Nonlinearity **23** (2010) 1–16 UNCORRECTED PROOF

On the reduction of the degree of linear differential operators

Marcin Bobieński and Lubomir Gavrilov

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Received 11 March 2010, in final form 15 November 2010
Published
Online at stacks.iop.org/Non/23

Recommended by A Chenciner

Abstract
Let L be a linear differential operator with coefficients in some differential field k of characteristic zero with algebraically closed field of constants. Let k^a be the algebraic closure of k. For a solution y_0, $Ly_0 = 0$, we determine the linear differential operator of minimal degree $\tilde{L}$ and coefficients in k^a, such that $\tilde{L}y_0 = 0$. This result is then applied to some Picard–Fuchs equations which appear in the study of perturbations of plane polynomial vector fields of Lotka–Volterra type.

Mathematics Subject Classification: 34C08, 34M03, 34M35

1. Introduction

Let $y_0 = y_0(t)$ be a solution of the linear differential equation

$$a_0(t)y^{(n)} + a_1(t)y^{(n-1)} + \cdots + a_n(t)y = 0, \tag{1}$$

where $a_i \in k = \mathbb{C}(t)$ are functions, rational in the independent variable t. We are interested in determining the equation of minimal degree $d \leqslant n$,

$$b_0(t)y^{(d)} + b_1(t)y^{(d-1)} + \cdots + b_d(t)y = 0 \tag{2}$$

such that

- y_0 is a solution,
- the coefficients b_i are algebraic functions in t.

Recall that a function $b(t)$ is said to be algebraic in t if there exists a polynomial P with coefficients in $k = \mathbb{C}(t)$, such that $P(b(t)) \equiv 0$.

We shall suppose that, more generally, k is an arbitrary differential field of characteristic zero with algebraically closed field of constants, k^a is its algebraic closure, and $a_i \in k$, $b_j \in k^a$. To find equation (2) we consider the differential Galois group G of (1) and its

JNL: NON PIPS: 350379 TYPE: PAP TS: NEWGEN DATE: 25/11/2010 EDITOR: TT SPELLING: UK

connected subgroup G^0 containing the unit element of G. Our first result, theorem 1, says that the orbit of y_0 under the action of G^0 spans the solution space of (2).

Particular attention is given further to the case in which (1) is of Fuchs or Picard–Fuchs type. Recall that (1) is said to be of Fuchs type, if every singular point, including the one at infinity, is a regular singularity. A Picard–Fuchs equation is a particular type of Fuchsian equation, whose solutions are of 'geometric origin'. The latter means here that the solutions can be expressed as complete Abelian integrals of the form

$$y(t) = \int_{\gamma(t)} \omega,$$

where ω is a polynomial one-form, and $\{\gamma(t)\}_t$ is a continuous family of closed loops, contained in the level sets

$$\{(x, y) \in \mathbb{C}^2 : F(x, y) = t\}$$

of the polynomial F, see [2]. The Galois group of a Fuchs-type equation is a Zariski closure of the monodromy group of the equation. Theorem 1 is re-formulated in terms of the action of the corresponding monodromy groups in theorems 2 and 3.

In the last part of the paper, section 3, we apply the general theory to some Abelian integrals appearing in the study of perturbations of the Lotka–Volterra system. These integrals have the form

$$I(t) = \int_{\gamma(t)} \omega,$$

where

$$\gamma(t) \subset \{(x, y) \in \mathbb{C}^2 : F(x, y) = t\}$$

is a continuous family of ovals,

$$F(x, y) = x^p y^p (1 - x - y) \quad \text{or} \quad F(x, y) = x^p (y^2 - x - 1)^q, \qquad p, q \in \mathbb{N}$$

and ω is a suitable rational one-form on $\mathbb{C}^2$. In the first case the Abelian integral satisfies a Picard–Fuchs equation of order $2p + 2$. It has been shown by van Gils and Horozov [4], that $I(t)$ satisfies also a second-order differential equation whose coefficients are functions algebraic in t. This allows us to compute the zeros of $I(t)$ (by the usual Rolle's theorem for differential equations) and finally, to estimate the number of limit cycles of the perturbed plane foliation defined by

$$\mathrm{d}F + \varepsilon\tilde{\omega} = 0,$$

where $\tilde{\omega}$ is a real polynomial one-form on $\mathbb{R}^2$. By making use of theorem 1 we provide the theoretical explanation of the phenomenon observed first in [4], see section 3.3. Another interesting case, studied in the paper is when $F(x, y) = x^p(y^2 - x - 1)^q$ (p, q relatively prime). The Abelian integral $I(t)$ satisfies a Picard–Fuchs equation of order $p + q + 1$, which is the dimension of the first homology group of the generic fibre $F^{-1}(t)$. We show that the minimal order of equation (2) is $p + q + 1$ or $p + q$ or $p + q - 1$, and that the coefficients $b_i(t)$ are rational in t, see section 3.2. The meaning of this is that the differential Galois group of the Picard–Fuchs equation is connected and, in contrast to [4], there is no reduction of the degree, which may only drop by one or two, depending on whether ω has or has not residues 'at infinity'.

2. Statement of the result

Let k be a differential field of characteristic zero with algebraically closed field of constants C, $E \supset k$ be a Picard–Vessiot extension for the homogeneous monic linear differential operator L:

$$L(y) = y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_0 y,\ a_i \in k \tag{3}$$

and $y_0 \in E$ a solution, $L(y_0) = 0$. We denote by $k^a \supset k$ the algebraic closure of k which is also a differential field.

Definition 1. *A homogeneous monic linear differential operator $\tilde{L}$ with coefficients in k^a is said to be annihilator of y_0, provided that $\tilde{L}(y_0) = 0$. The annihilator $\tilde{L}$ is said to be minimal, provided that its degree is minimal.*

The definition has a sense, because the algebraic closure E^a of E is a differential field which contains E and k^a as differential subfields. The minimal annihilator obviously exists and is unique, its degree is bounded by the degree of L which is an annihilator of y_0.

We are interested in the following question

For a given solution y_0 as above, find the corresponding minimal annihilator $\tilde{L}$.

To answer this, consider the differential Galois group $G = \mathrm{Gal}(E/k)$, which is the group of differential automorphisms of E fixing k. Recall that G is an algebraic group over C, and let G^0 be the connected component of G, containing the unit element (the identity). The intermediate field $\tilde{k} = E^{G^0}$, $k \subset \tilde{k} \subset E$, of elements invariant under G^0 is then a finite algebraic extension of k. We denote it by $\tilde{k}$.

Let $y_0, y_1, \ldots, y_{d-1}$ be a basis of the C-vector space spanned by the orbit

$$G^0 y_0 = \{g(y_0) : g \in G^0\} \subset E$$

and consider the Wronskian determinant in s variables

$$W(y_1, y_2, \ldots, y_s) = \det \begin{pmatrix} y_1 & y_2 & \cdots & y_s \\ y_1' & y_2' & \cdots & y_s' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(s-1)} & y_2^{(s-1)} & \cdots & y_{s-1}^{(s-1)} \end{pmatrix}.$$

y_0 satisfies the differential equation

$$W(y, y_0, y_1, \ldots, y_{d-1}) = 0 \tag{4}$$

and because of the C-linear independence of y_i

$$W(y_0, y_1, \ldots, y_{d-1}) \neq 0.$$

Let $\tilde{L}$ be the monic linear differential operator defined by

$$\tilde{L}(y) = \frac{W(y, y_0, y_1, \ldots, y_{d-1})}{W(y_0, y_1, \ldots, y_{d-1})}. \tag{5}$$

Its coefficients are invariant under the action of G^0, and hence they belong to the differential field $\tilde{k} = E^{G^0}$.

