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Families of Painlevé VI Equations Having a Common Solution

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1 Introduction

Consider the Painlevé VI (PVI_α) equation

$$\begin{aligned}\frac{d^2\lambda}{dt^2} &= \frac{1}{2}\left(\frac{1}{\lambda}+\frac{1}{\lambda-1}+\frac{1}{\lambda-t}\right)\left(\frac{d\lambda}{dt}\right)^2-\left(\frac{1}{t}+\frac{1}{t-1}+\frac{1}{\lambda-t}\right)\frac{d\lambda}{dt}\\ &\quad+\frac{\lambda(\lambda-1)(\lambda-t)}{t^2(t-1)^2}\left[\alpha_0-\alpha_1\frac{t}{\lambda^2}+\alpha_2\frac{t-1}{(\lambda-1)^2}+\left(\frac{1}{2}-\alpha_3\right)\frac{t(t-1)}{(\lambda-t)^2}\right],\end{aligned} \tag{1.1}$$

parameterized by $\alpha = (\alpha_0, \alpha_1, \alpha_2, \alpha_3) \in \mathbb{C}^4$. Although any solution of PVI_α, for generic α_i, is transcendental (and even a "new transcendental function"), there is a large amount of solutions which are algebraic in t. Their general classification is still an open problem (e.g., see Manin [15]), except in the particular case $\alpha_1 = \alpha_2 = \alpha_3 = 0$ (see Dubrovin, Mazzocco [4], Mazzocco [16]). The present paper addresses the question of classifying *families* of algebraic solutions. The simplest case occurs when a given algebraic solution satisfies each member of a nontrivial family of PVI_α equations. By a nontrivial family of PVI_α equations we mean a set $\{PVI_\alpha\}_\alpha$ containing at least two distinct elements corresponding to, say, α' and α''. Then this solution satisfies the PVI_α equations corresponding to the affine line containing α' and α''. It follows that each nontrivial family as above corresponds to an affine plane in the parameter space $\mathbb{C}^4\{\alpha\}$. We classify all such affine spaces, together with their associated algebraic solutions (Theorem 2.1, Table 2.1). The proof of Theorem 2.1 does not use the notion of Picard-Fuchs equation. It turns out that

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the solutions $2A, 2B, \ldots, 5L$ in Table 2.1 coincide surprisingly with the solutions obtained earlier by Doran, who used deformations of elliptic surfaces with four singular fibers and the related Picard-Fuchs equations, see Theorem 3.1.

The second purpose of the paper is to give a partial explanation of the above coincidence. Recall that each solution $(\lambda(t), \alpha)$ of a given PVI_α equation governs the isomonodromy deformation of an appropriate 2×2 Fuchsian system with four singular points. We say that such a deformation is *geometric* if there is a fundamental matrix of solutions whose entries are Abelian integrals depending algebraically on the deformation parameter. A geometric deformation of a Fuchsian system is isomonodromic, and defines an algebraic solution $(\lambda(t), \alpha)$ of an appropriate PVI_α equation. When this holds true, we say that the algebraic solution $(\lambda(t), \alpha)$ of PVI_α is of *geometric origin*.

The solutions $(\lambda(t), \alpha')$ of geometric origin coming from the deformations of elliptic surfaces with four singular fibers were computed by Doran [3] (this result is summarized in Theorem 3.1 and Table 3.1). We will prove that to each such $(\lambda(t), \alpha')$, we may associate a parameter $\alpha'' \neq \alpha'$, such that $(\lambda(t), \alpha'')$ is still of geometric origin, and governs the deformation of a ramified cover of $\mathbb{P}^1$ with four ramification points (Theorem 3.3 and Table 3.2). On its turn this already implies that $\lambda(t)$ is a common solution of the family $\{PVI_\alpha\}_\alpha$, where α belongs to the affine line containing α' and α''. This explains why the solutions $\lambda(t)$ found by Doran reappeared in Theorem 2.1. Finally we note that the converse is also true (although there is no apparent reason for this). Namely, each affine space of PVI_α equations in Table 2.1, except the families $0A, \ldots, 1F$, is generated by points α' and α'' given in Tables 3.1 and 3.2, respectively.

2 Families of Painlevé VI equations having a common solution

Let $\lambda = \lambda(t)$ be a solution of the equations $PVI_{\alpha'}$, $PVI_{\alpha''}$, $\alpha' \neq \alpha''$. Then $\lambda(t)$ satisfies the implicit equation

$$\beta_0 - \beta_1 \frac{t}{\lambda^2} + \beta_2 \frac{t-1}{(\lambda-1)^2} - \beta_3 \frac{t(t-1)}{(\lambda-t)^2} = 0, \tag{2.1}$$

where $\beta = \alpha' - \alpha'' = (\beta_0, \beta_1, \beta_2, \beta_3)$, and hence it is an algebraic function. The function $\lambda(t)$ satisfies, moreover, the family $\{PVI_\alpha\}_\alpha$, where α belongs to the affine line

$$\left\{\alpha' + s(\alpha' - \alpha'') : s \in \mathbb{C}\right\} \subset \mathbb{C}^4. \tag{2.2}$$

It is seen from this that the set of all α such that PVI_α is satisfied by the function $\lambda(t)$ form an affine subspace of $\mathbb{C}^4$. We refer to the set of these PVI_α equations as a *family of Painlevé VI equations having a common solution*.

Table 2.1 List of all algebraic solutions satisfying families of PVI_α equations.

Name	Solution of PVI_α equation	Family of PVI_α equations
0A	$\lambda = t$	$\left(a, b, c, \frac{1}{2}\right)$
0B	$\lambda = 1$	$(a, b, 0, c)$
0C	$\lambda = 0$	$(a, 0, b, c)$
0D	$\lambda = \infty$	$(0, a, b, c)$
1A	$\lambda^2 - kt$	$\left(a, ka, \frac{1}{8}, \frac{1}{8}\right)$, $k = \text{const.} \neq 0$
1B	$(\lambda - 1)^2 + k(t - 1)$	$\left(a, \frac{1}{8}, ka, \frac{1}{8}\right)$, $k = \text{const.} \neq 0$
1C	$(\lambda - t)^2 - k\,t(t - 1)$	$\left(a, \frac{1}{8}, \frac{1}{8}, ka\right)$, $k = \text{const.} \neq 0$
1D	$-t(\lambda - 1)^2 + k(t - 1)\lambda^2$	$\left(\frac{1}{8}, a, ka, \frac{1}{8}\right)$, $k = \text{const.} \neq 0$
1E	$(\lambda - t)^2 + k(t - 1)\lambda^2$	$\left(\frac{1}{8}, a, \frac{1}{8}, ka\right)$, $k = \text{const.} \neq 0$
1F	$(\lambda - t)^2 - kt(\lambda - 1)^2$	$\left(\frac{1}{8}, \frac{1}{8}, a, ka\right)$, $k = \text{const.} \neq 0$
2A	$\lambda^2 - t$	(a, a, b, b)
2B	$\lambda^2 - 2\lambda + t$	(a, b, a, b)
2C	$\lambda^2 - 2\lambda t + t$	(b, a, a, b)
3A	$\lambda^4 - 6\lambda^2 t + 4\lambda t + 4\lambda t^2 - 3t^2$	$(a, 9a, a, a)$
3B	$3\lambda^4 - 4\lambda^3 - 4\lambda^3 t + 6\lambda^2 t - t^2$	$(9a, a, a, a)$
3C	$\lambda^4 - 4\lambda^3 + 6t\lambda^2 - 4t^2\lambda + t^2$	$(a, a, 9a, a)$
3D	$\lambda^4 - 4t\lambda^3 + 6t\lambda^2 - 4t\lambda + t^2$	$(a, a, a, 9a)$
4A	$\lambda^4 - 2t\lambda^3 - 2\lambda^3 + 6t\lambda^2$ $-2t^2\lambda - 2t\lambda + t^3 - t^2 + t$	$\left(a, \frac{1}{8}, a, a\right)$
4B	$\lambda^4 - 2t\lambda^3 + 2t^2\lambda - t^3$	$\left(a, a, \frac{1}{8}, a\right)$
4C	$\lambda^4\left(t^2 - t + 1\right) - 2\lambda^3 t(t + 1)$ $+6t^2\lambda^2 - 2\lambda t^2(t + 1) + t^3$	$\left(\frac{1}{8}, a, a, a\right)$
4D	$\lambda^4 - 2\lambda^3 + 2t\lambda - t$	$\left(a, a, a, \frac{1}{18}\right)$
5A	$-2\lambda^3 + 3t\lambda^2 + 3\lambda^2 - 6t\lambda + t^2 + t$	$\left(4a, \frac{1}{8}, a, a\right)$
5B	$\lambda^3 - 3\lambda^2 + 3t\lambda - 2t^2 + t$	$\left(a, \frac{1}{18}, 4a, a\right)$
5C	$\lambda^3 - 3t\lambda^2 + 3t\lambda + t^2 - 2t$	$\left(a, \frac{1}{18}, a, 4a\right)$

Table 2.1 Continued.

