \cong \mathbf{A}_\mathbf{C} \cup D_\infty$. The real invariant manifold is embedded linearly in $\widetilde{\mathbf{A}}_\mathbf{C}$, and hence its closure consists of real two-dimensional tori. It turns out that each of these tori coincides with a stationary component of the antiholomorphic involution τ on $\widetilde{\mathbf{A}}_\mathbf{C}$. In order to determine whether a given stationary component corresponds to a

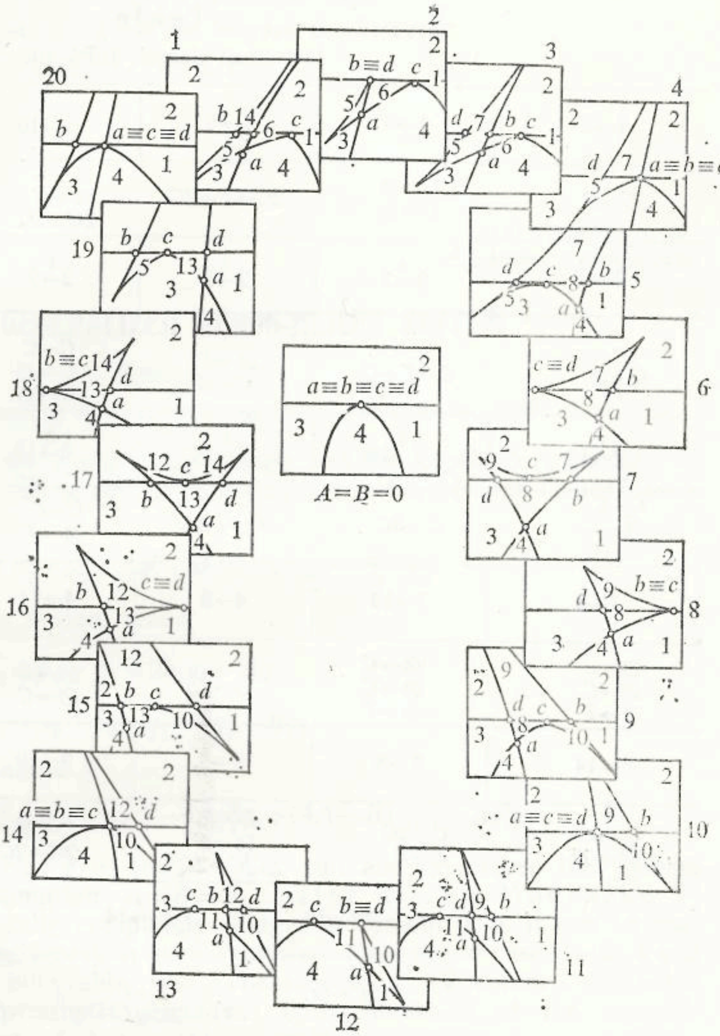

Fig. 1.

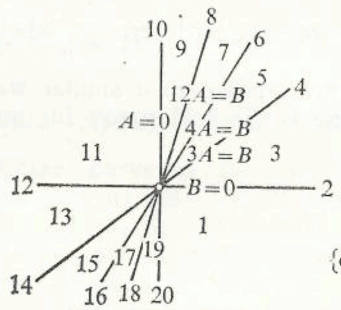

Fig. 2.

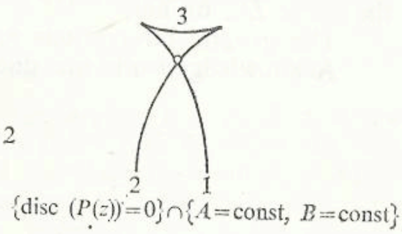

Fig. 3.

Table

List of all Generic Bifurcations

$1\to 2$	$1\to 4$	$1\to 8$	$1\overset{2}{\to}10$	$1\overset{3}{\to}10$	$1\to 13$
$2C\to C$	$C\to\varnothing$ $C\to\varnothing$	$2C\to T+2C$	$\varnothing\to C$ $\varnothing\to C$ $2C\sim 2C$	$C\to 2C$ $C\to 2C$	$2C\to T+2C$
$2\to 6$	$2\overset{2}{\to}7$	$2\overset{3}{\to}7$	$2\overset{1}{\to}9$	$2\overset{3}{\to}9$	$2\to 11$
$C\to\varnothing$	$C\to T+C$	$\varnothing\to T$ $C\sim C$	$C\sim C$ $C\to\varnothing$ $C\to\varnothing$	$C\to 3C$	$C\to 2C$
$2\overset{1}{\to}12$	$2\overset{3}{\to}12$	$2\overset{2}{\to}14$	$2\overset{3}{\to}14$	$3\to 4$	$3\to 5$
$C\sim C$ $\varnothing\to C$ $\varnothing\to C$	$C\to 3C$	$C\to T+C$	$C\sim C$ $\varnothing\to T$	$\varnothing\sim\varnothing$	$\varnothing\sim\varnothing$
$3\to 8$	$3\to 13$	$4\to 6$	$4\to 11$	$5\to 6$	$5\to 7$
$\varnothing\sim C$ $\varnothing\to C$ $\varnothing\to T$	$\varnothing\to C$ $\varnothing\to C$ $\varnothing\to T$	$\varnothing\sim\varnothing$	$\varnothing\to C$ $\varnothing\to C$	$\varnothing\to\varnothing$	$\varnothing\to C$ $\varnothing\to T$
$5\to 14$	$7\to 8$	$8\to 9$	$9\to 10$	$10\to 11$	$10\to 12$
$\varnothing\to C$ $\varnothing\to T$	$C\to 2C$ $T\to T$	$T\to\varnothing$ $\varnothing\to C$ $2C\sim 2C$	$C\to 2C$ $2C\to 2C$	$2C\sim 2C$ $C\to\varnothing$ $C\to\varnothing$	$2C\to 2C$ $2C\to C$

$12\to 13$	$13\to 14$	$2\to 3$
$2C\sim 2C$ $C\to\varnothing$ $\varnothing\to T$	$T\to T$ $2C\to C$	$C\to\varnothing$

torus or to a cylinder in $\mathbf{A}_{\mathbf{R}}$, we have to determine whether this component intersects the curve D_∞ or not.

The generic bifurcations are studied in a similar way (after going to the limits).

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