Our first result is the following

Theorem 1. *The differential operator $\tilde{L}$ (5) is the minimal annihilator of the solution y_0.*

Proof. Let $L_{\min}$ be the unique differential operator of minimal degree with coefficients in some algebraic extension $k_{\min}$ of k, such that $L_{\min}(y_0) = 0$. Denote by $E_{\min}$ the Picard–Vessiot extension for $L_{\min}$.

As a first step, we shall show that $E_{\min}$ can be identified to a differential subfield of the Picar–Vessiot extension E for L. The algebraic closure E^a of E is a differential field which contains E and every algebraic extension of k (hence it contains $k_{\min}$). Therefore the compositum $k_{\min} E$ of $k_{\min}$ and E, that is to say the smallest field containing E and $\tilde{k}$, is well defined [6]. The differential automorphisms group $Gal(k_{\min} E/k)$ acts on the compositum $k_{\min} E$ and leaves E invariant. Therefore $\mathrm{Gal}(k_{\min} E/k_{\min}) \subset \mathrm{Gal}(k_{\min} E/k)$ leaves E invariant too, and the orbit $\mathrm{Gal}(k_{\min} E/k_{\min}) y_0$ is contained in E. Let $y_0, y_1, \ldots, y_{m-1}$ be a basis of the C-vector space spanned by this orbit. Then y_0 satisfies the differential equation

$$\frac{W(y, y_0, y_1, \ldots, y_{m-1})}{W(y_0, y_1, \ldots, y_{m-1})} = 0 \tag{6}$$

and the coefficients of the corresponding monic linear homogeneous differential operator belong to $k_{\min}$.

Consider the ring of differential polynomials

$$k_{\min}\{Y\} = k_{\min}[Y^{(i)} : i = 0, 1, 2, \ldots]$$

in formal variables $Y^{(i)}$. Identifying differential operators on $\tilde{k}$ to polynomials (the derivatives $y^{(i)}$ correspond to variables $Y^{(i)}$), we may consider the ideal I generated by homogeneous linear differential operators with coefficients in $k_{\min}$ which annihilate y_0. This is obviously a linear ideal which, according to the general theory (see [7, proposition 1.8]), is principal in the following sense. There exists a linear differential operator with coefficients in $k_{\min}$, which generates I. Clearly the generator of I is the operator $L_{\min}$ defined above. It follows that the solution space of $L_{\min}$ can be identified to a C-vector subspace of the solution space of the operator defined by (6), which implies

$$k \subset k_{\min} \subset E_{\min} \subset E \tag{7}$$

(the first two inclusions hold by definition).

In the second step of the proof we shall show that $\deg L_{\min} = \deg \tilde{L}$. Indeed, the automorphisms group G^0 leaves fixed the elements of E which are algebraic on k. In particular, the elements of $k_{\min}$ are fixed by G^0 and hence G^0 induces differential automorphisms of the Picard–Vessiot extension $E_{\min}$. This shows that the solution space of $L_{\min}$ contains the solution space of $\tilde{L}$ and

$$\deg L_{\min} \geqslant \deg \tilde{L}.$$

Reciprocally, if we consider (by the construction above) the ideal in $\tilde{k}\{Y\}$ generated by all linear homogeneous differential operators with coefficients in $\tilde{k}$, which annihilate y_0, then this ideal is linear and principal. The generator of the ideal corresponds to the operator $L_{\min}$, and hence

$$\deg L_{\min} \leqslant \deg \tilde{L}.$$

Theorem 1 is proved. □

Towards the end of this section we apply theorem 1 to Fuchs and Picard–Fuchs differential operators. The minimal annihilator of a solution is described in terms of the action of the monodromy group.

Let L be a Fuchsian differential operator of order n on the Riemann sphere $\mathbb{P}^1$, $\Delta = \{t_1, \ldots, t_s, \infty\}$ be the set of its singular points. The field of constants is $C = \mathbb{C}$, the coefficients of L belong to the field of rational functions $k = \mathbb{C}(t)$. Denote by $S \cong \mathbb{C}^n$ the complex vector space of solutions of $L = 0$. The monodromy group $\mathcal{M}$ of L is the image of the homomorphism (monodromy representation)

$$\pi_1(\mathbb{P}^1 \setminus \Delta, *) \to GL(S),$$

where $*$ is some fixed point on $\mathbb{P}^1 \setminus \Delta$. The Zariski closure of $\mathcal{M}$ in $GL(S)$ is the differential Galois group G of L:

$$\overline{\mathcal{M}} = G.$$

A vector subspace $V \subset S$ is invariant under the action of G if and only if it is invariant under the action of $\mathcal{M}$. A subspace $V \subset S$ is said to be *virtually invariant*, provided that it is invariant under the action of identity component G^0 of G, or equivalently, under the action of $\mathcal{M} \bigcap G^0$. For an automorphism $g \in G$ the set $g(V) \subset S$ is a vector subspace of the same dimension. Thus G acts on the Grassmannian space $\mathrm{Gr}(d, S)$

$$G \times \mathrm{Gr}(d, S) \to \mathrm{Gr}(d, S) : (g, V) \mapsto g(V).$$

and for every plane $V \in \mathrm{Gr}(d, S)$ the orbit

$$G(V) = \{g(V) : g \in G\} \subset \mathrm{Gr}(d, S)$$

is well defined.

Lemma 1. *A plane $V \in \mathrm{Gr}(d, S)$ is virtually invariant, if and only if the orbit $G(V) \subset \mathrm{Gr}(d, S)$ is finite.*

Proof. We have

$$\overline{\mathcal{M}(V)} = \overline{\mathcal{M}}(V) = G(V) \supset G^0(V).$$

If the orbit $\mathcal{M}(V)$ is finite, then $\mathcal{M}(V) = \overline{\mathcal{M}(V)}$ and hence $G^0(V)$ is finite. As G^0 is a connected Lie group, then $G^0(V) = V$ and V is virtually invariant.

Suppose that V is virtually invariant. As G/G^0 is a finite group, then $G^0(V) = V$ implies that the orbit $\overline{\mathcal{M}}(V) = G(V) \subset \mathrm{Gr}(d, S)$ is finite and hence $\mathcal{M}(V) \subset \overline{\mathcal{M}}(V)$ is finite too. $\square$

Let L be a Fuchsian differential operator as above, and y_0 a solution, $L(y_0) = 0$. The minimal annihilator of y_0 is a differential operator $\tilde{L}$ of minimal degree with coefficients in some algebraic extension of $\mathbb{C}(t)$. Therefore, the coefficients of $\tilde{L}$ are meromorphic functions on an appropriate Riemann surface, which is a finite covering of $\mathbb{P}^1$. Thus $\tilde{L}$ is a Fuchsian operator too, but on a suitable compact Riemann surface realized as a finite covering of $\mathbb{P}^1$. Let $V_1, V_2 \subset S$ be two virtually invariant planes containing the solution y_0. Then $V_1 \cap V_2$ is a virtually invariant plane containing y_0. This shows the existence of a unique virtually invariant plane V of minimal dimension, containing y_0. We call such a plane minimal. According to lemma 1 and theorem 1 the minimal annihilator of y_0 is constructed as follows. Let $y_0, y_1, \ldots, y_{d-1}$ be a basis of the minimal virtually invariant plane V containing y_0. Consider the Fuchsian differential operator defined as in formula (5).