Name	Solution of PVI_α equation	Family of PVI_α equations
5D	$2\lambda^3 - 3t\lambda^2 + t^2$	$\left(4a, a, \frac{1}{18}, a\right)$
5E	$\lambda^3 - 3t\lambda + 2t^2$	$\left(a, 4a, \frac{1}{18}, a\right)$
5F	$\lambda^3 - 3t\lambda^2 + 3t\lambda - t^2$	$\left(a, a, \frac{1}{18}, 4a\right)$
5G	$\lambda^3(2-t) - 3t\lambda^2 + 3t^2\lambda - t^2$	$\left(\frac{1}{18}, a, 4a, a\right)$
5H	$\lambda^3(t+1) - 6t\lambda^2 + 3t(t+1)\lambda - 2t^2$	$\left(\frac{1}{18}, 4a, a, a\right)$
5I	$(1-2t)\lambda^3 + 3t\lambda^2 - 3t\lambda + t^2$	$\left(\frac{1}{18}, a, a, 4a\right)$
5J	$\lambda^3 - 3\lambda^2 + 3t\lambda - t$	$\left(a, a, 4a, \frac{1}{18}\right)$
5K	$\lambda^3 - 3t\lambda + 2t$	$\left(a, 4a, a, \frac{1}{18}\right)$
5L	$\lambda^3 - 3\lambda^2 + t$	$\left(4a, a, a, \frac{1}{18}\right)$

Theorem 2.1. The list of all families of Painlevé VI equations having a common solution, together with the corresponding solution, is shown in Table 2.1. □

Remark 2.2. The meaning of the solutions 0A–0D is as follows. If we write down the PVI_α equation as an equivalent hamiltonian nonautonomous system on $\mathbb{C}^2$ (e.g., [10, Theorem 1.5.2]), then $\lambda = 0, 1, t$ defines a solution of this system if and only if $\alpha_1 = 0$, $\alpha_2 = 0$, and $\alpha_3 = 1/2$. One may further complete canonically $\mathbb{C}^2$ to a surface Σ (the so-called "space of initial conditions"), such that PVI_α induces a foliation on $\Sigma \times \{\mathbb{P}^1 \setminus \{0, 1, \infty\}\}$ which is uniform with respect to the trivial fibration $\Sigma \times \{\mathbb{P}^1 \setminus \{0, 1, \infty\}\} \to \mathbb{P}^1 \setminus \{0, 1, \infty\}$ (see Okamoto [17] for details). Then $\lambda = 0, 1, t, \infty$ correspond to leaves of this foliation, for example, [17, pages 45–47] if and only if $\alpha_1 = 0$, $\alpha_2 = 0$, $\alpha_3 = 1/2$, or $\alpha_4 = 0$, respectively.

Remark 2.3. Each solution $\lambda(t)$ can be defined by a relation $P(\lambda(t), t) \equiv 0$ where P is an irreducible polynomial. This polynomial is given in Table 2.1. The solutions in each of the 6 series of families in Table 2.1 are equivalent up to an S_4-symmetry of Painlevé VI equation (see Section 2.2.1).

Remark 2.4. The solutions $1A, 1B, \ldots, 1F$ depend on the parameter $k \in \mathbb{C}$. Therefore they are solutions in a different sense from all the others. However, for every *fixed* value of k they are common solutions of a family (affine line) of PVI_α equations. It turns out that they are Okamoto equivalent to the solutions 2A, 2B, 2C which do not contain a parameter. More precisely, the solution 1A,

$$\lambda^2 - kt = 0, \qquad \alpha = \left(a, ka, \frac{1}{8}, \frac{1}{8}\right), \tag{2.3}$$

is equivalent, after applying the transformation w_2 of Okamoto [18, page 363], to the solution $\lambda^2 - t = 0$ where the parameter α equals to

$$\left(\left(\frac{\sqrt{2a}-\sqrt{2b}}{2\sqrt{2}}\right)^2, \left(\frac{\sqrt{2a}-\sqrt{2b}}{2\sqrt{2}}\right)^2, \left(1-\frac{\sqrt{2a}-\sqrt{2b}}{2\sqrt{2}}\right)^2, \left(1-\frac{\sqrt{2a}-\sqrt{2b}}{2\sqrt{2}}\right)^2\right), \tag{2.4}$$

and $b = ka$. Thus, up to Okamoto equivalence, the families of Painlevé VI equations having a common solution are represented, for instance, by the five families 0A, 2A, 3A, 4A, and 5A in Table 2.1.

2.1 Outline of the proof

Denote by Γ_β the compactified and normalized algebraic curve defined by (2.1), with affine model

$$\Gamma_\beta^{\mathrm{aff}} = \left\{(\lambda, t) \in \mathbb{C}^2 : N(\lambda, t) = 0\right\}, \tag{2.5}$$

where

$$\begin{aligned} N(\lambda, t) = {} & \beta_0\lambda^2(\lambda-1)^2(\lambda-t)^2 - \beta_1 t(\lambda-1)^2(\lambda-t)^2 \\ & + \beta_2(t-1)\lambda^2(\lambda-t)^2 - \beta_3 t(t-1)\lambda^2(\lambda-1)^2 = 0. \end{aligned} \tag{2.6}$$

In the case when Γ_β is irreducible, the relation $\{N(\lambda, t) = 0\}$ defines an algebraic function $\lambda(t)$. If this function were a solution of some PVI_α equation, then the only ramification points of $\lambda(t)$ would be at $t = 0, 1, \infty$ (because PVI_α satisfies the so-called Painlevé property [10]). Equivalently, the pair (Γ_β, t) is a Belyi pair, which means that the only possible critical values of the map

$$\pi : \Gamma_\beta \longrightarrow \mathbb{CP}^1 : (\lambda, t) \longrightarrow t \tag{2.7}$$

are 0, 1, or ∞. This means also that if $\Delta(t)$ is the discriminant of $N(\lambda, t)$ with respect to λ, then it is a polynomial whose only roots are at $t = 0$ and $t = 1$. A direct computation shows that this is impossible. The more difficult case is when $N(\lambda, t)$ is reducible over $\mathbb{C}$. Then Γ_β defines several algebraic functions and we have to apply the above to each of them. Finally we have to check whether the obtained function is actually a solution of some PVI_α equation. To check whether a given polynomial $N(\lambda, t)$ is reducible over $\mathbb{C}$ is a difficult task in general. We will make use of the action of the symmetric group $\mathcal{S}_4$ (see Section 2.2.1) on the set of curves Γ_β, parameterized by $\beta \in \mathbb{CP}^3$.

It turns out that, first, curves Γ_β with a trivial stabilizer under the action of $\mathcal{S}_4$ cannot produce a solution of PVI_α. The stabilizer of a curve acts on it as a group of automorphisms (symmetries) which imposes additional restrictions on β.

The second ingredient of the proof is the study of the Puiseux expansion of $\lambda(t)$ in a neighborhood of $t = 0, 1, \infty$ (Section 2.2.2). These expansions depend on the stabilizer of Γ_β only and imply the possible topological types of the solution $\lambda(t)$. Equivalently, to each solution $\lambda(t)$ we associate a Belyi pair, and the Puiseux expansions determine their possible *dessin d'enfant*. The algebraic functions which we obtain in this way are a posteriori the solutions of the PVI_α presented in Table 2.1.

