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The infinitesimal 16th Hilbert problem in dimension zero

Lubomir Gavrilov [a], Hossein Movasati [b,*]

[a] *Université Paul Sabatier, Laboratoire Emile Picard, MIG, CNRS UMR 5580, 31062 Toulouse Cedex 9, France*
[b] *Ochanomizu University, Department of Mathematics, 2-1-1 Otsuka, Bunkyo-ku, Tokyo 112-8610, Japan*

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Abstract

We study the analogue of the infinitesimal 16th Hilbert problem in dimension zero. Lower and upper bounds for the number of the zeros of the corresponding Abelian integrals (which are algebraic functions) are found. We study the relation between the vanishing of an Abelian integral $I(t)$ defined over $\mathbb{Q}$ and its arithmetic properties. Finally, we give necessary and sufficient conditions for an Abelian integral to be identically zero.

Résumé

Nous étudions l'analogue du 16ème problème de Hilbert infinitesimal en dimension zéro. Nous calculons des bornes supérieurs et inférieurs pour le nombre des zéros des intégrales abéliennes (qui sont des fonctions algébriques) associées. Nous étudions les relations entre l'annulation des intégrales abéliennes définies sur $\mathbb{Q}$ et leurs propriétés arithmétiques. Finalement, nous déduisons des conditions suffisantes et nécessaires pour qu'une intégrale abélienne soit identiquement nulle.

Keywords: Abelian integral; Infinitesimal 16th Hilbert problem

* Corresponding author. The author's address after 2.8.2006: Instituto de Matemática Pura e Aplicada, IMPA, Estrada Dona Castorina, 110, 22460-320, Rio de Janeiro, RJ, Brazil.

E-mail address: movasati@cc.ocha.ac.jp (H. Movasati).

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1. Introduction

Let $f : X \to Y$ be a morphism of complex algebraic varieties which defines a locally trivial topological fibration. Let $\gamma(a) \in H_q(f^{-1}(a), \mathbb{Z})$ be a continuous family of q-cycles and ω be a regular q-form on X which is closed on each fiber $f^{-1}(a)$ (the latter is always true if $q = \dim f^{-1}(a)$). By Abelian integral (depending on a parameter) we mean a complex multivalued function of the form

$$I : Y \to \mathbb{C} : a \to I(a) = \int\limits_{\gamma(a)} \omega.$$

Through the paper we shall also suppose that the varieties X, Y, the morphism f and the q-form ω are defined over a subfield $\mathsf{k} \subset \mathbb{C}$. In the case

$$\mathsf{k} = \mathbb{R}, \quad Y = \mathbb{C} \setminus S, \quad X = f^{-1}(Y) \subset \mathbb{C}^2, \quad \omega = P\,dx + Q\,dy, \quad f, P, Q \in \mathbb{R}[x, y]$$

where S is the finite set of atypical values of f, the zeros of $I(a)$ on a suitable open real interval are closely related to the limit cycles of the perturbed foliation on the real plane $\mathbb{R}^2$ defined by

$$df + \epsilon\omega = 0, \quad \epsilon \sim 0.$$

Recall that the second part of the 16th Hilbert problem asks to determine the maximal number and positions of the limit cycles of a polynomial plane vector field (or foliation) of a given degree. The infinitesimal 16th Hilbert problem asks then to *find the exact upper bound $Z(m, n)$ for the number of the zeros of $I(a)$ on an open interval, where* $\deg f \leqslant m$, $\deg P, Q \leqslant n$ [15]. It is only known that $Z(m, n) < \infty$ [16,24] and $Z(3, 2) = 2$ [9].

More generally, let X and Y be Zariski open subsets in $\mathbb{C}^{q+1}$ and $\mathbb{C}$ respectively, f a polynomial and ω a polynomial q-form in $\mathbb{C}^{q+1}$, all these objects being defined over a subfield $\mathsf{k} \subset \mathbb{C}$. *What is the exact upper bound $Z(m, n, \mathsf{k}, q)$ for the number of the zeros $a \in \mathsf{k} \cap \mathcal{D}$ of the Abelian integral I?* Here $\mathcal{D}$ is any simply connected domain in Y.

The present paper addresses the above question in the simplest case $q = 0$. The Abelian integral I is then an algebraic function over $\mathsf{k}[a]$ and every algebraic function over $\mathsf{k}[a]$ is an Abelian integral defined over k. We prove in Theorem 1 that

$$n - 1 - \left[\frac{n}{m}\right] \leqslant Z(m, n, \mathsf{k}, 0) \leqslant \frac{(m-1)(n-1)}{2}. \tag{1}$$

The lower bound in this inequality is given by the dimension of the vector space of Abelian integrals

$$V_n = \left\{ \int\limits_{\gamma(a)} \omega,\ \deg \omega \leqslant n \right\}$$

where f is a fixed general polynomial of degree m, while the upper bound is a reformulation of Bezout's theorem. When $m = 3$ we get $Z(m, m-1, \mathsf{k}, 0) = 1$. We give some evidence in Proposition 6 that, in the case $\mathsf{k} = \mathbb{R}$, $m = 4$, $n = 3$, the upper bound of (1) is strictly bigger than $Z(4, 3, \mathbb{R}, 0)$. This proposition also suggests that

$$\lim_{d \to \infty} \frac{Z(d, d-1, \mathbb{R}, 0)}{d} = 1$$

or, in other words, the space of Abelian integrals V_d is Chebishev, possibly with some accuracy. Recall that V_n is said to be Chebishev with accuracy c if every $I \in V_n$ has at most $\dim V_n - 1 + c$

zeros in the domain $\mathcal{D}$. The Chebishev property with some accuracy (if satisfied) would mean that the infinitesimal 16th Hilbert problem in dimension zero is a problem of real algebraic geometry (as opposed to Bezout's theorem which is a result of complex algebraic geometry).[1]

To the rest of the paper we explore some arithmetic properties of Abelian integrals. When k is a number field and $q = 1$ (so $f^{-1}(a)$ is a smooth curve), the Abelian subvariety theorem applied on the Jacobian variety of $f^{-1}(a)$ (see for instance [4]) gives necessary and sufficient conditions for the vanishing of an Abelian integral $I(a)$ defined over k. We formulate the 0-dimensional analogue of this result (Theorem 2). Its proof uses the relation between the vanishing of an Abelian integral of dimension zero and the Galois group of the splitting field of $f - a$. Finally we make use of the monodromy of f to obtain two additional results. The first one improves the upper bound for the number of the zeros of an Abelian integral with fixed f (Theorem 3). The second one gives necessary and sufficient conditions for an Abelian integral $I(a)$ to be *identically* zero. The analogue of this result for $q = 1$ is not known, although it is essential for computing the so called higher order Poincaré–Pontryagin functions [10,11].