Theorem 2. *The differential operator $\tilde{L}$ is the minimal annihilator of the solution y_0. The degree of $\tilde{L}$ equals the dimension of the minimal virtually invariant plane containing y_0.*

Suppose finally that L is a linear differential operator of Picard–Fuchs (and hence of Fuchs) type. We shall adapt theorem 2 to this particular setting.

Let $F : \mathbb{C}^2 \to \mathbb{C}$ be a bivariate non-constant polynomial. It is known that there is a finite number of atypical points $\Delta = \{t_1, \ldots, t_n\}$, such that the fibration defined by F

$$F : \mathbb{C}^2 \setminus F^{-1}(\Delta) \to \mathbb{C} \setminus \Delta \tag{8}$$

is locally trivial. The fibres $F^{-1}(t)$, $t \notin \Delta$ are open Riemann surfaces, homotopy equivalent to a bouquet of a finite number of circles. Consider also the associated homology and co-homology bundles with fibres $H_1(F^{-1}(t), \mathbb{C})$ and $H^1(F^{-1}(t), \mathbb{C})$, respectively. Both of these vector bundles carry a canonical flat connection. Choose a locally constant section $\gamma(t) \in H_1(F^{-1}(t), \mathbb{C})$ and consider the Abelian integral

$$I(t) = \int_{\gamma(t)} \omega, \tag{9}$$

where ω is a meromorphic one-form on $\mathbb{C}^2$ which restricts to a holomorphic one-form on the complement $\mathbb{C}^2 \setminus F^{-1}(\Delta)$. The Milnor fibration (8) induces a representation

$$\pi_1(\mathbb{C} \setminus \{\Delta\}, *) \to \mathrm{Aut}(H_1(F^{-1}(t), \mathbb{C})) \tag{10}$$

which implies the monodromy representation of the Abelian integral $I(t)$.

Let $V_t \subset H_1(F^{-1}(t), \mathbb{C})$ be a continuous family of complex vector spaces obtained by a parallel transport. The space V_t can be seen as a point of the Grassmannian variety $\mathrm{Gr}(d, H_1(F^{-1}(t), \mathbb{C}))$. Therefore the representation (10) induces an action of the fundamental group $\pi_1(\mathbb{C} \setminus \{\Delta\}, *)$ on $\mathrm{Gr}(d, H_1(F^{-1}(t), \mathbb{C}))$.

Definition 2. *We say that a complex vector space $V_t \subset H_1(F^{-1}(t), \mathbb{C})$ of dimension d is virtually invariant, provided that its orbit in the Grassmannian $\mathrm{Gr}(d, H_1(F^{-1}(t), \mathbb{C}))$ under the action of $\pi_1(\mathbb{C} \setminus \{\Delta\}, *)$ is finite. A virtually invariant space V_t is said to be irreducible, if it does not contain non-trivial proper virtually invariant subspaces.*

Let $\gamma(t)$ be a locally constant section of the homology bundle defined by F. As the intersection of virtually invariant vector spaces $V_t \subset H_1(F^{-1}(t), \mathbb{C})$ containing $\gamma(t)$ is virtually invariant again, then such an intersection is the minimal virtually invariant space containing $\gamma(t)$. Clearly a virtually invariant minimal space containing $\gamma(t)$ need not be irreducible: it might contain a virtually invariant subspace not containing $\gamma(t)$.

Consider the Abelian integral $I(t) = \int_{\gamma(t)} \omega$, where $\gamma(t)$ is a locally constant section of the homology bundle and ω is a meromorphic one-form as above. Denote by V_t the minimal virtually invariant vector space containing $\gamma(t)$.

Theorem 3. *If V_t is irreducible, then either the Abelian integral $I(t)$ vanishes identically, or its minimal annihilator is a linear differential operator of degree $d = \dim\ V_t$.*

Proof. Let S_t be the complex vector space of germs of analytic functions in a neighbourhood of t, obtained from $I(t)$ by analytic continuation along a closed path in $\mathbb{C} \setminus \Delta$. It suffices to check that V_t is isomorphic to S_t. Equivalently, for every locally constant section $\delta(t) \in V_t$ we must show that $\int_{\delta(t)} \omega \not\equiv 0$. Indeed, the vector space of all locally constant sections $\delta(t)$ with $\int_{\delta(t)} \omega \equiv 0$ is an invariant subspace of V_t. As V_t is supposed to be irreducible, then this space is trivial. Theorem 3 follows from theorem 2. □

The above theorem is easily generalized. For instance, the coefficients of the minimal annihilator of I are rational functions of t if and only if the minimal virtually invariant space V_t containing γ is monodromy invariant, i.e. its orbit in the Grassmannian consists of a single point. Further, it might happen that V_t is reducible. Let V_t^0 be a proper virtually invariant subspace of V_t. If the factor space V_t/V_t^0 is irreducible (does not contain proper virtually invariant subspaces), then theorem 3 still holds true, but the minimal annihilator of $I(t)$ is of order equal to $\dim\ V_t - \dim\ V_t^0$. Multidimensional Abelian integrals (along k-cycles) are studied in a similar way.

3. Examples of Abelian integrals related to perturbation of the Lotka–Volterra system

Let F be a real polynomial and $\omega = P\,\mathrm{d}x + Q\,\mathrm{d}y$ a real polynomial differential one-form in $\mathbb{R}^2$. Consider the perturbed real foliation in $\mathbb{R}^2$ defined by

$$\mathrm{d}F + \varepsilon\omega = 0. \tag{11}$$

The infinitesimal 16th Hilbert problem asks for the maximal number of limit cycles of (11) when $\varepsilon \sim 0$ as a function of the degrees of F, P, Q. Let $\gamma(t) \subset F^{-1}(t)$ be a continuous family of closed orbits of (11). The zeros of the Abelian integral $I(t) = \int_{\gamma(t)} \omega$ approximate limit cycles (at least far from the atypical points of F) in the following sense. If $I(t_0) = 0$, $I'(t_0) \neq 0$, then a limit cycle of (11) tends to the oval $\gamma(t_0)$, when ε tends to t_0. The question of explicit computing the number of zeros of Abelian integrals remains open (although a substantial progress has recently been achieved, see [3,5] and references therein). Generically an Abelian integral satisfies a Picard–Fuchs differential equation

$$I^{(d)} + a_1 I^{(d-1)} + \cdots + a_d I = 0, \qquad a_i \in \mathbb{R}(t)$$

of order equal to the dimension of the homology group of the typical fibre $F^{-1}(t)$. We are interested in the possibility of reducing the degree of this equation, assuming that the coefficients of the equation are algebraic in t, $a_i \in \mathbb{C}(t)^a$. The most interesting situation is probably when the reduced degree equals two. Indeed, the zeros of the solutions of a second-order equation are easily bounded, in terms of the zeros of the coefficients of this equation (by Rolle's theorem).