2.2 Proof of Theorem 2.1

2.2.1 The action of $\mathcal{S}_4$. The set of automorphisms of the projective line $\mathbb{CP}^1$ which send four distinct points $(0, 1, t, \infty)$ to the points $(0, 1, \tilde{t}, \infty)$ ($\tilde{t} = \tilde{t}(t)$ is uniquely defined) form a group isomorphic to $\mathcal{S}_4$ generated by the transpositions

$$x^1 : s \longmapsto 1 - s, \qquad x^2 : s \longmapsto \frac{1}{s}, \qquad x^3 : s \longmapsto \frac{t-s}{t-1}. \tag{2.8}$$

Each x^i sends an isomonodromic family of Fuchsian systems with singular points at 0, 1, t, ∞ to an isomonodromic family of such systems with singular points at 0, 1, $\tilde{t}$, ∞. Therefore x^i induce an action of $\mathcal{S}_4$ on the set of PVI_α equations, and hence on the set of curves Γ_β. Explicitly we have

$$\begin{aligned} &x^i : \Gamma_\beta \longrightarrow \Gamma_{x^i_*(\beta)} : (\lambda, t) \longrightarrow \left(x^i(\lambda), x^i(t)\right), \quad i = 1, 2, \\ &x^3 : \Gamma_\beta \longrightarrow \Gamma_{x^3_*(\beta)} : (\lambda, t) \longrightarrow \left(x^3(\lambda), x^3(0)\right) = \left(\frac{t-\lambda}{t-1}, \frac{t}{t-1}\right), \end{aligned} \tag{2.9}$$

where

$$\begin{aligned} x^1_* &: (\beta_0, \beta_1, \beta_2, \beta_3) \longrightarrow (\beta_0, \beta_2, \beta_1, \beta_3), \\ x^2_* &: (\beta_0, \beta_1, \beta_2, \beta_3) \longrightarrow (\beta_1, \beta_0, \beta_2, \beta_3), \\ x^3_* &: (\beta_0, \beta_1, \beta_2, \beta_3) \longrightarrow (\beta_0, \beta_3, \beta_2, \beta_1), \end{aligned} \tag{2.10}$$

which is the standard representation of $\mathcal{S}_4$ on $\mathbb{C}^4$ (upon identifying $\infty, 0, 1, t$ to β_0, β_1, β_2, β_3, resp.). The proof of the above facts is straightforward, see [10] for details.

2.2.2 The topological type of the projection $\Gamma_\beta \to \mathbb{CP}^1$ *in a neighborhood of the preimage of* $t = 0, 1, \infty$*.* Let Γ_β be the compactified and normalized curve defined by (2.5) (it is a disjoint union of Riemann surfaces). In this section, we determine the topological type of the projection (2.7):

$$\Gamma_\beta \longrightarrow \mathbb{CP}^1 \tag{2.11}$$

in a neighborhood of the preimage of $t = 0, 1, \infty$ in Γ_β. In the projective space $\mathbb{CP}^3$ with coordinates $[\beta_0 : \beta_1 : \beta_2 : \beta_3]$, consider the complex polyhedron W formed by the ten planes (2-faces)

$$W = \bigcup_{i \neq j} \{\beta_i = \beta_j\} \cup_k \{\beta_k = 0\}. \tag{2.12}$$

It has also 45 1-faces (projective lines) and 120 0-faces (points). We will see in the process of the proof that the topological type of the projection in a neighborhood of the preimage of $t = 0, 1, \infty$ is one and the same when β belongs to a given i-face, but does not belong to any other j-face with $j < i$. For this reason we will use, until the end of this paper, the following convention. *When we say that a point* β *belongs to a given face (satisfies some set of relations (2.12)), then this will mean that it does not belong to any other face of smaller dimension (does not satisfy any other relation from the list (2.12)).* The topological type in a neighborhood of the preimage of any point is determined by a partition of the degree of the map which is 6. Thus a partition $(1+1+1+1+2)$ means that we have 5 preimages and that the multiplicity of π at each preimage is 1, 1, 1, 1, 2, respectively. Similarly, a partition $(1+1+2+2)$ means that we have 4 preimages with multiplicities 1, 1, 2, 2, respectively, and so forth. To formulate the result we note that the symmetric group $\mathcal{S}_4$ acts on the polyhedron W by its standard representation (2.10), as well on the set of curves Γ_β by (2.9). The subgroup $\mathcal{S}_3$ generated by x^1, x^2 permutes the ramification points 0, 1, ∞ according to (2.8) without changing the topological type of the projection π over each of these points.

Proposition 2.5. The topological type of the projection (2.7) in a neighborhood of the preimage of $t = 0, 1, \infty$ is one and the same when β belongs to a given face of the polyhedron W or it does not belong to W. This topological type is shown in Table 2.2 (one representative for each orbit of $\mathcal{S}_3 = \langle x^1, x^2 \rangle$). □

Proof. The birational transformations x^1, x^2 defined by (2.9) are compatible with the projection π and permute the points $t = 0, 1, \infty$. Therefore it suffices to consider the preimage of 0. Let us consider in detail the "generic" case, when $\beta \notin W$. It follows from the Newton polygon of $N(\lambda, t)$, shown in Figure 2.1, that there are at least three Puiseux series in

Table 2.2 Multiplicity of π at the preimages of $t = 0, 1, \infty$.

Face of W	Stabilizer	$t = 0$	$t = 1$	$t = \infty$
$\beta_i \neq \beta_j$	S_4	$(1+1+1+1+2)$	$(1+1+1+1+2)$	$(1+1+1+1+2)$
$\beta_0 = \beta_2$	$S_2 \times S_2$	$(1+1+1+3)$	$(1+1+1+1+2)$	$(1+1+1+1+2)$
$\beta_0 = \beta_2, \beta_1 = \beta_3$	D_4	$(1+1+2+2)$	$(1+1+1+1+2)$	$(1+1+1+1+2)$
$\beta_0 = \beta_1 = \beta_2$	S_3	$(1+1+1+3)$	$(1+1+1+3)$	$(1+1+1+3)$
$\beta_0 = \beta_1 = \beta_2 = \beta_3$	S_4	$(1+1+2+2)$	$(1+1+2+2)$	$(1+1+2+2)$
$\beta_3 = 0$	S_3	$(1+1+2)$	$(1+1+2)$	$(1+1+2)$
$\beta_0 = \beta_2, \beta_3 = 0$	S_2	$(1+3)$	$(1+1+2)$	$(1+1+2)$
$\beta_0 = \beta_1 = \beta_2, \beta_3 = 0$	S_3	$(1+3)$	$(1+3)$	$(1+3)$
$\beta_2 = \beta_3 = 0$	$S_2 \times S_2$	(2)	$(1+1)$	(2)
$\beta_2 = \beta_3 = 0, \beta_0 = \beta_1$	$S_2 \times S_2$	(2)	$(1+1)$	(2)

a neighborhood of $(0, 0)$ (for the terminology, see, e.g., Kirwan [12]). The first two correspond to the line segment $[(3, 0), (1, 2)]$ and have nonequivalent leading terms

$$\lambda = c_1 t + \dots, \qquad \lambda = c_2 t + \dots, \tag{2.13}$$

where

$$(\beta_3 - \beta_1) c_{1,2}^2 + 2\beta_1 c_{1,2} - \beta_1 = 0 \tag{2.14}$$

provided that

$$\beta_1 \neq \beta_3, \qquad \beta_1^2 + \beta_1(\beta_3 - \beta_1) \neq 0. \tag{2.15}$$

The third one corresponds to the line segment $[(1, 2), (0, 4)]$ and has leading term

$$\lambda = c_3 t^{1/2} + \dots, \tag{2.16}$$

where

$$(\beta_0 - \beta_2) c_3^2 + \beta_3 - \beta_1 = 0 \tag{2.17}$$

provided that

$$\beta_0 \neq \beta_2, \qquad \beta_1 \neq \beta_3. \tag{2.18}$$

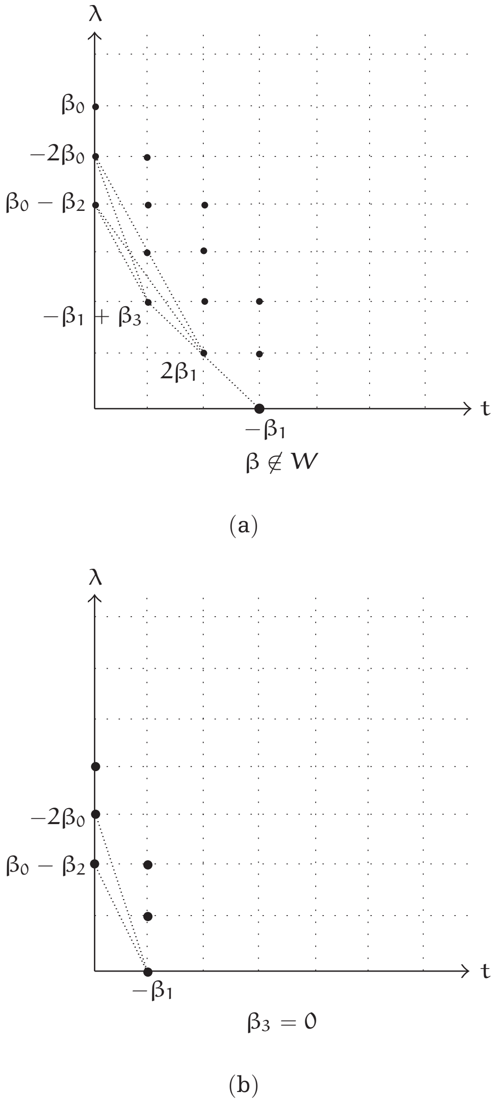

Figure 2.1 The Newton polygon of (a) $N(\lambda, t)$ and (b) $N^0(\lambda, t)$.