The paper is organized as follows. In Sections 2 and 3 we summarize, for convenience of the reader, the basic properties of zero dimensional Abelian integrals and their algebraic counterpart: the global Brieskorn module. The canonical connection of the (co)homology bundle is explained on a simple example of a polynomial f of degree three. In Section 4 we prove the Bezout type estimate for $Z(m, n, \mathsf{k}, 0)$ and consider the examples $\mathsf{k} = \mathbb{R}$, $m = 3, 4$. The arithmetic aspects of the problem are treated in Section 5, and the monodromy group of f in Section 6.

2. Zero dimensional Abelian integrals

In this section we introduce the necessary notations and prove, for convenience of the reader, some basic facts about the Abelian integrals of dimension zero.

Let $M = \{x_1, x_2, \ldots, x_d\}$ be a discrete topological space and G an additive Abelian group. By abuse of notation we denote by $H_0(M, G)$ the reduced homology group

$$H_0(M, G) = \left\{ \sum_{i=1}^{d} n_i x_i \colon n_i \in G, \sum_{i=1}^{d} n_i = 0 \right\}.$$

It is a free G-module of rank $d - 1$ generated by

$$x_1 - x_d,\ x_2 - x_d,\ \ldots,\ x_{d-1} - x_d$$

and its dual space will be denoted by $H^0(M, G)$. To the polynomial

$$f(x, a) = x^d - a_1 x^{d-1} - \cdots - a_d, \quad a = (a_1, a_2, \ldots, a_d)$$

we associate the surface

$$V = \left\{ (x, a) \in \mathbb{C}^{d+1} \colon f(x, a) = 0 \right\}$$

and the (singular) fibration

$$V \to \mathbb{C}^d : (x, a) \mapsto a \tag{2}$$

with fibers $L_a = \{x \in \mathbb{C} \colon f(x, a) = 0\}$. The polynomial $f(x, a)$ with $a_1 = 0$ is a versal deformation of the singularity $f(x, 0) = x^d$ of type A_{d-1}. We denote by $\Delta(a)$ the discriminant of

[1] When the paper was written, D. Novikov showed us some numerical simulations, showing the fact that the Chebishev property with accuracy 0 does not hold for V_4. However, the complete description of $Z(d, d-1, \mathbb{R}, 0)$ still remains open.

$f(x,a)$ with respect to x. The corresponding discriminant locus $\Sigma = \{a \in \mathbb{C}^d\colon\ \Delta(a) = 0\}$ is the set of parameters a, such that $f(x,a)$ has a multiple root (as a polynomial in x).

The map (2) induces homology and co-homology bundles with base $\mathbb{C}^d \setminus \Sigma$, and fibers $H_0(L_a, \mathbb{Z})$ and $H^0(L_a, \mathbb{C})$. The continuous families of cycles

$$\gamma_{ij}(a) = x_i(a) - x_j(a) \in H_0(L_a, \mathbb{Z})$$

generate a basis of locally constant sections of a unique connection in the homology bundle (the so called Gauss–Manin connection).

Let $\mathsf{k} \subset \mathbb{C}$ be a field. To define the connection algebraically we need the global Brieskorn module (relative co-homology) of f which is defined as

$$H = \frac{\mathsf{k}[x,a]}{f.\mathsf{k}[x,a] + \mathsf{k}[a]}. \tag{3}$$

This is a $\mathsf{k}[a]$-module in an obvious way. The basic properties of such modules in the local multi-dimensional case ($x \in \mathbb{C}^{n+1}, a \in \mathbb{C}$) when $\mathsf{k}[a]$ is replaced by $\mathbb{C}\{a\}$ were studied by Brieskorn [6] and Sebastiani [22]. The first results in the global one-dimensional case ($x \in \mathbb{C}^2$) were proved in [8]. For arbitrary n see [5,7,19,23]. In the zero-dimensional case the main properties of H are rather obvious and are summarized in Propositions 1 and 2 bellow.

Proposition 1. *H is a free $\mathsf{k}[a]$-module of rank $d-1$ generated by $x, x^2, \ldots, x^{d-1}$. More precisely, for every $m \geqslant d$ the following identity holds in H*

$$x^m = \sum_{i=1}^{d-1} p_i(a) x^i$$

where $p_i(a) \in \mathsf{k}[a]$ are suitable weighted homogeneous polynomials of degree $m-i$, and $\mathrm{weight}(a_i) = i$.

Proof. The proof is by induction on m and is left to the reader. □

Let $\gamma(.)$ be a locally constant section of the homology bundle of f. Every $\omega = \omega(x,a) \in \mathsf{k}[x,a]$ defines a global section of the co-homology bundle by the formula

$$I(a) = \int_{\gamma(a)} \omega = \sum_i \omega\big(x_i(a), a\big) \tag{4}$$

where $\gamma(a) = \sum_i n_i x_i(a)$, $\sum_i n_i = 0$.

Definition 1. An Abelian integral of dimension zero over the field k is a function $I(a)$, $a \in \mathbb{C}^d$, of the form (4), where $f, \omega \in \mathsf{k}[x,a]$ and $\gamma(a)$ is a continuous family of cycles.

Remark 1. A (multivalued) function $I(a)$ is an Abelian integral if and only if it is an algebraic function. Indeed, let $x = x(a)$ be the algebraic function defined by $g(x,a) \equiv 0$, $g \in \mathsf{k}[x,a]$. Then it is an Abelian integral $I(a)$ defined by either

$$\omega = x, \quad f = xg(x,a), \quad \gamma(a) = x(a) - 0$$

or

$$\omega = x, \quad f = g(2x,a)g(-2x,a), \quad \gamma(a) = \frac{x(a)}{2} - \left(-\frac{x(a)}{2}\right).$$

In the next two propositions we shall suppose, however, that

$$f(x,a) = x^d - a_1 x^{d-1} - \cdots - a_d, \quad a = (a_1, a_2, \ldots, a_d).$$

Proposition 2. *The polynomial $\omega = \omega(x,a) \in \mathsf{k}[x,a]$ defines the zero section of the canonical co-homology bundle of f, if and only if ω represents the zero equivalence class in the global Brieskorn module* (3).