In this section we study Abelian integrals which appear in the perturbations of foliations $\mathrm{d}F = 0$ with $F = x^p(y^2 + x - 1)^q$ and $F(x, y) = (xy)^p(x + y - 1)$, where p, q are positive integers. The corresponding foliation $\mathrm{d}F = 0$ is a special Lotka–Volterra system.

3.1. *Toy example $F = x^p y^q$*

Consider first the fibration defined by the polynomial $F = x^p y^q$. We assume that p, q are *relatively prime*. The base of the fibration is the punctured plane $B = \mathbb{C} \setminus \{0\}$. Each fibre is a sphere with two points removed. The homology bundle is one dimensional with trivial monodromy representation. We investigate the monodromy representation on the *relative homology* bundle. It will be a basic ingredient of the monodromy investigation in more complicated cases.

Consider a set of marked points B_t on the complex fibre $F^{-1}(t)$:

$$B_t = (F^{-1}(t) \cap \{x = L\}) \cup (F^{-1}(t) \cap \{y = L\}),$$

where $L > 1$ is a real number. The relative homology $H_1(F^{-1}(t), B_t)$ is a free group with $p + q$ generators. A convenient model for the pair $(F^{-1}(t), B_t)$ consists of a cylinder with some strips attached; marked points are located at the ends of these strips.

Note that there exists a unique pair of positive integers (m, n) satisfying the following relation:

$$pm + qn = 1, \qquad |m| < q, \quad |n| < p. \tag{12}$$

Let $S \subset \mathbb{C}$ be the strip in complex plane around the real segment $[1, L]$. Let $C(r, R) \subset \mathbb{C}$ be a ring (homeomorphic to a cylinder) where radii r and R satisfy relations $L^{-1} < r < 1 < R < L$. The model M is a surface constructed with three charts U_x, U_y, U_c:

$$\begin{aligned} U_x &= \{(x, \nu) : \ x \in S, \ \nu \in \mathbb{Z}/q\}, \\ U_y &= \{(y, \mu) : \ y \in S, \ \mu \in \mathbb{Z}/p\}, \\ U_c &= \{u \in C(r, R)\} \end{aligned} \tag{13}$$

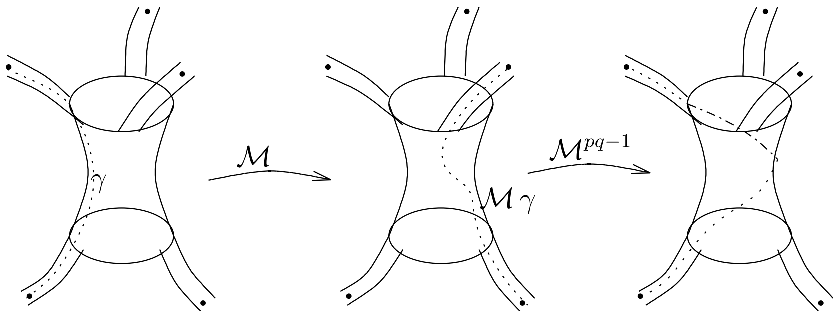

Figure 1. Monodromy transformation of the model surface M and a relative cycle γ.

with the following transition functions (strips U_x are attached to the external circle of radius R and strips U_y are attached to the internal boundary of U_c):

$$u(x, \nu) = x^{1/q} e^{2\pi i/q\,(-\nu m)}, \qquad u(y, \mu) = y^{-1/p} e^{2\pi i/p\,(\mu n)}. \tag{14}$$

The marked points are $\{x = L\}$ and $\{y = L\}$ at the end of strips. To construct a map ψ_t we will use a bump function $\varphi \in C^\infty([0, 1])$ which is 0 near $s = 0$ and 1 near $s = 1$. The map $\psi_t : M \to \mathbb{C}^2$ reads

$$\psi_t : \begin{cases} \psi_t(x, \nu) = (x,\ t^{1/q} x^{-p/q} e^{2\pi i/q\,\nu}) \\ \psi_t(y, \mu) = (t^{1/p} y^{-q/p} e^{2\pi i/p\,\mu},\ y) \\ \psi_t(u) = \left(u^q \exp\left(\dfrac{\log t}{p}\, \varphi\left(\dfrac{|u| - r}{R - r}\right)\right),\ t^{1/q} u^{-p} \exp\left(-\dfrac{\log t}{q}\, \varphi\left(\dfrac{|u| - r}{R - r}\right)\right)\right). \end{cases} \tag{15}$$

Lemma 2. *The surface M and the map ψ_t provide model of fibre for fibration defined by $F = x^p y^q$. The monodromy transformation $\mathcal{M} : M \to M$ around $t_0 = 0$ reads*

$$\mathcal{M} : \begin{cases} \mathcal{M}(x, \nu) = (x, \nu + 1) \\ \mathcal{M}(y, \mu) = (y, \mu + 1) \\ \mathcal{M}(u) = u \exp\left(2\pi i \left(-\dfrac{m}{q} + \dfrac{1}{pq} \varphi\left(\dfrac{|u| - r}{R - r}\right)\right)\right) \end{cases} \tag{16}$$

The surface M and its monodromy transformation described in the above lemma are drawn in figure 1.

Proof. Complex level curves $F^{-1}(t)$ intersect line at infinity in two points: $[1 : 0 : 0]$ and $[0 : 1 : 0]$. The neighbourhood of any of them is a punctured disc. Thus, there exists an isotopy of the level curve $F^{-1}(t)$ shrinking it to the region $\{|x| \leqslant R,\ |y| \leqslant R\}$ for sufficiently big R.

We will assume that t is sufficiently close to 0. The intersection of $F^{-1}(t)$ with the neighbourhood $\{|x| \leqslant r,\ |y| \leqslant r\}$ of $(0, 0)$ is a cylinder parametrized by the formula

$$u \mapsto (g^q u^q,\ g^{-p} u^{-p} t^{1/q}), \tag{17}$$

where $g(t, u)$ is a function which will be fixed later.

The intersection of $F^{-1}(t)$ with set $\{|x| \leqslant R,\ |y| \leqslant R\} \setminus \{|x| \leqslant r,\ |y| \leqslant r\}$ decomposes into two connected components V_x and V_y; one is located close to the x-plane and the other to the y-plane, respectively. The component V_x is a graph of multi-valued (q-valued) function $y = t^{1/q} x^{-p/q}$ defined over the ring $\{r \leqslant |x| \leqslant R\}$. Marked points are images of point $x = L$ located on the real axis. We deform this domain by isotopy to the strip S along the real

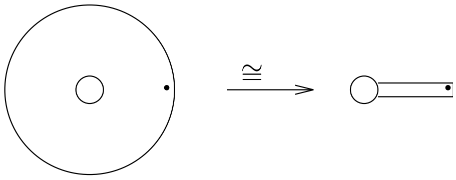

Figure 2. Deformation of domain to the strip S.