Taking into consideration that

$$N(\lambda, 0) = \lambda^4\left(\beta_0\lambda^2 - 2\beta_0\lambda + \beta_0 - \beta_2\right), \tag{2.19}$$

we conclude that we have at least 5 preimages of multiplicities at least 1, 1, 2, 1, 1, respectively. As the degree of the map π is six, then its topological type is exactly $(1+1+1+1+2)$. The topological type of the projection π over 0 and 1 is obtained by acting with the group $\mathcal{S}_3$ generated by x^1, x^2.

In a similar way one verifies that when $\beta_0 = \beta_2$ or $\beta_1 = \beta_3$ the multiplicities are $(1+1+1+3)$. If $\beta_0 = \beta_2$ and $\beta_1 = \beta_3$ the multiplicities are $(1+1+2+2)$. The case $\beta_0 = \beta_1 = \beta_3$ is the same as $\beta_0 = \beta_2$ and the multiplicity is $(1+1+1+3)$. The case $\beta_0 = \beta_1 = \beta_2 = \beta_3$ is of the same type as $\beta_0 = \beta_2$ and $\beta_1 = \beta_3$. The multiplicities of π over 1 and ∞ are obtained as before. This completes the study of faces of W for which $\beta_i \neq 0$. In the case $\beta_3 = 0$ we consider the curve

$$\Gamma^0_\beta = \left\{(\lambda, t) \in \mathbb{C}^2 : \beta_0 - \beta_1 \frac{t}{\lambda^2} + \beta_2 \frac{t-1}{(\lambda-1)^2} = 0,\ \lambda \neq 0, 1\right\}. \tag{2.20}$$

The polynomial $N(\lambda, t)$ is replaced by

$$N^0(\lambda, t) = \beta_0 \lambda^2 (\lambda-1)^2 - \beta_1 t(\lambda-1)^2 + \beta_2 (t-1)\lambda^2 \tag{2.21}$$

whose Newton polygon is shown in **Figure 2.1**. It follows that there is at least one Puiseux expansion with leading term

$$\lambda = ct^{1/2} + \dots, \qquad \beta_1 + \big(\beta_2 - \beta_0\big)c^2 = 0 \tag{2.22}$$

provided that $\beta_1 \neq 0$, $\beta_2 \neq \beta_0$. As

$$N^0(\lambda, 0) = \lambda^2 \big(\beta_0(\lambda-1)^2 - \beta_2\big), \tag{2.23}$$

then $t = 0$ has at least three preimages, provided that $\beta_0\beta_2 \neq 0$. We conclude that $t = 0$ has exactly three preimages with multiplicities $2, 1, 1$, respectively, provided that β belongs to the 2-face $\beta_3 = 0$. The remaining 1-faces and 0-faces are studied in the same way. The result is summarized in **Table 2.2**. It is worth noting that in all cases the computing of the leading term of the Puiseux expansion suffices to deduce the result. ■

We conclude this section by the following elementary claim which will be often useful in the computations.

Proposition 2.6. Let $N_1(\lambda, t)$ be a polynomial of nonzero degree with respect to λ and of nonzero degree with respect to t, which divides $N(\lambda, t)$, and $\beta_1\beta_2\beta_3 \neq 0$. Then

$$\begin{aligned} N_1(0, t) &= c_0 t^{n_0}, \quad c_0 \neq 0,\ 1 \leq n_0 \leq 3, \\ N_1(1, t) &= c_1 (t-1)^{n_1}, \quad c_1 \neq 0,\ 1 \leq n_1 \leq 3, \\ N_1(t, t) &= c_2 t^{m_0} (t-1)^{m_1}, \quad c_2 \neq 0,\ 1 \leq m_0,\ 1 \leq m_1,\ m_0 + m_1 \leq 3. \end{aligned} \tag{2.24}$$

□

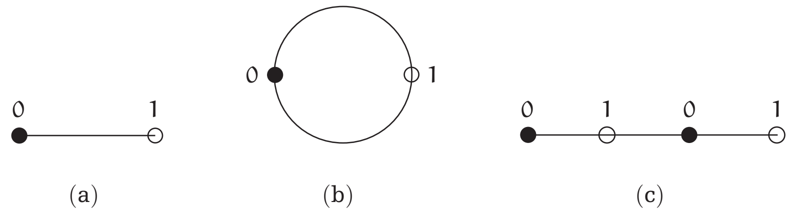

Figure 2.2 Dessins of degrees 1, 2, and 3.

Proof. We have $N(0,t) = -\beta_1 t^3$. For a fixed $\lambda = c \sim 0$ the polynomial $N(c,t) \in \mathbb{C}[t]$ has exactly three roots which tend to zero when c tends to zero. Therefore the polynomial $N_1(c,t) \in \mathbb{C}[t]$ has at least one and at most three roots which tends to zero when c tends to zero, which proves the claim concerning $N_1(0,t)$. The claim concerning $N_1(1,t)$ is proved in the same way. As $N_1(0,0) = N_1(1,1) = 0$ then $N_1(t,t)$ is divided by $t(t-1)$ but also divides $N(t,t) = -\beta_3 t^3 (t-1)^3$. ■

We are ready to compute the solutions of PVI_α corresponding to the faces of W. Let Γ be the Riemann surface of an irreducible component of Γ_β, which defines a solution of some PVI_α equation. Then the only ramification points of the induced map

$$\pi : \Gamma \longrightarrow \mathbb{CP}^1 : (\lambda, t) \longrightarrow t \tag{2.25}$$

are at $0, 1, \infty$, and Γ is connected. The pair (Γ, π) is called a Belyi pair, to which we associate a dessins d'enfant, which is the graph obtained as a preimage of the segment $[0,1]$ under the map π. The degree of the dessin is the degree of π (see [20]). The dessin d'enfant will be useful when describing the topological type of the projection π.

2.2.3 The case $\beta \notin W$. We suppose that $\lambda(t)$ is an algebraic function, such that $N(\lambda(t),t) \equiv 0$, and consider the corresponding Belyi pair (Γ, π), (2.25). Let $\{\lambda_1, \dots, \lambda_d\} = \pi^{-1}(t_0)$ where $t_0 \neq 0, 1, \infty$. The loops originating from t_0 and going clockwise once around $0, 1, \infty$ induce permutations $\sigma_0, \sigma_1, \sigma_\infty$ of the points $\lambda_1, \dots, \lambda_d$, such that $\sigma_0\sigma_1\sigma_\infty = 1$. According to Table 2.2, σ_0, σ_1, and σ_∞ are transpositions unless one of them is the identity permutation. In the former case $(\sigma_0\sigma_1)^2 = 1$ and hence $\sigma_0\sigma_1 = \sigma_1\sigma_0$. Thus σ_∞ is a product of two disjoint transpositions, which contradicts Table 2.2. On the other hand, if one of the permutations σ_0, σ_1, or σ_∞ is the identity, the group generated by them is either Z_2 or is trivial. This shows that the degree of π is either two (because the covering (2.25) is connected) or one. The corresponding dessin d'enfants are shown in Figures 2.2a and 2.2b (in particular $N(\lambda, t) \in \mathbb{C}[\lambda, t]$ is reducible). If the dessin is of degree one, then the solution is defined as $\lambda = P(t)$, where P is a polynomial (the coefficient of λ^6 in the polynomial $N(\lambda, t)$ is $\beta_0 \neq 0$). By Proposition 2.6 we conclude that either $\lambda = t$, $\lambda = t^2$, or $\lambda = t^3$. But $\lambda - t$, $\lambda - t^2$, or $\lambda - t^3$ cannot divide $N(\lambda, t)$, provided that $\beta_i \neq 0$. If the dessin is of degree two, then $\lambda(t)$ is defined by $\lambda^2 + 2p(t)\lambda + q(t) = 0$. The functions p, q are polynomials in t, because the coefficient of λ^6 in the polynomial $N(\lambda, t)$ is $\beta_0 \neq 0$. Further, we may suppose (acting with an appropriate symmetry x^i on Γ, see Section 2.2.1) that $\lambda(t)$ is ramified over 0 and ∞ only. By Proposition 2.6 $q(t)$ is a nonconstant polynomial which divides t^3. As $p(t)^2 - q(t)$ is a nonconstant monomial of odd degree, then $p(t) \equiv 0$. Proposition 2.6 implies that we have either $\lambda^2 = t$ or $\lambda^2 = t^3$. The polynomial $\lambda^2 - t^3$ cannot divide $N(\lambda, t)$ while $\lambda^2 - t$ divides $N(\lambda, t)$ if and only if $\beta_0 = \beta_1$ and $\beta_2 = \beta_3$ (this case is excluded, as $\beta \notin W$). *The curve Γ_β does not define a solution.*

2.2.4 The face $\beta_1 = \beta_2$. The possible dessins d'enfant are determined as above. Namely, when one of the permutations σ_0, σ_1, or σ_∞ is identity, the dessin is of degree one or two. Up to a symmetry it is equivalent to the one shown in Figure 2.2a or Figure 2.2b. Reasoning as in the case $\beta \notin W$ we conclude that Γ_β does not define a solution.

