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ON THE GEOMETRY OF GORJATCHEV-TCHAPLYGIN TOP

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1. Introduction. The present paper concerns a special case of complete integrability of the motion of a rigid body around a fixed point — the so-called Gorjatchev-Tchaplygin top [1, 2]. Recently it was proved [5,4] that the complex integral manifolds of Kowalewski top may be completed into Abelian surfaces on which the flows are straight-line motions, i. e. the system is algebraically completely integrable. It turns out that the Gorjatchev-Tchaplygin top is not algebraically completely integrable. Nevertheless, we prove that the complex integral manifolds are double covers of $\mathrm{Jac}(\Gamma)\setminus D$ where D is a divisor on $\mathrm{Jac}(\Gamma)$ and $\mathrm{Jac}(\Gamma)$ is the Jacobian of a hyperelliptic curve of genus two Γ. The projection map induces flows on $\mathrm{Jac}(\Gamma)$ which are straight-line motions. This is the main result reported in the paper and it is formulated in detail in Theorem 1. The geometry of the pole divisor on $\mathrm{Jac}(\Gamma)$ is studied in 3. It should be noted that our technique uses essentially (in contrast to [4, 5]), the explicite solutions of the Gorjatchev-Tchaplygin top [7].

2. On the Geometry of the Integral Manifold. Consider the Euler-Poisson equations in the Gorjatchev-Tchaplygin case [1, 2].

$$\begin{aligned}
\dot m_1 &= 3m_2 \,.\, m_3\\
\dot m_2 &= -3m_1 \,.\, m_3 - 2\gamma_2\\
\dot m_3 &= 2\gamma_2 \qquad\qquad (1)\\
\dot\gamma_1 &= 4m_3 \,.\, \gamma_2 - m_2 \,.\, \gamma_3\\
\dot\gamma_2 &= m_1 \,.\, \gamma_3 - 4m_3\gamma_1\\
\dot\gamma_3 &= m_2 \,.\, \gamma_1 - m_1 \,.\, \gamma_2
\end{aligned}$$

Further, the above system will be regarded as a system of complex differential equations. It possesses three integrals of motion

$$\begin{aligned}
H_1 &= (m_1^2+m_2^2)/4+m_3^2-\gamma_1\\
H_2 &= m_1 \,.\, \gamma_1+m_2 \,.\, \gamma_2+m_3 \,.\, \gamma_3\\
H_3 &= \gamma_1^2+\gamma_2^2+\gamma_3^2
\end{aligned}$$

and on the hypersurface $\{H_2=0\}$ there is a fourth integral of motion

$$H_4=m_3 \,.\, (m_1^2+m_2^2)+2m_1 \,.\, \gamma_3.$$

The complex affine variety A where

$$\mathrm{A}=\{H_1=c_1,\ H_2=0,\ H_3=1,\ H_4=c_4\}$$

is called an integral manifold (enventually singular) of the system (1). Let $\Gamma: y^2=\Phi(x)$ be a hyperelliptic curve of genus two where $\Phi(x)=x^2-(x^3-C_1 . x-C_4/4)^2$ $\mu:\Gamma\to \mathrm{Jac}(\Gamma)$ be the Abel map [6], ∞^+ and ∞^- the two 'infinite, points on Γ, $\Theta_{\infty+}=$ $=\mu(\Gamma)+\mu(\infty^+)$ and $\Theta_{\infty-}=\mu(\infty^-)$. Denote by Δ the discriminant of $\Phi(x)$. The following theorem is our main result.