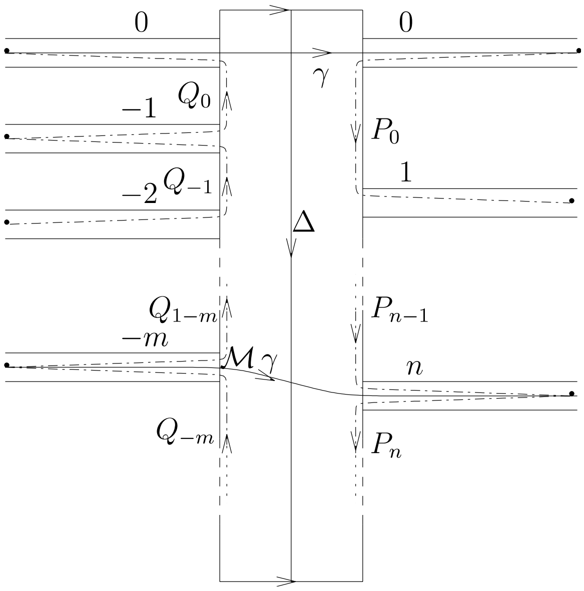

Figure 3. The model surface M with generators of the relative homology.

line, attached to a 'small' annulus, as in figure 2. The values (leaves) of function $x^{-p/q}$ are numbered by $\nu \in \mathbb{Z}/q$. Thus, the domain U_x and the map ψ_t are defined as in lemma.

The model of V_y is constructed in an analogous way. To glue the above map together with parametrization (17) of the disc around zero, we use the auxiliary function g. It must be equal to 1 near the internal circle of the ring $C(r, R)$ (i.e. $|u| = r$) and $t^{1/pq}$ near the exterior boundary ($|u| = R$). It is easy to check that $g = \exp(\frac{1}{pq} \log t\, \varphi(\frac{|u|-r}{R-r}))$ solves the problem.

Formula (16) for the monodromy around $t = 0$ is a direct consequence of formula (15). □

A two-dimensional version of figure 1 presenting the model surface M is drawn in figure 3. It is obtained from figure 1 by cutting the cylinder along a vertical line. We will use this planar style of drawing models in subsequent, more complicated cases.

The cylinder shown in figure 1 is represented in figure 3 by a rectangle with upper and lower sides identified. Strips U_x and U_y are enumerated by integers $\frac{q}{2\pi} \arg u$ and $\frac{p}{2\pi} \arg u$, respectively; the argument $\arg u$ is calculated in point $u \in U_c$ which is glued with point $1 \in S$ according to relations (14). Generators of the relative homology of M are also marked.

Proposition 1. *The relative homology is $H_1(F^{-1}(t), B_t)$of the complex fibre $F^{-1}(t)$ has dimension $p + q$. It is generated by cycles*

$$\gamma, \Delta, Q_0, \ldots, Q_{q-1}, P_0, \ldots, P_{p-1}$$

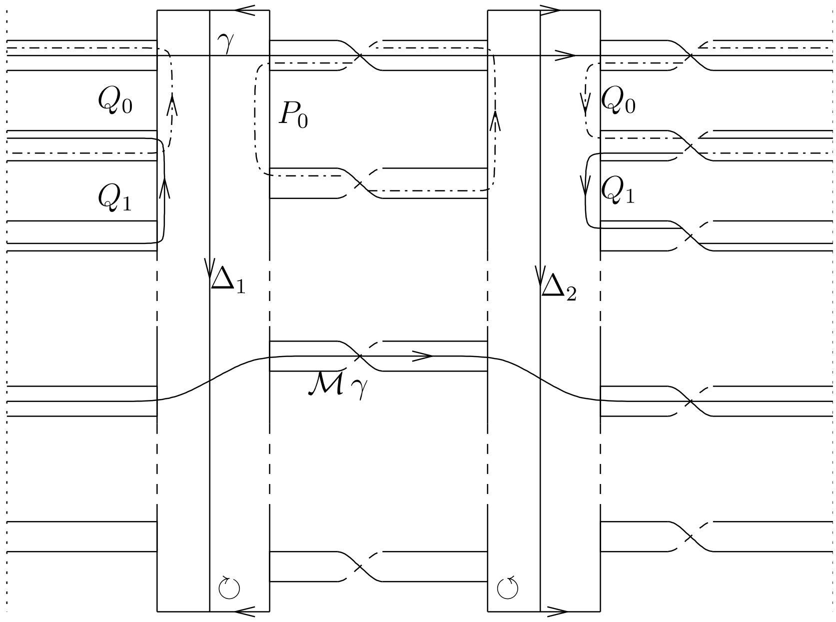

Figure 4. Model of the level curve $F^{-1}(h)$ for $F = x^p(y^2 + x - 1)^q$.

with the relations:

$$Q_0 + \cdots + Q_{q-1} = -\Delta, \qquad P_0 + \cdots + P_{p-1} = \Delta.$$

The monodromy transfomation on the relative homology space reads

$$\begin{aligned} &\mathcal{M}Q_j = Q_{j-m}, \qquad \mathcal{M}P_k = P_{k+n}, \quad \mathcal{M}\Delta = \Delta \\ &\mathcal{M}\gamma = \gamma + Q_0 + \cdots + Q_{-m+1} + P_0 + \cdots + P_{n-1}. \end{aligned} \tag{18}$$

The proposition is a direct consequence of lemma 2.

3.2. The parabolic case

Consider the fibration given by a polynomial $F = x^p(y^2 + x - 1)^q$, where p, q is a pair of positive, relatively prime integer numbers. Thus, they satisfy relation (12) with a pair of integers m, n. They must be of opposite signs; we assume $m > 0$ and so $n \leqslant 0$.

The base of locally trivial fibration in this case is a plane with two points removed $B = \mathbb{C} \setminus \{0, c\}$, where $c = (\frac{p}{p+q})^p(\frac{-q}{p+q})^q$ corresponds to a centre $(\frac{p}{p+q}, 0)$ of the Hamiltonian vector field X_F. The cycle $\gamma(t)$ for $t \in (0, c)$ is an oval (compact component) of the real level curve $F^{-1}(t)$.

The model of the complex fibre is presented in figure 4. It consists of two cylinders and $p + q$ strips glued together as shown in the figure. Cylinders are drawn as rectangles, with horizontal sides identified. To simplify the combinatorial structure, there are opposite orientations on these two cylinders. Vertical, dotted lines mark another identification.

Lemma 3. *The surface shown in the figure 4 provides a model M for the complex fibre $F^{-1}(t)$. The homology group $H_1(M)$ has dimension $p + q + 1$ and is generated by cycles $\gamma, \Delta_1, \Delta_2, Q_0, \ldots, Q_{q-1}, P_0, \ldots, P_{p-1}$ with the following relations:*

$$Q_0 + \cdots + Q_{q-1} = \Delta_2 - \Delta_1, \qquad P_0 + \cdots + P_{p-1} = \Delta_1 - \Delta_2. \tag{19}$$

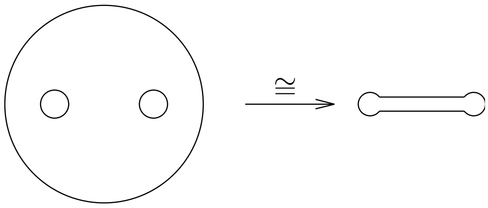

Figure 5. Deformation of domain U_l to the strip S.