If, on the other hand, σ_0, σ_1 are nontrivial transpositions, we have one more case compared to Section 2.2.3: σ_3 is cyclic of order three and σ_0, σ_1 are nondisjoined permutations. Taking into account that the covering (2.25) is connected, we conclude that the dessin is of degree three. Up to a symmetry, this dessin is shown in Figure 2.2c. The function $\lambda(t)$ satisfies $N_1(\lambda(t), t) \equiv 0$ where N_1 is an irreducible polynomial of degree three in λ, dividing the polynomial $N(\lambda, t)$ defined in (2.6). We denote

$$\begin{aligned} &N = N_1 N_2, \qquad \Gamma_1^{\mathrm{aff}} = \{N_1(\lambda, t) = 0\}, \\ &\Gamma_2^{\mathrm{aff}} = \{N_2(\lambda, t) = 0\}, \qquad \Gamma_\beta^{\mathrm{aff}} = \Gamma_1^{\mathrm{aff}} \cup \Gamma_2^{\mathrm{aff}}. \end{aligned} \tag{2.26}$$

As before, let Γ_1, Γ_2, Γ_β be the corresponding compactified and normalized curves. The symmetry x^1 is an automorphism of Γ_β and hence either it is an automorphism of the curves Γ_1 and Γ_2 or it permutes these curves (we used that Γ_1 is irreducible). Suppose first that x^1 is an automorphism of Γ_1. Then the rational function

$$\frac{N_1(\lambda, t)}{\lambda(\lambda - 1)(\lambda - t)} = 1 + \frac{A_1}{\lambda} + \frac{B_1}{\lambda - 1} + \frac{C_1}{\lambda - t} \tag{2.27}$$

is invariant under the action of x^1 too. Here A_1, B_1, C_1 are polynomials in t which divide t, $t-1$, and $t(t-\lambda)$, respectively (see (2.1)), and hence we have

$$A_1(1-t) = -B_1(t), \qquad B_1(1-t) = -A_1(t), \qquad C_1(1-t) = -C_1(t). \tag{2.28}$$

Similarly, if

$$\frac{N_2(\lambda,t)}{\lambda(\lambda-1)(\lambda-t)} = 1 + \frac{A_2}{\lambda} + \frac{B_2}{\lambda-1} + \frac{C_2}{\lambda-t}, \tag{2.29}$$

then

$$A_2(1-t) = -B_2(t), \qquad B_2(1-t) = -A_2(t), \qquad C_2(1-t) = -C_2(t). \tag{2.30}$$

We conclude that $C_1(t) = c_1 t(t-1)$, $C_2(t) = c_2 t(t-1)$, which contradicts $C_1C_2 = -\beta_3 t(t-1)/\beta_1$, $\beta_1, \beta_3 \neq 0$. Suppose now that the map x^1 exchanges the curves Γ_1 and Γ_2. Then we have

$$A_1(1-t) = -B_2(t), \qquad B_1(1-t) = -B_2(t), \qquad C_1(1-t) = -C_2(t). \tag{2.31}$$

The polynomial $N(\lambda, t)$ is of degree three with respect to t and

$$A_1A_2 = -\frac{\beta_1}{\beta_0}t, \qquad B_1B_2 = -\frac{\beta_2}{\beta_0}(t-1), \qquad C_1C_2 = -\frac{\beta_3}{\beta_0}t(t-1). \tag{2.32}$$

Therefore without loss of generality, we may suppose that $C_1(t) = c_1 t$, $C_2 = c_1(t-1)$ and $A_1(t) = a_1 t$, $B_2 = b_2(t-1)$ or $A_2(t) = a_2 t$, $B_1 = b_1(t-1)$, where $a_i, b_j \neq 0$. In both cases, the polynomials $N_1(\lambda,t)$, $N_2(\lambda,t)$ are of degree two in t, in contradiction to the fact that the degree of $N(\lambda,t)$ with respect to t is three. *We conclude that the curve Γ_β does not define a solution.*

2.2.5 The face $\beta_0 = \beta_2$, $\beta_1 = \beta_3$. We have the identity

$$N(\lambda,t) = \left(\lambda^2 - 2\lambda + t\right)\left(\beta_0\lambda^2(\lambda-t)^2 - \beta_1 t^2(\lambda-1)^2\right). \tag{2.33}$$

Indeed,

$$\lambda^2 - 2\lambda + t = 0 \tag{2.34}$$

defines a solution of PVI_α, for example, [3], Table 2.2, solution 2B. Its dessin is equivalent to the one in Figure 2.2b. The function $\lambda(t)$ defined by

$$\lambda(\lambda - t) - ct(\lambda - 1) = 0, \quad c = \pm\sqrt{\frac{\beta_1}{\beta_0}}, \tag{2.35}$$

is ramified over $0, 1, \infty$ only provided that $c = 0, \pm 1$. This is, however, impossible as $\beta_0 \neq \beta_1$, $\beta_i \neq 0$. *The curve* Γ_β *defines the solution (2.34).*

2.2.6 The face $\beta_0 = \beta_1 = \beta_2$. According to Table 2.2 each of the permutations σ_0, σ_1, σ_∞ is either the identity or is a cycle of length three.

If one of the permutations $\sigma_0, \sigma_1, \sigma_\infty$ is the identity, then the group generated by $\sigma_0, \sigma_1, \sigma_\infty$ is either $\mathbb{Z}_3$ or the trivial one $\{1\}$, and hence the degree of the corresponding dessin is one or three. The case of degree one does not lead to a solution (see Section 2.2.3). The case of degree three is studied as in Section 2.2.4 and does not lead to a solution too (provided that $\beta_i \neq 0$).

If neither of the permutations σ_0, σ_1, or σ_∞ is the identity, then they are disjoint three-cycles. As the symmetric group $\mathbb{S}_3$ contains only two three-cycles we conclude that the degree of the projection π is at least four. Suppose that $\lambda(t)$ is defined by the polynomial N_1, $N_1(\lambda(t), t) \equiv 0$, where $N_1 \in \mathbb{C}[\lambda, t]$ is irreducible of degree four in λ. Then x^1, x^2 are automorphisms of

$$\Gamma_1 = \left\{N_1(\lambda, t) = 0\right\}. \tag{2.36}$$

It follows that the curve

$$\Gamma_2 = \left\{N_2(\lambda, t) = 0\right\} \tag{2.37}$$

defined by the polynomial $N_2 = N/N_1$ is also invariant. We have

$$\frac{N_2(\lambda, t)}{\lambda(\lambda - 1)} = 1 + \frac{A}{\lambda} + \frac{B}{\lambda - 1}, \tag{2.38}$$

where A, B are polynomials in t of degree at most three. The $x^{1,2}$ invariance of the above expression implies

$$A(t)A\left(\frac{1}{t}\right) = 1, \qquad A\left(\frac{1}{t}\right)B(t) = -B\left(\frac{1}{t}\right), \qquad B(t) = -A(1-t). \tag{2.39}$$

with solutions

$$
\begin{aligned}
&A(t) = t, \qquad B(t) = t - 1;\\
&A(t) = t^3, \qquad B(t) = (t-1)^3;\\
&A(t) = -t^2, \qquad B(t) = (t-1)^2.
\end{aligned}
\tag{2.40}
$$

The case $A(t) = t^3$, $B(t) = (t-1)^3$ does not lead to a solution as $N_1(\lambda, t)$ depends on t. The case $A(t) = -t^2$, $B(t) = (t-1)^2$ implies $N_2(\lambda, t) = (\lambda - t)^2$, and hence $\beta_3 = 0$. Finally, in the case $A(t) = t$, $B(t) = t - 1$ we have $N_2(\lambda, t) = \lambda^2 - 2\lambda + 2\lambda t - t$ (which does not define a solution). The condition that $N_2(\lambda, t)$ divides $N(\lambda, t)$ leads to $\beta_3 = 9\beta_0$ and we get

$$
N(\lambda, t) = \left(\lambda^2 - 2\lambda + 2\lambda t - t\right)\left(t^2 - 4\lambda^3 t + 6\lambda^2 t - 4\lambda t + \lambda^4\right). \tag{2.41}
$$

The function $\lambda(t)$ defined by

$$
t^2 - 4\lambda^3 t + 6\lambda^2 t - 4\lambda t + \lambda^4 = 0 \tag{2.42}
$$

is indeed a solution of PVI_α, for example, [3], Table 2.2, solution 3D. In the case when the dessin corresponding to $\lambda(t)$ is of degree five, we conclude that the polynomial $N_2(\lambda, t)$ is linear in λ. By Proposition 2.6 we get $N_2(\lambda, t) = \lambda - t^2$ which implies $\beta_1 = 0$. To resume, *the curve* Γ_β *defines a solution, provided that* $\beta_0 = \beta_1 = \beta_2 = \beta_3/9$.