Theorem 1. If $\Delta\neq 0$ then **A** is a connected smooth complex manifold which is a double cover of the manifold $\mathrm{Jac}(\Gamma)\setminus\{\Theta_{\infty+}\cup\Theta_{\infty-}\}$. The projection map $\pi:\mathbf{A}\to\mathrm{Jac}(\Gamma)$ is given explicitely as follows

$$(m_1, m_2, m_3, \gamma_1, \gamma_2, \gamma_3)\xrightarrow{\pi}[f_0, f_1, \ldots, f_8]\in \mathbf{CP}^8$$

where

$$f_0=1$$

$$f_1=\gamma_2+i\gamma_1$$

$$f_2=(m_2+im_1)\,.\,\gamma_3$$

$$f_3=f_2\,.\,m_3-2f_1\,.\,(m_3^2-\gamma_1)+i\gamma_3^2$$

$$f_4=f_1\,.\,\gamma_3^2$$

$$f_5=f_1\,.\,m_3$$

$$f_6=f_1\,.\,(m_3^2-\gamma_1)$$

$$f_7=f_2(m_3^2-\gamma_1)+f_3\,.\,m_3$$

$$f_8=2f_1\,.\,(2c_1\,.\,f_1+f_3)-f_2^2$$

Moreover, the flows on **A** (run with complex time) induce via the projection map global flows on $\mathrm{Jac}(\Gamma)$ which are straight-line motions.

Further we give a sketch of the proof of Theorem 1. According to [7], the functions $f_0, f_1, \ldots, f_8$ may be considered as meromorphic on $\mathrm{Jac}(\Gamma)$. We prove that they form a basis of $\mathscr{L}(3\Theta_{\infty+})$ (for definition see [6] and hence, they provide an embedding of $\mathrm{Jac}(\Gamma)$ into $\mathbf{CP}^8$. To prove that $f_0, f_1, \ldots, f_8$ form a basis of $\mathscr{L}(3\Theta_{\infty+})$, we study the asymptotic expansions of the generic solutions of (1). A procedure, similar to the one used by Adler a. van Moerbeke [4], leads to the following result

$$m_1=\alpha\,.\,t^{-3/2}+\lambda\,.\,t^{-1/2}+\ldots$$

$$m_2=\varepsilon\alpha t^{-3/2}+\varepsilon\lambda\,.\,t^{-1/2}+\ldots$$

$$m_3=\varepsilon\,.\,t^{-1}/2-\varepsilon\lambda/3\alpha+\ldots$$

(2)

$$\gamma_1=-t^{-2}/4-\lambda^2/9\alpha^2-c_1/3+\ldots$$

$$\gamma_2=-\varepsilon\,.\,t^{-2}/4-\varepsilon\lambda^2/9\alpha^2-\varepsilon\,.\,c_1/3+\ldots$$

$$\gamma_3=\lambda\,.\,\varepsilon\,.\,t^{-1/2}/3\alpha-\lambda^2\,.\,\varepsilon\,.\,t^{1/2}/3\alpha^3+\ldots$$

where $\varepsilon=\pm i$ and the parameters α and λ satisfy the following equality

(3)

$$-216\alpha^6+27\,.\,\varepsilon\,.\,c_4\,.\,\alpha^4+72\,.\,c_1\,.\,\alpha^3\,.\,\lambda+32\alpha\lambda^3-6\lambda^2=0$$

Using the asymptotic expansions (2), one shows that the functions f_i behave at worst like t^{-3} when $\varepsilon=i$ and that they have no poles at $t=0$ when $\varepsilon=-i$. As the solutions of (1) blow up exactly at the points of $\Theta_{\infty+}$ ($\varepsilon=i$ in (2)) and $\Theta_{\infty-}$ ($\varepsilon=-i$ in (2)), we conclude that $f_i\in\mathscr{L}(3\Theta_{\infty+})$. To prove that the functions f_i are linearly independent we compare the coefficients (which are rational functions on the curve (3)) of their Laurent power series. At last we note that $\dim\mathscr{L}(3\,\Theta_{\infty+})=9$ [8].