Proof. Indeed, if $[\omega] = 0$ in H, the claim is obvious. If ω defines the zero section of the co-homology bundle, then

$$\omega\big(x_i(a), a\big) = \omega\big(x_j(a), a\big), \quad \forall a, i, j.$$

According to Proposition 1 we may suppose that

$$\omega(x,a) = \sum_{i=1}^{d-1} p_i(a) x^i + f.p(x,a) + q(a)$$

and a is such that $x_i(a) \neq x_j(a)$ for $i \neq j$. Then it follows that the affine algebraic curves

$$\Gamma_\omega = \left\{(x,y) \in \mathbb{C}^2\colon \frac{\omega(x,a) - \omega(y,a)}{x - y} = 0\right\} \tag{5}$$

and

$$\Gamma_f = \left\{(x,y) \in \mathbb{C}^2\colon \frac{f(x,a) - f(y,a)}{x - y} = 0\right\} \tag{6}$$

have at least $d(d-1)$ distinct intersection points at the points$(x,y) = (x_i, x_j)$. But this contradicts the Bezout's theorem, as the degree of the curves (5) and (6) is (at most) $d-2$ and $d-1$ respectively. It follows that either Γ_f and Γ_ω have a common component, or $p_i(a) = 0$, $\forall a, i$. In the former case the algebraic curve is reducible which is impossible for generic values of a. We obtain finally that $\omega(x,a)$ represents the zero equivalence class in the Brieskorn module H. □

For a given $f \in \mathsf{k}[x,a]$ and a section $\gamma(a) \in H_0(L_a, \mathbb{Z})$, let $\mathcal{A}_f$ be the set of Abelian integrals (4), where a belongs to some simply connected sub-domain of $\mathbb{C}$. Then $\mathcal{A}_f$ is a $\mathsf{k}[a]$-module and moreover

Proposition 3. *$\mathcal{A}_f$ and the Brieskorn module H are isomorphic $\mathsf{k}[a]$-modules.*

Proof. The homomorphism

$$H \to \mathcal{A}_f : \omega \to \int_{\gamma(a)} \omega = \sum_i \omega\big(x_i(a), a\big)$$

is obviously surjective. As the monodromy group of the fibration defined by f is transitive, then $I(a) \equiv 0$ implies that $\int_{\gamma(a)} \omega \equiv 0$ for every section $\gamma(a) \in H_0(f^{-1}(a), \mathbb{Z})$. Proposition 2 implies that $\omega = 0 \in H$. □

Let $\mathsf{k}^q \to \mathsf{k}^d : b \to a$ be a polynomial map, and consider

$$H_b = \frac{\mathsf{k}[x,b]}{g.\mathsf{k}[x,b] + \mathsf{k}[b]} \tag{7}$$

where

$$g = x^d - a_1(b)x^{d-1} - \cdots - a_d(b).$$

As before H_b is a free $\mathsf{k}[b]$ module with generators $x, x^2, \ldots, x^{d-1}$. Of particular interest is the polynomial map

$$\mathsf{k} \to \mathsf{k}^d : t \to \left(a_1^0, a_2^0, \ldots, a_{d-1}^0, t\right).$$

The $\mathsf{k}[t]$ module H_t is then isomorphic to

$$H_t = \frac{\mathsf{k}[x]}{\mathsf{k}[g]}, \quad g = x^d - a_1^0 x^{d-1} - \cdots - a_{d-1}^0 x \tag{8}$$

with multiplication $t \cdot \omega = g(x)\omega(x) \in \mathsf{k}[x]$. The analogues of Propositions 1, 2 hold true for H_t, but not Proposition 3 (see Section 6).

3. The connection of *H*

Let $x(a)$ be a root of the polynomial f, $f(x(a), a) \equiv 0$. Then

$$\frac{\partial x(a)}{\partial a_i} \frac{\partial f(x,a)}{\partial x} \equiv x(a)^i. \tag{9}$$

There exist polynomials $p, q \in \mathbb{Q}[x, a]$ such that

$$p\frac{\partial f}{\partial x} + qf = \Delta(a) \tag{10}$$

and hence in H

$$\Delta(a) = p(x,a)\frac{\partial f}{\partial x}.$$

This combined with (9) suggests to define a connection on H as follows

$$\nabla_{\frac{\partial}{\partial a_i}} : H \to H_\Delta : x^m \mapsto \frac{m x^{m-1+i} p(x)}{\Delta}$$

where H_Δ is the localization of H on $\{\Delta, \Delta^2, \ldots\}$. The operator ∇ satisfies the Leibniz rule and so it is a connection on the module H. It follows from (9), (10) that

$$\frac{\partial}{\partial a_i} \int_{\gamma(a)} \omega = \int_{\gamma(a)} \nabla_{\frac{\partial}{\partial a_i}} \omega \tag{11}$$

where $\gamma(a)$ is a continuous family of cycles. Every element of H defines a section of the cohomology bundle of f. By (11) every continuous family of cycles is a locally constant section of the homology bundle, which means that ∇ coincides with the Gauss–Manin connection described previously

It is well known that ∇ is a flat (integrable) connection. Indeed, a fundamental matrix of solutions for this connection is given by

$$X(a) = \left(\int_{\gamma_j(a)} x^i \right)_{i,j=1,1}^{d-1,d-1}$$

where $\gamma_1(a), \gamma_2(a), \ldots, \gamma_{d-1}(a)$ is a basis of locally constant sections of the homology bundle. The connection form is therefore

$$\sum_{i=1}^{d} A_i(a)\, da_i, \quad \text{where } A_i(a) = \frac{\partial}{\partial a_i} X(a).X^{-1}(a)$$

and the family of (Picard–Fuchs) differential operators

$$\frac{\partial}{\partial a_i} - A_i, \quad i = 1, 2, \ldots, d,$$

commute. We end this section by a simple but significant example. Let

$$f = 4x^3 - g_2 x - g_3$$

with discriminant

$$\Delta(g_2, g_3) = g_2^3 - 27 g_3^2.$$

Suppose further that g_2, g_3 depend on a parameter z. A straightforward and elementary computation implies

Proposition 4. *In the Brieskorn module H the following identity holds*

$$\Delta(z)\nabla_{\frac{\partial}{dz}} \begin{pmatrix} x \\ x^2 \end{pmatrix} = \begin{pmatrix} \frac{\Delta'_z}{6} & -3\delta \\ -\frac{g_2\delta}{2} & \frac{\Delta'_z}{3} \end{pmatrix} \begin{pmatrix} x \\ x^2 \end{pmatrix}$$

where

$$\delta(z) = 3g_3 \frac{dg_2}{dz} - 2g_2 \frac{dg_3}{dz}.$$

If we introduce the Abelian integrals

$$\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} \frac{\int_{\gamma(z)} x}{\Delta^{1/4}} \\ \frac{\int_{\gamma(z)} x^2}{\Delta^{1/4}} \end{pmatrix}$$

then they satisfy the Picard–Fuchs system

$$\Delta(z)\frac{d}{dz} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} -\frac{\Delta'_z}{12} & -3\delta \\ -\frac{g_2\delta}{2} & \frac{\Delta'_z}{12} \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}. \tag{12}$$

It is interesting to compare the above to the Picard–Fuchs system associated to the "stabilization" $y^2 - 4x^3 + g_2(z)x + g_3(z)$ of f. Namely, let

$$\eta_1 = \int_{\gamma(z)} \frac{dx}{y}, \qquad \eta_2 = \int_{\gamma(z)} \frac{x\, dx}{y}$$

be complete elliptic integrals of first and second kind on the elliptic curve with affine equation

$$\Gamma_z = \{(x, y) \in \mathbb{C}^2 \colon\ y^2 = 4x^3 - g_2(z)x - g_3(z)\}$$

where $\gamma(z) \subset \Gamma_z$ is a continuous family of closed loops (representing a locally constant section $z \mapsto H_1(\Gamma_z, \mathbb{Z})$ of the homology bundle). Then η_1, η_2 satisfy the following Picard–Fuchs system (this goes back at least to [12, Griffiths], see [21, Sasai]).