2.2.7 The face $\beta_0 = \beta_1 = \beta_2 = \beta_3$. We have

$$
N(\lambda, t) = \beta_0\left(\lambda^2 - 2\lambda + t\right)\left(\lambda^2 - 2\lambda t + t\right)\left(\lambda^2 - t\right) \tag{2.43}
$$

and the three algebraic functions defined by $N(\lambda, t) = 0$ are solutions of suitable PVI_α equations, for example, [3], Table 2.2, solutions 2B, 2C, and 2A, respectively. *The curve* Γ_β *defines three solutions*.

2.2.8 The face $\beta_3 = 0$. Recall that in this case $N(\lambda, t) = (\lambda - t)^2 N^0(\lambda, t)$, where

$$
N_0(\lambda, t) = \beta_0\lambda^2(\lambda - 1)^2 - \beta_1 t(\lambda - 1)^2 + \beta_2(t - 1)\lambda^2. \tag{2.44}
$$

Indeed, $\lambda = t$ is the so-called constant solution 0B (because, up to a symmetry, it coincides with $\lambda = 0, 1, \infty$). To the end of this section the polynomial N bill be replaced by N^0 and the curve Γ_β by $\Gamma_\beta^0 = \{N_0(\lambda, t) = 0\}$. The same arguments as in Section 2.2.3 show that the corresponding dessin d'enfant is of degree one or two.

If the degree is one, then the solution is $\lambda = P(t)$ for some nonconstant polynomial P. Therefore $N^0(P(t), t) \not\equiv 0$ and $P(t)$ cannot be a solution.

If the degree is two, then $\lambda(t)$ has exactly two ramification points. Without loss of generality, we suppose that these points are 0 and ∞, and as in Section 2.2.3, we conclude that $\lambda(t)$ is defined by $\lambda^2+2p(t)\lambda+q(t) = 0$ for some $p, q \in \mathbb{C}[t]$. The polynomial $\lambda^2+2p(t)\lambda+q(t)$ divides

$$N_0(\lambda, t) = t\big(\beta_2\lambda^2 - \beta_1(\lambda-1)^2\big) + \beta_0\lambda^2(\lambda-1)^2 - \beta_2\lambda^2 \tag{2.45}$$

and hence $p(t) = c_1$ and $q(t) = c_2 t$ for some constants c_1, c_2. Without loss of generality, we suppose that the ramification points of $\lambda(t)$ are 0 and ∞ and hence the discriminant $4(p^2 - q)$ is a power of t. This implies that $c_1 = 0$. Finally, a direct computation shows that the identity $N^0(\sqrt{-c_2 t}, t) \equiv 0$ implies $\beta_2 = 0$ which is not true. *The curve* Γ^0_β *does not define a solution*.

2.2.9 The face $\beta_3 = 0, \beta_0 = \beta_2$. It is easier to analyze the face $\beta_3 = 0, \beta_1 = \beta_2$, which is equivalent to $\beta_3 = 0, \beta_0 = \beta_2$ after applying the transformation x^2. Suppose for a moment that $\beta_3 = 0, \beta_1 = \beta_2$. The dessin is of degree at most three, and hence $N^0(\lambda, t)$ is reducible. It follows that $\lambda - c$ divides $N^0(\lambda, t)$ for some constant c. As $N^0(\lambda, t)$ is linear in t, then $\lambda - c$ is deduced from the coefficient of t which equals $\beta_1(1 - 2\lambda)$. Thus $c = 1/2$ and the condition that $1 - 2\lambda$ divides $N^0(\lambda, t)$ leads to $\beta_0 = 4\beta_1 = 4\beta_2$, in which case

$$N^0(\lambda, t) = \beta_0(2\lambda - 1)\big(2\lambda^3 - 3\lambda^2 + t\big). \tag{2.46}$$

The function $\lambda(t)$ defined by $(2\lambda^3 - 3\lambda^2 + t) = 0$ is indeed a solution, see [3], Table 2.2, solutions 5L. Applying the transformation $(x^2)^{-1} = x^2$ of Section 2.2.1 we get the solution (see [3], Table 2.2, solution 5K)

$$\lambda^3 - 3\lambda t + 2t = 0 \tag{2.47}$$

defined by Γ^0_β with $\beta_3 = 0, \beta_1 = 4\beta_0 = 4\beta_2$. *The curve* Γ^0_β *defines a solution provided that* $\beta_3 = 0, \beta_1 = 4\beta_0 = 4\beta_2$.

2.2.10 The face $\beta_3 = 0, \beta_0 = \beta_1 = \beta_2$. *The polynomial* $N^0(\lambda, t)$ *is irreducible and defines a solution, see* [3], *Table 2.2, solution* 4D.

2.2.11 The face $\beta_3 = 0$, $\beta_2 = 0$. *The curve* Γ^0_β *defines the constant solution* $\lambda = 1$, 0B, *as well the solution* 1A$\beta_0\lambda^2 - \beta_1 t = 0$.

2.2.12 The face $\beta_3 = 0$, $\beta_2 = 0$, *and* $\beta_0 = \beta_1$. *The curve* Γ^0_β *defines the constant solution* $\lambda = 1$, 0B *and the solution* $\lambda^2 = t$, 2A.

The results are summarized in Table 2.1. Theorem 2.1 is proved.

3 Algebraic solutions of PVI_α and Picard-Fuchs equations

It was noted in Remark 2.4 that the families 1A, 1B, . . . , 1F are Okamoto equivalent to the families 2A, 2B, and 2C. To this end we consider the remaining 23 families 2A − 5L, see Table 2.1. To each of them corresponds an affine plane or line in the parameter space $\mathbb{C}^4\{\alpha\}$ which, as we will prove bellow, is generated by special points α of geometric origin, see Tables 3.1 and 3.2. Indeed, observe that exactly the same 23 solutions 2A−5L were already obtained by Doran, see Theorem 3.1 bellow, by making use of the deformations of elliptic surfaces with four singular fibers. The corresponding special values of the parameter α are given in Table 3.1. The main result of this section is that exactly the same list of solutions can be obtained from the deformations of ramified covers of $\mathbb{P}^1$ with four ramification points. The corresponding values of the parameters α are different and are shown in Table 3.2, see Theorem 3.3.

Recall that an elliptic surface is a complex compact analytic surface S with a projection $S \to \mathbb{P}^1$, such that the general fiber $f^{-1}(z) = \Gamma_z$ is an elliptic curve. Two elliptic surfaces are equivalent if there is a bianalytic map compatible with the projections, see Kodaira [14].

We may suppose that the fiber Γ_z is written in the Weierstrass form

$$\Gamma_z = \left\{(x, y) \in \mathbb{C}^2 : y^2 = 4x^3 - g_2(z)x - g_3(z)\right\} \tag{3.1}$$

and consider the complete elliptic integrals of first and second kind:

$$\eta_1 = \int_{\gamma(z)} \frac{dx}{y}, \qquad \eta_2 = \int_{\gamma(z)} \frac{x\,dx}{y}, \tag{3.2}$$

where $\gamma(z) \subset \Gamma_z$ is a continuous family of closed loops (representing a locally constant section $z \mapsto H_1(\Gamma_z, \mathbb{Z})$ of the associated homology bundle). Then η_1, η_2 satisfy the following

Table 3.1 Solutions $(\lambda(t), \alpha)$ of PVI_α equations related to the Picard-Fuchs system (3.3).