Intersection indices of γ and other generators of the homology group read

$$\begin{aligned}
&\gamma \cdot Q_0 = -1, \quad \gamma \cdot Q_{q-1} = -1, \quad \gamma \cdot Q_j = 0, \qquad \text{for } j = 1, \ldots, q-2,\\
&\gamma \cdot P_0 = +1, \quad \gamma \cdot P_{p-1} = +1, \quad \gamma \cdot P_j = 0, \qquad \text{for } j = 1, \ldots, p-2,\\
&\gamma \cdot \Delta_1 = +1, \quad \gamma \cdot \Delta_2 = -1.
\end{aligned} \tag{20}$$

The monodromy transformation associated with the critical value $0 \in \mathbb{C}$ takes the form:

$$\begin{aligned}
&\mathcal{M}_0 Q_j = Q_{j+m}, \qquad \mathcal{M}_0 P_k = P_{k-n}, \qquad \mathcal{M}_0 \Delta_j = \Delta_j\\
&\mathcal{M}_0 \gamma = \gamma + Q_0 + \cdots + Q_{m-1} + P_0 + \cdots + P_{-n+1}.
\end{aligned} \tag{21}$$

Proof. The idea of proof is similar to the proof of lemma 2. We shrink the level curve $F^{-1}(t)$ by isotopy to the region $\{|x| \leqslant R,\ |y| \leqslant R\}$. We take the value of t sufficiently close to 0. The intersections of $F^{-1}(t)$ with neighbourhoods of saddle points $(0, 1)$, $(0, -1)$ are cylinders; we parametrize them by formulae similar to (17). The remaining part of the fibre $F^{-1}(t)$ splits into two pieces: V_l and V_p, located close to the line $x = 0$ and close to the parabola $y^2 + x - 1 = 0$, respectively. The part V_l is the graph of p-valued function $x = t^{1/p}(y^2 + x - 1)^{-q/p}$ defined over the disc of radius R with small discs around points $y = \pm 1$ removed:

$$U_l = \{y : \ |y| \leqslant R,\ |y - 1| \geqslant r,\ |y + 1| \geqslant r\}.$$

We deform the domain U_l to the strip S along the real segment—see figure 5. Leaves of function over the strip S are numbered by $\mu \in \mathbb{Z}/p$. In an analogous way we deform V_p to the graph of the q-valued function defined over the strip along a real segment of the parabola $\{y^2 + x - 1 = 0\}$. Leaves of the function are numbered by $\nu \in \mathbb{Z}/q$.

We glue together both collections of strips with two cylinders in a way analogous to the toy example. Indeed, in sufficiently small neighbourhood of the point $(0, 1)$ the pair of functions $(x, y^2 + x - 1)$ defines a holomorphic chart. In this chart the function F takes the form as in the toy example. The same is true for the other saddle $(0, -1)$. Both cylinders are glued by p strips going along the line $x = 0$ and q strips going along the parabola $y^2 + x - 1$. The monodromy around zero permutes strips according to the rule $\nu \mapsto \nu + 1$, $\mu \mapsto \mu + 1$, which is compatible with formula (16) for monodromy in the toy example case. Thus, the monodromy acts on both cylinders around saddles as in the toy example.

The surface shown in figure 4 provides a model for the complex fibre $F^{-1}(t)$. Formulae (21) follow from the respective formulae (18) in the toy example. One can read relations (19) and intersection indices (20) from figure 4. □

Corollary 1.

$$\mathcal{M}_0^{pq} \gamma = \gamma + \Delta_2 - \Delta_1.$$

The critical value $t = c$ corresponds to a Morse critical point of F. The monodromy operator $\mathcal{M}_c$ around c is therefore described by the usual Picard–Lefschetz formula. Let $\gamma = \gamma(t)$ be the continuous family of cycles, vanishing c.

Corollary 2.

$$\begin{aligned}
&\mathcal{M}_c Q_0 = Q_0 - \gamma, \quad \mathcal{M}_c Q_{q-1} = Q_{q-1} - \gamma,\\
&\mathcal{M}_c Q_j = Q_j, \qquad \text{for } j = 1, \ldots, q-2,\\
&\mathcal{M}_c P_0 = P_0 + \gamma, \quad \mathcal{M}_c P_{p-1} = P_{p-1} + \gamma,\\
&\mathcal{M}_c P_j = P_j, \qquad \text{for } j = 1, \ldots, p-2,\\
&\mathcal{M}_c \Delta_1 = \Delta_1 + \gamma, \quad \mathcal{M}_c \Delta_2 = \Delta_2 - \gamma.
\end{aligned} \tag{22}$$

Theorem 4. *The related Abelian integral $I = \int_\gamma \omega$ is either identically zero, or it does not satisfy any differential equation with algebraic coefficients of order $k < p + q - 1$.*

Proof. The proof is based on theorem 3. Let H be a k-dimensional subspace of the (complex) homology space $H_1 = H_1(F^{-1}(t), \mathbb{C})$ and $\gamma \in H$. Assume that the monodromy orbit of H in the Grassmannian $G_k(H_1)$ is finite. We show that the dimension of H satisfies $\dim H \geqslant p + q$.

Let $\mathcal{M}_0$ be the operator of monodromy around $t = 0$ (i.e. along a loop winding once around $t = 0$); let $\mathcal{M}_c$ be a monodromy around the centre $t = c$. It follows formulae (22) that the $\mathcal{M}_c - \mathrm{Id}$ is a nilpotent operator and its image is one dimensional, generated by γ. The homology space H_1 splits into two-dimensional $\mathcal{M}_c$-invariant subspace N and $(\dim H_1 - 2)$-dimensional; the monodromy $\mathcal{M}_c$ restricted to the latter one is the identity. The matrix of the restricted monodromy operator $\mathcal{M}_c|_N$ in a basis (γ, δ) has the form

$$[\mathcal{M}_c|_N]_{(\gamma,\delta)} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Note that the subspace N is not defined uniquely. It is spanned by γ and any element $\delta \in H_1$ such that $\gamma \cdot \delta \neq 0$.

Consider the intersection $H \cap N$. The property that H has a finite π_1 orbit (see theorem 3) implies that the $\mathcal{M}_c$-orbit of $\mathcal{M}_0^k H$, $k \in \mathbb{Z}$, is finite. Thus, the intersection $HN_k = (\mathcal{M}_0^k H) \cap N$ has also finite $\mathcal{M}_c$ orbit in N. The form of $\mathcal{M}_c|_N$ implies that there are only three subspaces with a finite orbit:

$$HN_k = \{0\}, \qquad HN_k = \mathbb{C}\,\gamma, \qquad HN_k = N. \tag{23}$$

Note that all these subspaces are $\mathcal{M}_c$-invariant.