Proposition 5.

$$\Delta(z)\frac{d}{dz}\begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix}=\begin{pmatrix}-\frac{\Delta_z'}{12} & -\frac{3\delta}{2}\\ -\frac{g_2\delta}{8} & \frac{\Delta_z'}{12}\end{pmatrix}\begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix}. \tag{13}$$

Note the striking similarity of these two non-equivalent systems. The algorithms which calculate the Gauss–Manin connection can be implemented in any software for commutative algebra (see [18]). Using them one can obtain equalities like (13) for other families of varieties in arbitrary dimension.

4. The infinitesimal 16th Hilbert problem in dimension zero

It is well known that the number of the limit cycles of the perturbed real foliation

$$df+\varepsilon(P\,dx+Q\,dy)=0,\quad P,Q,R\in\mathbb{R}[x,y],\ \varepsilon\sim 0$$

is closely related to the number of the zeros of the Abelian integral (Poincaré–Pontryagin function)

$$I(t)=\int\limits_{\gamma(t)} P\,dx+Q\,dy$$

where $\gamma(t)\in H_1(f^{-1}(t),\mathbb{Z})$ is a continuous family of cycles. The problem on zeros of such Abelian integrals, in terms of the degrees of f, P, Q, is known as the infinitesimal 16th Hilbert problem: see the recent survey of Ilyashenko [15], as well [2, problem 1978-6]. This problem is still open (except in the case $\deg F\leqslant 3$, see [9,13]). One can further generalize, by taking $f\in\mathbb{R}[x_1,x_2,\dots,x_n]$, ω – a polynomial $n-1$ form, $\gamma(t)\in H_{n-1}(f^{-1}(t),\mathbb{Z})$ a locally constant section of the homology bundle of f, and

$$I(t)=\int\limits_{\gamma(t)}\omega.$$

In the present paper we solve (partially) the infinitesimal 16th Hilbert problem by taking $n=1$. The Abelian integral $I(t)$ is of dimension zero, in the sense explained in the preceding section. To our knowledge, such integrals appeared for a first time, in the context of the 16th Hilbert problem, in the Ilyashenko's pioneering paper [14], see [8].

To formulate the problem, let us denote $f\in\mathsf{k}[x]$ where $\mathsf{k}\subset\mathbb{C}$ is a field, and consider the singular fibration

$$f:\mathbb{C}\to\mathbb{C}: x\mapsto f(x)$$

with fibers $L_t=f^{-1}(t)$. Let $\mathcal{D}\subset\mathbb{C}\setminus\Sigma$ be simply connected set, where Σ is the set of critical values of f. A cycle $\gamma(t)\in H_0(L_t,\mathbb{Z})$ is said to be simple, if $\gamma(t)=x_i(t)-x_j(t)$, where $f(x_i(t))=f(x_j(t))=t$. Let $\gamma(t)$ be a continuous family of simple cycles, m,n two integers and k a field. The infinitesimal 16th Hilbert problem in dimension zero is

Find the exact upper bound $Z(m,n,\mathsf{k},0)$ of the number of the zeros

$$\{t\in\mathsf{k}\cap\mathcal{D}\colon\ I(t)=0,\ \deg f\leqslant m,\ \deg\omega\leqslant n,\ \mathcal{D}\subset\mathbb{C}\setminus\Sigma\}$$

where $\mathcal{D}$ is any simply connected complex domain.

In the present paper we are interested in the cases $\mathsf{k}=\mathbb{Q},\mathbb{R},\mathbb{C}$. When $\mathsf{k}=\mathbb{R}$ the problem has the following geometric interpretation. Consider the real plane algebraic curves

$$\Gamma_f^{\mathbb{R}}=\left\{(x,y)\in\mathbb{R}^2\colon\ \frac{f(x)-f(y)}{x-y}=0\right\},$$
$$\Gamma_\omega^{\mathbb{R}}=\left\{(x,y)\in\mathbb{C}^2\colon\ \frac{\omega(x)-\omega(y)}{x-y}=0\right\}.$$

The Abelian integral

$$\int\limits_{\gamma(t)}\omega=\omega\bigl(x_i(t)\bigr)-\omega\bigl(x_j(t)\bigr)$$

vanishes if and only if $(x_i(t),x_j(t))\in\Gamma_f^{\mathbb{R}}\cap\Gamma_\omega^{\mathbb{R}}$. If we suppose in addition that $\mathcal{D}\subset\mathbb{R}$ is an open interval, then $(x_i(t),x_j(t))\colon t\in\mathcal{D}$ is contained in some connected component of $\Gamma_f^{\mathbb{R}}$ which we denote by $\Gamma_{f,0}^{\mathbb{R}}$.

It is clear that the number of intersection points $\#(\Gamma_{f,0}^{\mathbb{R}},\Gamma_\omega^{\mathbb{R}})$ (counted with multiplicity) between $\Gamma_{f,0}^{\mathbb{R}}$ and the real algebraic curve $\Gamma_\omega^{\mathbb{R}}$ is an upper bound for the corresponding number $Z(m,n,\mathbb{R},0)$. On the other hand $\#(\Gamma_{f,0}^{\mathbb{R}},\Gamma_\omega^{\mathbb{R}})$ can be bounded by the Bezout's theorem. It is not proved, however, that

$$Z(m,n,\mathbb{R},0)=\#\bigl(\Gamma_{f,0}^{\mathbb{R}},\Gamma_\omega^{\mathbb{R}}\bigr)$$

and we discuss this at the end of the section. We have the following

Theorem 1.

$$n-1-\left[\frac{n}{m}\right]\leqslant Z(m,n,\mathbb{C},0)\leqslant\frac{(m-1)(n-1)}{2}.\tag{14}$$