Stabilizer of the solution	Name of the solution	PVI_α equation	Stabilizer of PVI_α equation
D_4	2A	$\left(0, 0, \frac{1}{18}, \frac{1}{18}\right)$ $\left(\frac{1}{18}, \frac{1}{18}, 0, 0\right)$	$S_2 \times S_2$
	2B	$\left(\frac{1}{18}, 0, \frac{1}{18}, 0\right)$ $\left(0, \frac{1}{18}, 0, \frac{1}{18}\right)$	
	2C	$\left(\frac{1}{18}, 0, 0, \frac{1}{18}\right)$ $\left(0, \frac{1}{18}, \frac{1}{18}, 0\right)$	
D_4	2A 2B 2C	$(0, 0, 0, 0)$	S_4
S_3	3A 3B 3C 3D	$(0, 0, 0, 0)$	S_4
S_3	4A	$\left(0, \frac{1}{8}, 0, 0\right)$	S_3
	4B	$\left(0, 0, \frac{1}{8}, 0\right)$	
	4C	$\left(\frac{1}{8}, 0, 0, 0\right)$	
	4D	$\left(0, 0, 0, \frac{1}{8}\right)$	
S_2	5A 5B 5C	$\left(0, \frac{1}{18}, 0, 0\right)$	S_3
	5D 5E 5F	$\left(0, 0, \frac{1}{18}, 0\right)$	

Table 3.1 Continued.

Stabilizer of the solution	Name of the solution	PVI_α equation	Stabilizer of PVI_α equation
S_2	5G		S_3
	5H	$\left(\frac{1}{18}, 0, 0, 0\right)$	
	5I		
	5J		
	5K	$\left(0, 0, 0, \frac{1}{18}\right)$	
	5L		

Picard-Fuchs system (this goes back at least to Griffiths [6], see Sasai [19]):

$$\triangle(z)\frac{d}{dz}\begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix} = \begin{pmatrix} -\dfrac{\triangle_z'}{12} & \dfrac{3\delta}{2} \\ -\dfrac{g_2\delta}{8} & \dfrac{\triangle_z'}{12}\end{pmatrix}\begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix}, \tag{3.3}$$

where

$$\begin{aligned} &\triangle(g_2, g_3) = g_2^3 - 27g_3^2, \\ &\delta(z) = 3g_3\frac{dg_2}{dz} - 2g_2\frac{dg_3}{dz}. \end{aligned} \tag{3.4}$$

The singular points of the system correspond to the singular fibers of the surface. The elliptic surfaces with four singular fibers were classified by Herfurtner [7] who obtained 50 distinct cases, but only 5 of them contain an additional parameter, see Table 3.3. They lead to nontrivial isomonodromic deformations of the above Picard-Fuchs system with four regular singular points. If we renormalize the singular points to be $0, 1, \infty, t$, then the zero λ of $\delta(z)$, considered as a function in t, is a solution of an appropriate PVI_α equation, see [18] for details. The result is summarized as follows.

Theorem 3.1 [3, Theorem 3.13]. All algebraic solutions $(\lambda(t), \alpha)$ of PVI_α equation coming from moduli of elliptic surfaces with four singular fibers are shown in Table 2.1. The corresponding values of α together with the stabilizer of the solution and the PVI_α equation under the action of the symmetric group S_4 are listed in Table 3.1. □

Table 3.2 Solutions $(\lambda(t), \alpha)$ of PVI_α equations related to the Picard-Fuchs system (3.9).

Stabilizer of the solution	Name of the solution	PVI_α equation	Stabilizer of PVI_α equation
D_4	2A	$\left(\frac{1}{8}, \frac{1}{8}, \frac{1}{18}, \frac{1}{18}\right)$ $\left(\frac{1}{18}, \frac{1}{18}, \frac{1}{8}, \frac{1}{8}\right)$	$S_2 \times S_2$
	2B	$\left(\frac{1}{18}, \frac{1}{8}, \frac{1}{18}, \frac{1}{8}\right)$ $\left(\frac{1}{8}, \frac{1}{18}, \frac{1}{8}, \frac{1}{18}\right)$	
	2C	$\left(\frac{1}{18}, \frac{1}{8}, \frac{1}{8}, \frac{1}{18}\right)$ $\left(\frac{1}{8}, \frac{1}{18}, \frac{1}{18}, \frac{1}{8}\right)$	
D_4	2A	$\left(\frac{1}{8}, \frac{1}{8}, \frac{1}{2}, \frac{1}{2}\right)$ $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{8}, \frac{1}{8}\right)$	$S_2 \times S_2$
	2B	$\left(\frac{1}{2}, \frac{1}{8}, \frac{1}{2}, \frac{1}{8}\right)$ $\left(\frac{1}{8}, \frac{1}{2}, \frac{1}{8}, \frac{1}{2}\right)$	
	2C	$\left(\frac{1}{2}, \frac{1}{8}, \frac{1}{8}, \frac{1}{2}\right)$ $\left(\frac{1}{8}, \frac{1}{2}, \frac{1}{2}, \frac{1}{8}\right)$	
S_3	3A	$\left(\frac{1}{8}, \frac{9}{8}, \frac{1}{8}, \frac{1}{8}\right)$	S_3
	3B	$\left(\frac{9}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}\right)$	
	3C	$\left(\frac{1}{8}, \frac{1}{8}, \frac{9}{8}, \frac{1}{8}\right)$	
	3D	$\left(\frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{9}{8}\right)$	
S_3	4A		S_4
	4B	$\left(\frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}\right)$	
	4C		
	4D		

Table 3.2 Continued.

Stabilizer of the solution	Name of the solution	PVI_α equation	Stabilizer of PVI_α equation
	5A	$\left(\frac{1}{2},\frac{1}{18},\frac{1}{8},\frac{1}{8}\right)$	
	5B	$\left(\frac{1}{8},\frac{1}{18},\frac{1}{2},\frac{1}{8}\right)$	
	5C	$\left(\frac{1}{8},\frac{1}{18},\frac{1}{8},\frac{1}{2}\right)$	
	5D	$\left(\frac{1}{2},\frac{1}{8},\frac{1}{18},\frac{1}{8}\right)$	
	5E	$\left(\frac{1}{8},\frac{1}{2},\frac{1}{18},\frac{1}{8}\right)$	
S_2	5F	$\left(\frac{1}{8},\frac{1}{8},\frac{1}{18},\frac{1}{2}\right)$	S_3
	5G	$\left(\frac{1}{18},\frac{1}{8},\frac{1}{2},\frac{1}{8}\right)$	
	5H	$\left(\frac{1}{18},\frac{1}{2},\frac{1}{8},\frac{1}{8}\right)$	
	5I	$\left(\frac{1}{18},\frac{1}{8},\frac{1}{8},\frac{1}{2}\right)$	
	5J	$\left(\frac{1}{8},\frac{1}{8},\frac{1}{2},\frac{1}{18}\right)$	
	5K	$\left(\frac{1}{8},\frac{1}{2},\frac{1}{8},\frac{1}{18}\right)$	
	5L	$\left(\frac{1}{2},\frac{1}{8},\frac{1}{8},\frac{1}{18}\right)$	

Remark 3.2. The Picard-Fuchs system (3.3) has generically an infinite monodromy group. More precisely, let $z \mapsto (g_2(z), g_3(z))$ be a curve intersecting transversally the discriminant locus $\{g_2^3 - 27g_3^2 = 0\}$ of Γ_z at some smooth point $(g_2(z_0), g_3(z_0))$. The Picard-Lefschetz formula implies that the monodromy of the first homology group $H_1(\Gamma_z, \mathbb{Z})$ along a small loop which makes one turn about z_0 is represented by the unipotent matrix

$$\begin{pmatrix} 1 & \pm 1 \\ 0 & 1 \end{pmatrix} \tag{3.5}$$

and hence the monodromy group of (3.3) is infinite. When the curve

$$z \longmapsto (g_2(z), g_3(z)) \tag{3.6}$$

is chosen as in Table 3.3, the result also follows from [7, Table 1].

Table 3.3 The Herfurtner list of “deformable” elliptic surfaces with four singular fibers.