Now we have two mappings. The first one is given implicitely by Tchaplygin [1] (see also [2] and for explicite formulae [7]). This mapping is two-valued and maps the points of $\mathrm{Jac}(\Gamma)\setminus\{\Theta_{\infty+}\cup\Theta_{\infty-}\}$ onto **A**. The second one is the projection map π. It is easy to see that these two mappings are locally inverse one to the other and hence, **A** is locally biholomorphic to a smooth manifold, i. e. **A** itself is a smooth manifold. Further, we see that $\deg\pi=2$ (π identifies the points $(m_1, m_2, m_3, \gamma_1, \gamma_2, \gamma_3)$ and $(-m_1, -m_2, m_3, \gamma_1, \gamma_2, -\gamma_3)$ and, consequently, **A** is a double cover of $\mathrm{Jac}(\Gamma)\setminus\{\Theta_{\infty+}\cup\Theta_{\infty-}\}$. Of course, the above reasonings are correct only under the assumption that the curve Γ is non-degenerate, i. e. $\Delta\neq 0$.

Remark 1. Our technique is also applicable to the Kowalewski top. In this way one could obtain new proofs of some results of Lesfari [5], Adler & van Moerbeke [6].

Remark 2. A result analogous to Theorem 1 holds for the gyrostat of Sretenskij [8] which is the natural generalization of the Gorjatchev-Tchaplygin top. It is interesting to note that the Kowalewski top may be generalized also. Using the terminology of [8] we suppose that $A=B=2C=2$, $y_0=z_0=\lambda_1=\lambda_2=0$, $x_0 . M.g=1$. In this case the equations describing the motion of the gyrostat possess a fourth first integral

$$H_4=(p^2-q^2+\nu_1)^2+(2pq+\nu_2)^2+4\lambda_3 . p . \nu_3+2\lambda_3 . (p^2+q^2) . (r-\lambda_3).$$

3. On the geometry of the pole divisor. The divisor $D=\Theta_{\infty+}\cup\Theta_{\infty-}$ on which the solution of (1) blow up is called a pole divisor [4,5]. Consider the curve (3). It parametrizes the solutions which run through the pole divisor. As the solutions of (1) are two-valued functions, the parameters α, λ and $-\alpha$, $-\lambda$ determine one point on D. The quotient of (3) by the involution $(\alpha,\lambda)\to(-\alpha, -\lambda)$ is a new curve

$$-216\ x^4+27 . \varepsilon . c_4 . x^3+72 . c_1 . x^2y+32y^3-6y^2=0 \tag{4}$$

which is called also a pole divisor. As the curves $\Theta_{\infty+}$ are isomorphic to Γ, then the curve (4) is isomorphic to Γ. Explicitely we have

Lemma 1. The mapping

$$y=3z . (iw+z^3-c_1 . z-c_4/4)/8$$
$$x=(iw+z^3-c_1 . z-c_4/4)/4\varepsilon$$

is an birational isomorphism between the curve Γ: $w^2=\Phi(z)$ and the curve (4).

The proof is trivial.

Lemma 2. The divisor D on $\mathrm{Jac}(\Gamma)$ consists of two smooth curves ($\Theta_{\infty+}$ and $\Theta_{\infty-}$) both isomorphic to Γ. The only common point of $\Theta_{\infty+}$ and $\Theta_{\infty-}$ is $\mu(\infty^+ + \infty^-)$ which is a point of tangency. The induced flows on $\mathrm{Jac}(\Gamma)$ are tangent to D exactly at $\mu(2\infty^+)$, $\mu(2\infty^-)$ and $\mu(\infty^+ + \infty^-)$ (see Fig.).

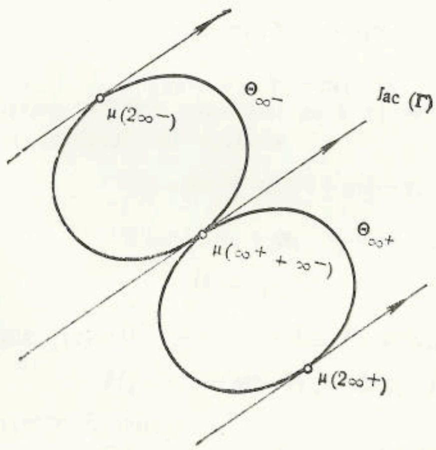

Fig.

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