Proof. Let $\gamma(t)=x(t)-y(t)$ be a continuous family of simple cycles. Then $I(t)=0$ for some $t\notin\Sigma$ if and only if $(x(t),y(t))$ is an isolated intersection point of the plane algebraic curves

$$\Gamma_f=\left\{(x,y)\in\mathbb{C}^2\colon\ \frac{f(x)-f(y)}{x-y}=0\right\},$$
$$\Gamma_\omega=\left\{(x,y)\in\mathbb{C}^2\colon\ \frac{\omega(x)-\omega(y)}{x-y}=0\right\}.$$

Indeed, if an intersection point were non isolated, this would mean that the curves Γ_f and Γ_ω have a common connected component and $I(t)\equiv0$. By Bezout's Theorem the number of isolated intersection points of Γ_f and Γ_ω, counted with multiplicity, is bounded by $(\deg f-1)\times(\deg\omega-1)$. Moreover if (x,y) is an isolated intersection point which corresponds to some $t_0\in\mathbb{C}\setminus\Sigma$, $\gamma(t_0)=x-y$, $I(t_0)=0$, then (y,x) is an isolated intersection point too. As $I(t)$ is single-valued in $\mathcal{D}$ then (y,x) does not correspond to any zero of $I(t)$ in $\mathcal{D}$. Thus the number of the zeros of $I(t)$ on $\mathcal{D}$ is bounded by $(\deg f-1)(\deg\omega-1)/2$.

Let

$$V_n=\left\{\int\limits_{\gamma(t)}\omega,\ \deg\omega\leqslant n\right\}$$

be a vector space of Abelian integrals defined in a simply connected domain for some fixed *generic* polynomial f of degree m. We have $\dim V_n - 1 \leqslant Z(m, n, \mathbb{C}, 0)$. On the other hand, if f is a generic polynomial, then the orbit of $\gamma(t)$ under the action of the monodromy group of the polynomial spans $H_1(f^{-1}(t), \mathbb{Z})$ (see Section 6). Therefore $I \in V_n$ is identically zero if and only if ω represents the zero co-homology class in $H^1(f^{-1}(t), \mathbb{C})$ which is equivalent (by Proposition 2) to $\omega \in \mathbb{C}[f]$. This shows that the vector space V_n is isomorphic to

$$\{\omega \in \mathbb{C}[x]\colon \deg \omega \leqslant n\} / \{p(f(x))\colon\ p \in \mathbb{C}[x]\colon \deg p(f(x)) \leqslant n\}.$$

The basis of this space is

$$\{x^i f^j\colon i + jm \leqslant n\}$$

and hence $\dim V_n = n - [\frac{n}{m}]$. The theorem is proved. □

The bound in the above theorem is probably far from the exact one. If one wants to count zeros in a not simply connected domain $\mathcal{D}$ then the bound is exact for the case $\deg(\omega) = d - 1$. For instance take $f = \prod_{i=1}^{d} (x - i)$ and $\omega = \prod_{i=1}^{d-1} (x - i)$. Then $\int_{\gamma_{ij}} \omega = 0$ for all $\gamma_{ij} = i - j \in H_0(\{f = 0\}, \mathbb{Z}), i, j = 1, 2, \ldots, d - 1$. In Section 6 we will give another approach using the monodromy representation of f.

Example 1. Let $\deg f = 3$, and V be the k-vector space of Abelian integrals generated by

$$\int\limits_{\gamma(t)} x, \qquad \int\limits_{\gamma(t)} x^2.$$

By Theorem 1 each $I \in V$ has at most one simple zero in $\mathcal{D}$. As $\dim V = 2$ this means that the bound cannot be improved, that is to say V is a Chebishev space in $\mathcal{D}$.

Example 2. Let $\deg f = 4$, and V be the k-vector space of Abelian integrals generated by

$$\int\limits_{\gamma(t)} x, \qquad \int\limits_{\gamma(t)} x^2, \qquad \int\limits_{\gamma(t)} x^3.$$

By Theorem 1 each $I \in V$ has at most three zeros in $\mathcal{D}$. As $\dim V = 3$ this does not imply that V is Chebishev.

Example 3. Consider the particular case $f = x^4 - x^2$ and take $\mathsf{k} = \mathbb{R}$. The set of critical values is $\Sigma = \{0, -1/4\}$ and let $\gamma(t)$ be a continuous family of simple cycles where $t \in (-1/4, 0)$.

Proposition 6. *The Vector space V_n of Abelian integrals $I(t) = \int_{\gamma(t)} \omega$, $\deg \omega \leqslant n$ is Chebishev. In other words, each $I \in V_n$ can have at most* $\dim V_n - 1$ *zeros in* $(-1/4, 0)$.

Proof. We shall give two distinct proofs. The first one applies only for the family of cycle vanishing at $t = 0$, but has the advantage to hold in a complex domain.

1) Suppose that $\gamma(t) = x_1(t) - x_2(t)$ is a continuous family of simply cycles vanishing as t tends to zero and defined for $t \in \mathbb{C} \setminus (-\infty, -1/4]$. Each integral $I \in V_n$ admits analytic continuation in $\mathbb{C} \setminus (-\infty, -1/4]$. Following Petrov [20], we shall count the zeros of $I(t)$ in $\mathbb{C} \setminus (-\infty, -1/4]$ by making use of the argument principle. Consider the function

$$F(t) = \frac{\int_{\gamma(t)} \omega}{\int_{\gamma(t)} x}$$

which admits analytic continuation in $\mathbb{C}\setminus(-\infty,-1/4]$. Let D be the domain obtained from $\mathbb{C}\setminus(-\infty,-1/4]$ by removing a "small" disc $\{z\in\mathbb{C}\colon |z|\leqslant r\}$ and a "big" disc $\{z\in\mathbb{C}\colon |z|\geqslant R\}$. We compute the increase of the argument of F along the boundary of D traversed in a positive direction. Along the boundary of the small disc the increase of the argument is close to zero or negative (provided that r is sufficiently small). Along the boundary of the big disc the increase of the argument is close to $(n-1)\pi/2$ or less than $(n-1)\pi/2$. Finally, along the coupure $(-\infty,-1/4)$ we compute the imaginary part of $F(t)$. Let $F^{\pm}(t),\gamma^{\pm}(t)$ be the determinations of $F(t),\gamma(t)$ when approaching $t\in(-\infty,-1/4)$ with $\operatorname{Im} t>0$ ($\operatorname{Im} t<0$). We have

$$\operatorname{Im} F(t)=\big(F^+(t)-F^-(t)\big)/2\sqrt{-1},\qquad \gamma^+(t)-\gamma^-(t)=\delta(t),$$

where $\delta(t)=x_3(t)-x_1(t)-(x_4(t)-x_2(t))$ and $x_3(t),x_4(t)$ are roots of $f(x)-t$ which tend to $x_1(t)$ and $x_2(t)$ respectively, as t tends to $-1/4$ along a path contained in $\mathbb{C}\setminus(-\infty,-1/4]$. We obtain

$$2\sqrt{-1}\operatorname{Im} F(t)=\beta\det\begin{pmatrix}\int_{\gamma(t)}\omega & \int_{\gamma(t)}x\\ \int_{\delta(t)}\omega & \int_{\delta(t)}x\end{pmatrix}\Big/\Big|\int\limits_{\gamma(t)}x\Big|^2.$$

Denote by $W=W_{\gamma,\delta}(\omega,x)$ the determinant in the numerator above.