Lemma 4. *Assume that the monodromy orbit of H in $G_k(H_1)$ is finite. If $u \cdot \gamma \neq 0$ for an element $u \in \mathcal{M}_0^l H$, $l \in \mathbb{Z}$, then $\gamma \in \mathcal{M}_0^l H$.*

Proof. Take $\delta = u$ and consider two-dimensional, $\mathcal{M}_c$ invariant space N spanned by γ and δ. The $\mathcal{M}_c$ orbit of the $\mathcal{M}_0^l H$ space is finite, so the intersection $HN_l = N \cap \mathcal{M}_0^l H$ has one of the three forms listed in (23). Since $\delta \in HN_l$ then $HN_l = N$ and so $\gamma \in \mathcal{M}_0^l H$. □

Consider the $\mathcal{M}_0$-orbit of $\gamma \in H$. Lemma 4 (for $l = -m$) and the property that the intersection number is preserved by the monodromy imply the following condition:

$$\gamma \cdot \mathcal{M}_0^m \gamma \neq 0 \Rightarrow \mathcal{M}_0^m \gamma \in H. \tag{24}$$

Consider an element $\mathcal{M}_0^{pq}\gamma = \gamma + (\Delta_2 - \Delta_1)$. Since $\gamma \cdot (\Delta_2 - \Delta_1) = -2$, then $(\Delta_2 - \Delta_1) \in H$. Consider elements $\mathcal{M}_0^l \gamma$ for $l = 1, \ldots, pq$. The intersection index $|\gamma \cdot (\mathcal{M}_0^l \gamma)| \leqslant N$. So, the cycles

$$\mathcal{M}_0^{pq\,N+q\,l}\gamma = \gamma + (N + lm)\,(\Delta_2 - \Delta_1) + \sum_{j=0}^{p} a_j(l) P_j, \qquad l = 1, \ldots, p \tag{25}$$

have non-zero intersection indices with γ. Since p, q are relatively prime the space spanned sums $\sum_{j=0}^{p} a_j(l) P_j,\ l = 1, \ldots, p$ coincide with the space generated by $(P_0 + \cdots + P_{-n-1})$, $(P_{-n} + \cdots + P_{-2n-1})$,...; the latter one is the full space generated by $P_0, \ldots, P_{p-1}$. Both claims follow the fact that p and n are also relatively prime (see (12)) and the following observation.

Lemma 5. *Let V be a vector space of dimension p and let q be an integer. Assume that p, q are relatively prime. Let $e_0, \ldots, e_{p-1}$ be a basis of V. Then the following sums*

$$(e_0 + \cdots + e_{q-1}),\ (e_q + \cdots + e_{2q-1}),\ \ldots (e_{(p-1)q} + \cdots + e_{pq-1}) \tag{26}$$

(all indices mod p *assumed) generate the whole space V.*

The proof of this lemma is based on the following observations. Since p, q are relatively prime, any sum of length q appears in a sequence (26). The difference of two sums has the form $e_j - e_{j+q}$, $j = 0, \ldots, p-1$; they generate a hyperplane orthogonal to vector $e_0 + \cdots + e_{p-1}$. Since the scalar product $(e_0 + \cdots + e_{p-1}) \cdot (e_0 + \cdots + e_{q-1}) = q$, the space generated by vectors (26) is a whole V.

Thus, it is proved that the subspace H must contain the subspace generated by $P_0, \ldots, P_{p-1}$. In a similar way we show that H contains the subspace generated by $Q_0, \ldots, Q_{q-1}$.

We have shown that the subspace of the homology group containing γ, with finite π_1-orbit must be necessarily π_1-invariant hyperplane in the homology space H_1. It proves the theorem for a *generic* 1-form ω (when the zero subspace $Z_\omega = \{0\}$). To finish the proof we show that either $\dim Z_\omega \leqslant 1$ or $Z_\omega = H$.

Consider an element

$$H \cap Z_\omega \ni v = a\,\gamma + \sum_{j=0}^{p-1} \alpha_j P_j + \sum_{i=0}^{q-1} \beta_i Q_i$$

and its images under the monodromy around $t = 0$: $\mathcal{M}_0^l v$. Since Z_ω is monodromy invariant, all elements $\mathcal{M}_0^l v \in Z_\omega$. If the intersection index $\gamma \cdot \mathcal{M}_0^{l_0} v \neq 0$, then monodromy around the centre $t = c$ adds a multiple of γ, so $\gamma \in Z_\omega$. Then, it follows from the previous analysis that $Z_\omega = H$. Assume now that all intersection indices $\gamma \cdot \mathcal{M}_0^l v = 0$. The coefficient a must vanish then, otherwise $\mathcal{M}_0^{pq}$ adds the cycle $\Delta_2 - \Delta_1$ which realizes intersection index -2. Consider monodromies $\mathcal{M}_0^{ql} v, l = 0, \ldots, p-1$. It preserves the expression $\sum_{i=0}^{q-1} \beta_i Q_i$. Vanishing of the intersection indices $\gamma \cdot \mathcal{M}_0^{ql} v$ implies equations

$$\alpha_j + \alpha_{j+1} = \beta_0 + \beta_{q-1}, \qquad j = 0, 1, \ldots, p-1. \tag{27}$$

The solution of (27) depends on the parity of p. If p is odd then all coefficients α_j are equal: $\alpha_j = \alpha = \frac{1}{2}(\beta_0 + \beta_{q-1})$. If p is even the solution of (27) reads

$$\alpha_{2l} = \alpha_0, \qquad \alpha_{2l+1} = \alpha_1, \qquad \alpha_0 + \alpha_1 = \beta_0 + \beta_{q-1}.$$

We repeat then the analogous analysis with iterations of $\mathcal{M}_0^p$. We obtain the following form of Z_ω:

$$Z_\omega \cap H \subset \begin{cases} \{0\} & \text{for } p, q \text{ odd}, \\ \mathrm{Span}(2\sum_{j=1}^{p/2} P_{2j} + (\Delta_2 - \Delta_1)) & \text{for } p \text{ even and } q \text{ odd}, \\ \mathrm{Span}(2\sum_{j=1}^{q/2} Q_{2j} - (\Delta_2 - \Delta_1)) & \text{for } q \text{ even and } p \text{ odd}, \end{cases} \qquad (28)$$

Thus, dim $Z_\omega \cap H \leqslant 1$ and so the theorem is proved. □

Corollary 3. *We have actually shown that the Abelian integral does not satisfy any differential equation with algebraic coefficients of order lower than the Fuchs-type equation with rational coefficients which follows the general theory.*

3.3. The special Lotka–Volterra case

Consider a fibration given by a polynomial $F(x, y) = (xy)^p(x + y - 1)$. It defines a locally trivial fibration defined over plane with two points removed $B = \mathbb{C} \setminus \{0, c\}$, where $c = F(\frac{p}{1+2p}, \frac{p}{1+2p})$ corresponds to a centre. The cycle γ_t for $t \in (0, c)$ is an oval (compact component) of the real level curve $F^{-1}(t)$. Note, that the fibration has a Morse-type singularity at $t = c$ and γ_t is a vanishing cycle at the centre.

Below we investigate the fibration and the monodromy representation on the sufficiently small neighbourhood of $t = 0$: $|t| < \varepsilon_0$. The monodromy around centre $t = c$ follows the Picard–Lefshetz formula. Thus, to determine the monodromy representation it is enough to investigate the monodromy around $t = 0$ and intersection indices with the cycle γ.

The model of complex fibre is presented in figure 6 which should be understood as follows. Each rectangle represents a cylinder, with sides pasted according to the arrows. Another identification is assumed on vertical, dotted lines.