Name	Deformation
1	$g_2(z,a) = 3(z-1)(z-a^2)^3$ $g_3(z,a) = (z-1)(z-a^2)^4(z+a)$
2	$g_2(z,a) = 12z^2(z^2+az+1)$ $g_3(z,a) = 4z^3(2z^3+3az^2+3az+2)$
3	$g_2(z,a) = 12z^2(z^2+2az+1)$ $g_3(z,a) = 4z^3(2z^3+3(a^2+1)z^2+6az+2)$
4	$g_2(z,a) = 3z^3(z+a)$ $g_3(z,a) = z^5(z+1)$
5	$g_2(z,a) = 3z^3(z+2a)$ $g_3(z,a) = z^4(z^2+3az+1)$

We will deduce a Picard-Fuchs system closely related to (3.3), but having a finite monodromy. Consider a ramified covering $\Gamma \to \mathbb{P}^1$ of degree three with branching locus consisting of four points, where Γ is a Riemann surface. We choose an affine model $\Gamma^{\mathrm{aff}} = \{(x,z) \in \mathbb{C}^2 : f(x,z) = 0\}$ of Γ, where $f(x,z) = 4x^3 - g_2x - g_3$, and $g_2 = g_2(z)$, $g_3 = g_3(z)$ are suitable polynomials. Moreover, without loss of generality, we suppose that the covering $\Gamma \to \mathbb{P}^1$ is induced from the projection

$$\{(x,z) \in \mathbb{C}^2 : f(x,z) = 0\} \longrightarrow \mathbb{C} : (x,z) \longmapsto z. \tag{3.7}$$

Let $x_1(z), x_2(z)$ be two distinct roots of f. Then $\gamma(z) = x_1(z) - x_2(z)$ is a 0-cycle of the fiber $\{x : f(x,z) = 0\}$ and the Abelian integrals above are replaced by the algebraic functions

$$\eta_1(z) = \int_{\gamma(z)} x = x_1(z) - x_2(z), \qquad \eta_2(z) = \int_{\gamma(z)} x^2 = x_1^2(z) - x_2^2(z). \tag{3.8}$$

A straightforward computation shows that η_1, η_2 satisfy the following Picard-Fuchs system (see [5]):

$$\triangle(z)\frac{d}{dz}\begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix} = \begin{pmatrix}\frac{\triangle_z'}{6} & -3\delta\\ -\frac{g_2\delta}{2} & \frac{\triangle_z'}{3}\end{pmatrix}\begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix}. \tag{3.9}$$

As before, if we renormalize the system (3.9) to have singular points at $0, 1, t, \infty$, then the root $\lambda(t)$ of $\delta(z)$ is a solution of a suitable PVI_α equation, provided that the deformation is isomonodromic. The last property holds, strictly speaking, in the case when the system is nonresonant. In our case it holds too, because the deformation is isoprincipal in the sense of [11]. This can be also checked by a direct computation. Thus, if we consider the fibration (3.7) and take for g_2, g_3 the expressions found by Herfutner, see Table 3.3, we get the 23 algebraic solutions 2A–5L shown in Table 2.1. The corresponding values for α are different because the monodromy group of (3.9) is finite, see Remark 3.2. They are computed in Table 3.2. In this way we proved the following theorem.

Theorem 3.3. The algebraic solutions $(\lambda(t), \alpha)$ of PVI_α equation coming from the deformations of the covering (3.7) with g_2, g_3 as on the Herfutner list, Table 3.3, are shown in Table 2.1. The corresponding values of α together with the stabilizer of the solution and the PVI_α equation under the action of the symmetric group S_4 are listed in Table 3.2. □

Remark 3.4. Particular cases of the Picard-Fuchs system (3.9), in a more or less explicit way, were considered by many authors, for example, Boalch [2], Dubrovin and Mazzocco [4], Hitchin [8, 9], and Kitaev [13].

To this end, for the convenience of the reader, we explain how the solution 4A with $\alpha = (1/8, 1/8, 1/8, 1/8)$ follows from (3.9) and Table 3.3. The remaining solutions in Table 3.2 are computed in a similar way.

The Picard-Fuchs system (3.9) implies that the Abelian integral of first kind η_1 satisfies the following equation:

$$p_0(z,a)\eta_1'' + p_1(z,a)\eta_1' + p_2(z,a)\eta_1 = 0, \tag{3.10}$$

where

$$\begin{aligned} p_0(z,a) &= 144\delta\Delta^2, \\ p_1(z,a) &= 144\Delta\left(\delta\frac{d\Delta}{dz} - \Delta\frac{d\delta}{dz}\right), \\ p_2(z,a) &= 12\delta\frac{d^2\Delta}{dz^2} - 216\delta^3 g_2 - 12\Delta\frac{d\delta}{dz}\frac{d\Delta}{dz} - \delta\left(\frac{d\Delta}{dz}\right)^2. \end{aligned} \tag{3.11}$$

Consider, for instance, the deformation 2 from the Herfurtner list (Table 3.3)

$$\begin{aligned} g_2 &= g_2(z,a) = 3z^3(z+a), \\ g_3 &= g_3(z,a) = z^5(z+1). \end{aligned} \tag{3.12}$$

We have

$$\begin{aligned} \Delta &= \Delta(g_2, g_3) = 27z^9\big((3a-2)z^2 + \big(3a^2-1\big)z + a^3\big), \\ \delta &= \delta(z,a) = -3z^7\big((3a-2)z + a\big). \end{aligned} \tag{3.13}$$

The Picard-Fuchs equation (3.10) takes the form

$$\begin{aligned} &144z^2\big((3a-2)z+a\big)\big((3a-2)z^2 + \big(3a^2-1\big)z + a^3\big)^2\eta_1'' \\ &\quad + 144z\big((3a-2)z^2 + \big(3a^2-1\big)z + a^3\big) \\ &\quad \times \big(3(3a-2)^2z^3 + 2(3a-2)(3a-1)(a+1)z^2 \\ &\qquad + a\big(3a^3+7a^2-3\big)z + 2a^4\big)\eta_1' \\ &\quad + \big[135(3a-2)^3z^5 + (3a-2)^2\big(468a^2+267a-164\big)z^4 \\ &\qquad + 2(3a-2)\big(189a^4+522a^3-48a^2-208a+10\big)z^3 \\ &\qquad - 2a\big(270a^5-1269a^4+252a^3+460a^2-70\big)z^2 \\ &\qquad - a^4\big(243a^3-666a^2+176\big)z + 27a^7\big]\eta_1 = 0 \end{aligned} \tag{3.14}$$

and has four regular singular points at ∞ and the roots of $(3a-2)z^2 + (3a^2-1)z + a^3$ (the roots of Δ), as well one apparent singularity at the root of $(3a-2)z + a$ (which is a root of δ). Renormalizing the singular points to $0, 1, t, \infty$, we get

$$\lambda = \frac{a^2-a+1}{a^2(2-a)}, \quad t = \frac{2a-1}{a^3(2-a)}, \quad a \in \mathbb{C}. \tag{3.15}$$

The parameter a defines an algebraic isomonodromic deformation of the Picard-Fuchs equation (3.10) with the Riemann schema

$$\begin{pmatrix} 0 & 1 & t & \lambda & \infty \\ -\frac{3}{4} & \frac{1}{4} & \frac{1}{4} & 0 & \frac{5}{4} \\ -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & 2 & \frac{3}{4} \end{pmatrix}. \tag{3.16}$$

Therefore the algebraic function $\lambda = \lambda(t)$ determined implicitly by (3.15) is an algebraic solution of PVI_α equation with

$$\alpha_0 = \frac{1}{8}, \qquad \alpha_1 = \frac{1}{8}, \qquad \alpha_2 = \frac{1}{8}, \qquad \alpha_3 = \frac{1}{8}. \tag{3.17}$$

(see [10] for details). Eliminating a from (3.15), we get

$$\lambda^4 - 2t\lambda^3 - 2\lambda^3 + 6t\lambda^2 - 2t^2\lambda - 2t\lambda + t^3 - t^2 + t = 0 \tag{3.18}$$

which is an equation for the solution $4A$ with $\alpha = (1/8, 1/8, 1/8, 1/8)$. Together with the Doran point $\alpha = (0, 1/8, 0, 0)$, this implies that the solution $\lambda(t)$ satisfies also the implicit equation (2.1):

$$1 + \frac{t-1}{(\lambda-1)^2} - \frac{t(t-1)}{(\lambda-t)^2} = 0 \tag{3.19}$$

corresponding to the affine line through $(1/8, 1/8, 1/8, 1/8)$ and $(0, 1/8, 0, 0)$ described in Table 2.1, $4A$. The solution (3.15) with $\alpha = (1/8, 1/8, 1/8, 1/8)$ was found by Hitchin [8, Section 6.1] and [9, (34)].

Remark 3.5. If we repeat the same computation, but making use of the Picard-Fuchs system (3.3), then of course we obtain the same algebraic solution but with $\alpha = (1/8, 0, 0, 0)$. This value has been erroneously computed by Doran [3] to be $(1/18, 0, 0, 0)$. This led him to the wrong conclusion that the solution $4C$ is equivalent by an Okamoto transformation to the "cubic" solution B_3 of Dubrovin and Mazzocco [4, page 140] with $\alpha = (25/18, 0, 0, 0)$, see [3, Remark 7]. As the Okamoto transformations of PVI_α act within the ring $\mathbb{Z}[1/2, \sqrt{2\alpha_1}, \sqrt{2\alpha_1}, \sqrt{2\alpha_1}, \sqrt{2\alpha_1}]$, then no solution of $PVI_{(25/18,0,0,0)}$ is equivalent to a solution of $PVI_{(1/8,0,0,0)}$. Mazzocco kindly informed us for a missprint in the formula for the B_3-solution, [4, page 140]. The corrected formula is reproduced in [3, Formula (3.1)].

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