It is easily seen that W^2 is univalued and hence rational in t. Moreover it has no poles, vanishes at $t=0,-1/4$ and as t tends to infinity it grows no faster than $t^{(n+1)/2}$. Therefore W^2 is a polynomial of degree at most $[(n+1)/2]$ which vanishes at 0 and $-1/4$, and hence the imaginary part of F along $(-\infty,-1/4)$ has at most $[[(n+1)/2]/2-1]$ zeros. Summing up the above information we conclude that the increase of the argument of $F(t)$ along the boundary of D is close to $([(n+1)/2]-1)2\pi$ or less. Therefore F and hence the Abelian integral I has at most $[(n+1)/2]-1$ zero in D (and hence in $(-1/4,0)$). It is seen from this proof that the dimension of V_n should be at least $[(n+1)/2]$. Indeed

$$\int\limits_{\gamma(t)}x^{2k}\equiv 0,\quad \forall k$$

and

$$\int\limits_{\gamma(t)}x,\quad \int\limits_{\gamma(t)}x^3,\quad \ldots$$

form a basis of V_n (this follows from Proposition 3), which shows that V_n is Chebishev.

2) Suppose now that $\gamma(t)=x_1(t)-x_3(t)$ is a cycle vanishing as t tends to $-1/4$, and defined on the interval $(-1/4,0)$. The dimension of V_n equals to n and the preceding method does not work. The curve Γ_f is, however, reducible

$$\Gamma_f=\big\{(x,y)\in\mathbb{C}^2\colon\ (x+y)\big(x^2+y^2-1\big)=0\big\}$$

and the family $\gamma(t)$, $t\in(-1/4,0)$ corresponds to a piece of the oval $x^2+y^2-1=0$. This oval intersects Γ_ω in at most $2(n-1)$ points (by Bezout's theorem). The points (x,y) and (y,x) correspond to $\gamma(t)$ and $-\gamma(t)$ respectively. This shows that each integral $I\in V_n$ can have at most $n-1$ zeros in $(-1/4,0)$. The proposition is proved. $\square$

It seems to be difficult to adapt some of the above methods to the case of a general polynomial f of degree four.

5. Arithmetic zero dimensional Abelian integrals

In this section k is an arbitrary field of characteristic zero and we work with polynomials in $\mathsf{k}[x]$. The reader may follow this section for $\mathsf{k} = \mathbb{Q}$. The main result of this section is Theorem 2 which will be used in Section 6 for the functional field $\mathsf{k} = \mathbb{C}(t)$.

For polynomials $f, \omega \in \mathsf{k}[x]$ we define the discriminant of f

$$\Delta_f := \prod_{1 \leqslant i, j \leqslant d} (x_i - x_j) \in \mathsf{k}$$

and the following polynomial

$$\omega * f(x) := \big(x - \omega(x_1)\big)\big(x - \omega(x_2)\big) \cdots \big(x - \omega(x_d)\big) \in \mathsf{k}[x] \tag{15}$$

where $f(x) = (x - x_1)(x - x_2) \cdots (x - x_d)$. Note that $(\omega * f) \circ \omega(x_i) = 0$, $i = 1, 2, \ldots, d$ and the multiplicity of $(\omega * f) \circ \omega$ at x_i is at least the multiplicity of f at x_i. Therefore

$$f \mid (\omega * f) \circ \omega. \tag{16}$$

For $\omega, \omega_1, \omega_2, f, f_1, f_2 \in \mathsf{k}[x]$ we have the following trivial identities:

$$\omega_1 * (\omega_2 * f) = (\omega_1 \circ \omega_2) * f, \ \ \omega * (f_1 \cdot f_2) = (\omega * f_1) \cdot (\omega * f_2). \tag{17}$$

Proposition 7. *For an irreducible $f \in \mathsf{k}[x]$ and arbitrary $\omega \in \mathsf{k}[x]$, we have $\omega * f = g^k$ for some $k \in \mathbb{N}$ and irreducible polynomial $g \in \mathsf{k}[x]$. Moreover, if for some simple cycle $\gamma \in H_0(\{f = 0\}, \mathbb{Z})$ we have $\int_\gamma \omega = 0$ then $k \geqslant 2$.*

Proof. Let

$$f = (x - x_1)(x - x_2) \cdots (x - x_d), \quad d := \deg(f), \quad I := \{x_1, x_2, \ldots, x_d\}.$$

We define the equivalence relation $\sim$ on I:

$$x_i \sim x_j \Leftrightarrow \omega(x_i) = \omega(x_j).$$

Let G_f be the Galois group of the splitting field of f. For $\sigma \in G_f$ we have

$$x_i \sim x_j \Rightarrow \sigma(x_i) \sim \sigma(x_j). \tag{18}$$

Since f is irreducible over k, the action of G_f on I is transitive (see for instance [17] Proposition 4.4). This and (18) imply that G_f acts on $I/\sim$ and each equivalence class of $I/\sim$ has the same number of elements as others. Let $I/\sim = \{v_1, v_2, \ldots, v_e\}$, $e \mid d$ and $c_i := \omega(v_i)$. Define

$$g(x) := (x - c_1)(x - c_2) \cdots (x - c_e).$$

We have

$$g^k = f * \omega \in \mathsf{k}[x]$$

where $k = \frac{n}{e}$. By calculating the coefficients of g in terms of the coefficients of the right hand side of the above equality, one can see easily that $g \in \mathsf{k}[x]$. Since G_f acts transitively on the roots of g, we conclude that g is irreducible over k. $\square$

Let $f, g, \omega \in \mathsf{k}[x]$ such that

$$f \mid g \circ \omega. \tag{19}$$

We have the morphism

$$\{f=0\} \stackrel{\alpha_\omega}{\to} \{g=0\}, \quad \alpha_\omega(x)=\omega(x)$$

defined over k. Let $\gamma \in H_0(\{f=0\},\mathbb{Z})$ such that $(\alpha_\omega)_*(\gamma)=0$, where $(\alpha_\omega)_*$ is the induced map in homology. For instance, if $\deg(g)<\deg(f)$ then because of (19), there exist two zeros x_1, x_2 of f such that $\int_\gamma \omega = \omega(x_1)-\omega(x_2)=0$ and so the topological cycle $\gamma := x_1 - x_2$ has the desired property. Note that and the 0-form ω on $\{f=0\}$ is the pull-back of the 0-form x by α_ω. The following theorem discusses the inverse of the above situation:

Theorem 2. *Let $f, \omega \in \mathsf{k}[x]$ be such that such that*

$$\int_\gamma \omega = 0 \tag{20}$$

for some simple cycle $\gamma \in H_0(\{f=0\},\mathbb{Z})$. Then there exists a polynomial $g \in \mathsf{k}[x]$ such that

1. $\deg(g) < \deg(f)$;
2. *the degree of each irreducible components of g divides the degree of an irreducible component of f;*
3. *$f \mid g\circ\omega$, the morphism $\alpha_\omega : \{f=0\} \to \{g=0\}$ defined over k is surjective and $(\alpha_\omega)_*(\gamma)=0$.*

Proof. Let $f = f_1^{\alpha_1} f_2^{\alpha_2}\cdots f_r^{\alpha_r}$ (resp. $\omega * f = g_1^{\beta_1} g_2^{\beta_2}\cdots g_s^{\beta_s}$) be the decomposition of $f \in \mathsf{k}[x]$ (resp. $\omega * f$) into irreducible components. By Proposition 7 and the second equality in (17), we have $s \leqslant r$ and we can assume that $\omega * f_i = g_i^{k_i}$ for $i=1,2,\ldots,s$ and some $k_i \in \mathbb{N}$. The polynomial $g = g_1^{\alpha_1} g_2^{\alpha_2}\cdots g_s^{\alpha_s}$ is the desired one. Except the first item and $(\alpha_\omega)_*(\gamma)=0$, all other parts of the theorem are satisfied by definition.

Let $\gamma = x_1 - x_2$. We consider two cases: First let us assume that x_1 and x_2 are two distinct roots of an irreducible component of f, say f_1. By Proposition 7 we have $\omega * f_1 = g_1^{k_1}$, $k_1>1$ and so $\deg(g)<\deg(f)$. Now assume that x_1 is a zero of f_1 and x_2 is a zero of f_2. Let $\omega * f_1 = g_1^{k_1}$, $\omega * f_2 = g_2^{k_2}$, $k_1, k_2 \in \mathbb{N}$. The number $\omega(x_1)=\omega(x_2)$ is a root of both g_i, $i=1,2$, and G_f acts transitively on the roots of both g_i, $i=1,2$. This implies that $g_1=g_2$ and so $\deg(g)<\deg(f)$. $\square$

Let m be a prime number and $n<m$. Theorem 2 with $\mathsf{k}=\mathbb{Q}$ implies that for an irreducible polynomial $f \in \mathbb{Q}[x]$ of degree m the integral $\int_\gamma \omega$, $\omega \in \mathbb{Q}[x]\backslash\mathbb{Q}$, $\deg(\omega)=n$ never vanishes. Therefore, the number $Z(m,n,\mathbb{Q},0)$ cannot be reached by irreducible polynomials.

Remark 2. Let k be a subfield of $\mathbb{C}$ and $f, \omega \in \mathsf{k}[x]$. Any $\sigma \in \mathrm{Gal}(\bar{\mathsf{k}}/\mathsf{k})$ induces a map

$$\sigma : H_0(L_t,\mathbb{Z}) \to H_0(L_{\sigma(t)},\mathbb{Z})$$

in a canonical way and so if $\int_{\gamma(t)}\omega = 0$ then $\int_{\sigma(\gamma(t))}\omega=0$. This means that $\mathrm{Gal}(\bar{\mathsf{k}}/\mathsf{k})$ acts on the set $\{t \in \mathbb{C} \mid \exists\gamma \in H_0(L_t,\mathbb{Z}) \text{ s.t. } \int_\gamma \omega = 0\}$.

6. Monodromy group

Let k be a subfield of $\mathbb{C}$, $f \in \mathsf{k}[x]$ and C be the set of its critical values. We fix a regular value b of f. The group $\pi_1(\mathbb{C}\backslash C, b)$ acts on $L_b := \{x_1, x_2, \dots, x_d\}$ from left. We define the monodromy group

$$G := \pi_1(\mathbb{C}\backslash C, b)/\{g \in \pi_1(\mathbb{C}\backslash C, b) \mid g(x) = x,\ \forall x \in L_b\} \subset S_d$$

where S_d is the permutation group in d elements $x_1, x_2, \dots, x_d$. Since the two variable polynomial $f(x) - t$ is irreducible, the action of G on L_b is also irreducible. However, the action of G on simple cycles $\mathsf{S} \subset H_0(L_b, \mathbb{Z})$ may not be irreducible. For instance for $f = x^d,\ b = 1$ the group G is generated by the shifting map $1 \mapsto \zeta_d \mapsto \cdots \mapsto \zeta_d^{d-1} \mapsto 1$, where $\zeta_d = e^{\frac{2\pi i}{2}}$.

Let

$$\mathsf{S} = \mathsf{S}_1 \cup \mathsf{S}_2 \cup \cdots \cup \mathsf{S}_m$$

be the partition of S obtained by the action of G, i.e. the partition obtained by the equivalence relation $\gamma_1 \sim \gamma_2$ if $\gamma_1 = g\gamma_2$ for some $g \in G$. For $\omega \in \mathsf{k}[x]$ with $\deg(\omega) = n$ the functions

$$R_{\omega,i}(t) := \prod_{\gamma \in \mathsf{S}_i/\pm 1} \frac{\int_\gamma \omega}{\int_\gamma x}, \quad \Delta_i(t) := \prod_{\gamma \in \mathsf{S}_i} \int_\gamma x, \quad i = 1, 2, \dots, m,$$

are well-defined in a neighborhood of b. They extend to one valued functions in $\mathbb{C}\backslash C$ and by growth conditions at infinity and critical values of f, we conclude that they are polynomials in t with coefficients in the algebraic closure $\bar{\mathsf{k}}$ of k in $\mathbb{C}$. Without lose of generality we assume that for $1 \leqslant m' \leqslant m$ we have $\mathsf{S}_i \neq -\mathsf{S}_j$ for all $1 \leqslant i, j \leqslant m'$ and for $m \geqslant i > m'$, $\mathsf{S}_i = -\mathsf{S}_j$ for some $1 \leqslant j \leqslant m'$. Let

$$R_\omega := \prod_{i=1}^{m'} R_{\omega,i} = \prod_{\gamma \in \mathsf{S}/\pm 1} \frac{\int_\gamma \omega}{\int_\gamma x}, \quad \Delta := \prod_{i=1}^{m} \Delta_i = \prod_{\gamma \in \mathsf{S}_i} \int_\gamma x.$$