Lemma 6. *The complex level curve $F^{-1}(t)$ is a surface of genus $p - 1$ with 3 points removed (intersection with the line at infinity). The surface shown in figure 6 provides a model M for $F^{-1}(t)$. The homology group $H_1(M)$ has dimension $2p + 2$; it is generated by cycles $\gamma, \Delta_1, \Delta_2, P_0, \dots, P_{p-1}, \delta_0, \dots, \delta_{p-1}$ with the following relation:*

$$P_0 + \cdots + P_{p-1} = \Delta_1 - \Delta_2 + \delta_0.$$

Intersection indices of γ with other generators of the homology group read

$$\begin{aligned} &\gamma \cdot P_{p-1} = -1, \qquad \gamma \cdot P_j = 0, \qquad \text{for } j = 0, \dots, p-2, \\ &\gamma \cdot \delta_0 = -1, \qquad \gamma \cdot \delta_j = 0, \qquad \text{for } j = 1, \dots, p-1, \\ &\gamma \cdot \Delta_1 = -1, \qquad \gamma \cdot \Delta_2 = -1. \end{aligned} \qquad (29)$$

The monodromy transformation associated with the singular value $0 \in \mathbb{C}$ acts as follows:

$$\begin{aligned} &\mathcal{M}\Delta_j = \Delta_j, \qquad \mathcal{M}\delta_j = \delta_{j+1}, \qquad \mathcal{M}\gamma = \gamma + P_0, \\ &\mathcal{M}P_j = P_{j+1}, \qquad \text{for } j = 0, \dots, p-2, \\ &\mathcal{M}P_{p-1} = P_0 + \delta_1 - \delta_0. \end{aligned} \qquad (30)$$

Proof. The proof is analogous to the proofs of lemmas 2 and 3. We modify the level curve $F^{-1}(t)$ by isotopy to the part contained in the compact region $|x| \leqslant R$, $|y| \leqslant R$. We consider points t sufficiently close to 0. We cut the level curve $F^{-1}(t)$ into pieces lying close to lines $x = 0$, $y = 0$, $x + y - 1 = 0$ and close to saddles $(0, 0)$, $(1, 0)$, $(0, 1)$. The analysis of pieces of level curve $F^{-1}(t)$ close to the saddles $(1, 0)$ and $(0, 1)$ and close to the line $x + y - 1 = 0$

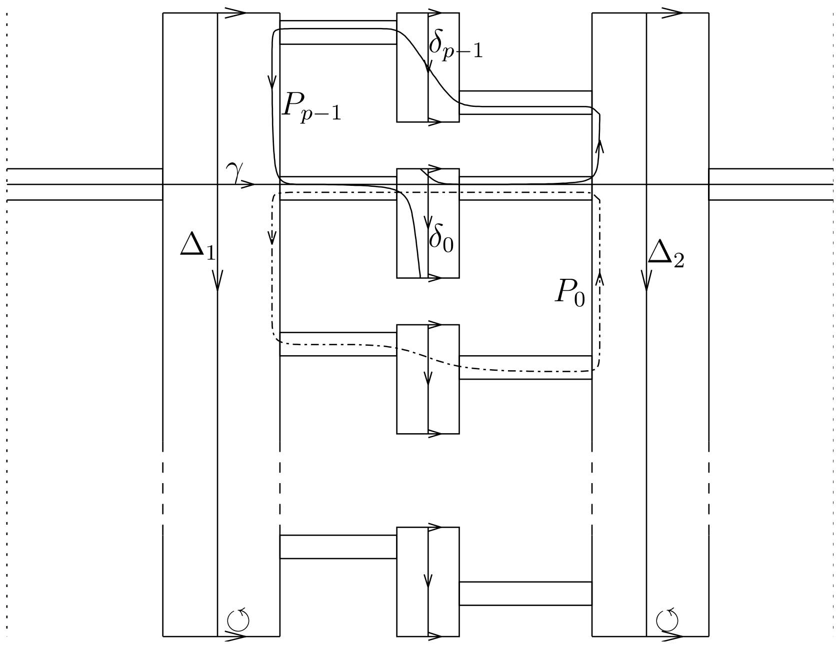

Figure 6. Model of the level curve $F^{-1}(h)$ for $F = (xy)^p(x + y - 1)$.

is completely analogous to the parabola case—see the proof of lemma 3. The model of this part of the level curve consists of two cylinders joined by a single strip.

Now we consider the region which is at a finite distance from the line $x + y - 1 = 0$. The level curve $F^{-1}(t)$ outside the line $x + y - 1 = 0$ splits into p components defined by the equation

$$xy\,(x + y - 1)^{1/p} = t^{1/p}\varepsilon_p^{\nu}, \qquad \nu = 0, 1, \ldots, p - 1.$$

Any of these components coincide with the toy example with $p = q = 1$. Thus, it is isotopic to the cylinder with two strips attached. As t winds around zero $t \mapsto \mathrm{e}^{2\pi\mathrm{i}}t$ we rotate components according to the rule $\nu \mapsto \nu + 1 \bmod p$. The pth power of the monodromy (winding p times around zero) $\mathcal{M}^p$ corresponds to the usual monodromy in the toy example; it follows from formula (18) (for $p = q = 1$) that $\mathcal{M}^p$ adds the generator δ_j.

It proves that the combinatorial structure of model of the level curve $F^{-1}(t)$, defining how cylinders and strips are glued, must be as that shown in figure 6. □

Proposition 2. *Let H be the following two-dimensional subspace*

$$H = \mathrm{Span}(\gamma,\ \Delta_1 - \Delta_2 + \delta_0)$$

of the (complex) homology space $H_1(F^{-1}(t))$. The orbit of H under the monodromy representation $\pi_1 \cdot H$ in Grassmannian $\mathrm{Gr}_2(H_1)$ consists of p elements and hence is finite.

Proof. Denote by $\mathcal{M}$ and $\mathcal{M}_c$ the monodromy around $t = 0$ and around the centre critical value $t = c$. By lemma 6, we have

$$\mathcal{M}^k H = \begin{cases} \mathrm{Span}\big(\gamma + P_0 + \cdots + P_{k-1},\ \Delta_1 - \Delta_2 + \delta_k\big) & \text{for } k = 1, \ldots, p-1 \\ \mathrm{Span}\big(\gamma + \Delta_1 - \Delta_2 + \delta_0,\ \Delta_1 - \Delta_2 + \delta_0\big) = H & \text{for } k = p. \end{cases} \tag{31}$$

The crucial observation is that subspaces $\mathcal{M}^k H$ for $k \in \mathbb{Z}$ are $\mathcal{M}_c$-invariant. Indeed, the subspace H is $\mathcal{M}_c$-invariant since $\gamma \in H$ is a vanishing cycle corresponding to the centre critical value $t = c$. We calculate (using formulae (29)) the intersection indices of γ and generators of $\mathcal{M}^k H$ for $k = 1, \dots, p-1$:

$$\gamma \cdot \big(\gamma + (P_0 + \cdots + P_{k-1})\big) = 0, \qquad \gamma \cdot \big(\Delta_1 - \Delta_2 + \delta_k)\big) = 0.$$

Thus, both generators are $\mathcal{M}_c$-invariant. This proves that the orbit $\pi_1 \cdot H$ in Grassmannian $\mathrm{Gr}_2(H_1)$ consists of p subspaces given in formula (31). □

Corollary 4. *Proposition 2 provides a geometric explanation of the phenomenon described in [4]. According to the general theory given in theorem 3 and calculations of monodromy given in lemma 6 and proposition 2 the Abelian integral along cycle γ satisfies a linear second-order equation with algebraic coefficients in variable t.*

Acknowledgments

Part of the paper was written while the first author was visiting the University Paul Sabatier of Toulouse. He is obliged for the hospitality. This work was supported by the Polish MNiSzW Grant No N N201 397937.

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