We define $\tilde{f} := f - t$ and consider it as a polynomial in $\mathsf{k}(t)[x]$. In this way the polynomial $\Delta(t)$ is equal to $\Delta_{\tilde{f}}$, the discriminant of $\tilde{f}$, and

$$R_\omega^2 = \frac{\Delta_{\omega * \tilde{f}}}{\Delta_{\tilde{f}}}. \tag{21}$$

Considering $f = x^d - a_1 x^{d-1} - \cdots - a_d$ in parameters a_i with $\text{weight}(a_i) = i$, we know that $\Delta_{\omega * f}$ (resp. Δ_f) is a polynomial (resp. homogeneous polynomial) of degree $(n-1)d(d-1)$ (resp. $d(d-1)$) in parameters a_i and with coefficients in k (resp. in $\mathbb{Q}$). This implies that we have $R_\omega, \Delta \in \mathsf{k}[t]$ and by (21)

$$\sum_{i=1}^{m'} \deg(R_{\omega,i}) = \deg(R_\omega) \leqslant \frac{(n-1)(d-1)}{2}. \tag{22}$$

We summarize the above discussion in the following theorem:

Theorem 3. *Let $\mathsf{k} = \mathbb{C}$ and $\gamma(t)$ be a continuous family of simple cycles, $\gamma(b) \in \mathsf{S}_i$ and $\omega \in \mathbb{C}[x]$. If the Abelian integral $I(t) = \int_{\gamma(t)} \omega$ does not vanish identically, then the number of its complex zeros in any simply connected set $\mathcal{D} \subset \mathbb{C} \setminus \Sigma$ with $b \in \mathcal{D}$ is bounded by* $\deg(R_{\omega,i}) \leqslant \frac{(n-1)(d-1)}{2}$.

The proof follows from the definition of $R_{\omega,i}$ and (22). This theorem generalize Theorem 1 and, roughly speaking, it says that as much as the action of the monodromy group on $\gamma(b)$ produces less cycles, so far we expect less zeros for $I(t)$.

In Theorems 1 and 3 we have assumed that $I(t)$ is not identically zero. Now the natural question is that if $I(t) \equiv 0$ then what one can say about f and ω. For instance, if there is a polynomial $g(x,t) \in \mathbb{C}[t][x]$ such that $\deg_x(g) < \deg(f)$ and $g(\omega(x), f(x)) \equiv 0$ then there is a continuous family of simple cycles $\gamma(t)$ such that $\int_{\gamma(t)} \omega \equiv 0$.

Theorem 4. *Let $\gamma(t)$ be a continuous family of simple cycles in the fibers of $f \in \mathbb{C}[x]$ and $\omega \in \mathbb{C}[x]$. If the Abelian integral $I(t) = \int_{\gamma(t)} \omega$ vanishes identically, then there is a polynomial $g(x,t) \in \mathbb{C}[t][x]$ such that*

1. $\deg_x(g) \mid \deg(f)$ *and* $\deg_x(g) < \deg(f)$;
2. $g(\omega(x), f(x)) \equiv 0$;
3. *If the action of the monodromy group on a regular fiber of f is irreducible then $\omega = p(f)$ for some $p \in \mathbb{C}[x]$.*

Proof. We consider $\tilde{f} = f - t$ as a polynomial in $\mathsf{k}[x]$ with $\mathsf{k} = \mathbb{C}(t)$. The assumption of the theorem is translated into $\int_\gamma \omega = 0$ for some simple cycle $\gamma \in H_0(\{\tilde{f} = 0\}, \mathbb{Z})$. We apply Theorem 2 and we conclude that there are polynomials $g, s \in \mathsf{k}[x]$, $\deg(g) < \deg(f)$, $\deg(g) \mid \deg(f)$ such that $s \cdot (f - t) = g(\omega(x), t)$. Note that $\tilde{f}$ is irreducible over k. After multiplication with a certain element in $\mathbb{C}[t]$, we can assume that $s, g \in \mathbb{C}[x,t]$. We replace t with $f(x)$ and in this way the item 1 and 2 are proved.

The third part of the theorem follows from Proposition 2. We give an alternative proof as follows: We identify the elements of the splitting field of $\tilde{f}$ with holomorphic functions in a neighborhood of b in $\mathbb{C}$. In this way we can identify the monodromy group G with a subset of the Galois group $G_{\tilde{f}}$ of the splitting field of $\tilde{f}$ over k. If the action of G on simple cycles is irreducible then by Theorem 2 (in fact its proof) we have $g = x - p(t)$ for some $p(t) \in \mathbb{C}[t]$. $\square$

Remark 3. The classification of all polynomials $f \in \mathbb{C}[x]$ such that the two variable polynomial $F_f := \frac{f(x)-f(y)}{(x-y)}$ has an irreducible factor of degree $n = 1, 2$ has been done recently in [3]. This gives us a complete classification of polynomials f for which $\int_{\gamma_t} \omega \equiv 0$ for some continuous family of cycles γ_t and polynomial ω with $\deg(\omega) = n + 1$ (F_ω identically vanishes on some irreducible component of $F_f = 0$). Note that in $n = 1$ the mentioned classifications are equivalent but in the case $n \geqslant 2$ they are not equivalent (in Example 3 there is no polynomial $\omega(x)$ with $\deg(\omega) = 3$ such that $x^2 + y^2 - 1$ divides F_ω).

The space of polynomials with $d - 1$ distinct critical values can be identified with a quasi-affine subset T of $\mathbb{C}^d$. We claim that for $f \in T$, the action of the monodromy group G on S is irreducible. Since our assertion is topological and T is connected, it is enough to prove our assertion for an example of $f \in T$; for instance take an small perturbation $\tilde{f}$ of $f = (x-1)(x-2)\cdots(x-d)$ which has $d-1$ non-zero distinct critical values $\tilde{c}_1, \tilde{c}_2, \ldots, \tilde{c}_{d-1}$. Let $b = 0$. We take a system of distinguished paths s_i, $i = 1, 2, \ldots, d-1$, in $\mathbb{C}$ (see [1]) such that $\gamma_i = \tilde{i} - (\widetilde{i+1})$, $1 \leqslant i \leqslant d-1$, vanishes along s_i in the critical point associated to $\tilde{c}_i$, where $\tilde{i} \in \tilde{f}^{-1}(0)$ is near $i \in f^{-1}(0)$. Now, the intersection graph of γ_i's (known as Dynkin diagram of f) is a line graph, and so it is connected. By Picard–Lefschetz formula in dimension zero we conclude that the action of the monodromy group on simple cycles of $H_0(\tilde{f}^{-1}(b), \mathbb{Z})$ is irreducible.

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