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Two-dimensional Fuchsian systems and the Chebyshev property ☆

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Abstract

Let $(x(t), y(t))^\top$ be a solution of a Fuchsian system of order two with three singular points. The vector space of functions of the form $P(t)x(t) + Q(t)y(t)$, where P, Q are real polynomials, has a natural filtration of vector spaces, according to the asymptotic behavior of the functions at infinity. We describe a two-parameter class of Fuchsian systems, for which the corresponding vector spaces obey the Chebyshev property (the maximal number of isolated zeros of each function is less than the dimension of the vector space). Up to now, only a few particular systems were known to possess such a non-oscillation property. It is remarkable that most of these systems are of the type studied in the present paper. We apply our results in estimating the number of limit cycles that appear after small polynomial perturbations of several quadratic or cubic Hamiltonian systems in the plane.

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1. Introduction

In many bifurcation problems the main difficulty is to estimate the number of isolated zeros of certain functions of the form

$$I(h) = p_1(h)I_1(h) + p_2(h)I_2(h), \quad h \in \Sigma, \tag{1}$$

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where $p_1(h)$ and $p_2(h)$ are polynomials, and the vector function $\mathbf{I}(h) = (I_1(h), I_2(h))^\top$ satisfies a two-dimensional first-order Fuchsian system

$$\mathbf{I}(h) = \mathbf{A}(h)\mathbf{I}'(h), \quad ' = d/dh, \tag{2}$$

with a first-degree polynomial matrix $\mathbf{A}(h)$. Typically, $I_1(h)$ and $I_2(h)$ are complete Abelian integrals along the ovals $\delta(h)$ within a continuous (in h) family of ovals contained in the level sets of a fixed real polynomial $H(x, y)$ (called the Hamiltonian), and $\Sigma \subset \mathbb{R}$ is the maximal open interval of existence of such ovals $\delta(h)$, see Table 1.

In the present paper, our main assumptions on (2) are the following:

(H1) $\mathbf{A}'$ is a constant matrix having real distinct eigenvalues.
(H2) The equation $\det \mathbf{A}(h) = 0$ has real distinct roots h_0, h_1 and the identity $\operatorname{trace} \mathbf{A}(h) \equiv (\det \mathbf{A}(h))'$ holds.
(H3) $\mathbf{I}(h)$ is analytic in a neighborhood of h_0.

The conditions that $\mathbf{A}'$ is a constant matrix and $\det \mathbf{A}(h)$ has distinct roots imply that the singular points of the system

$$\mathbf{I}'(h) = \mathbf{A}^{-1}(h)\mathbf{I}(h)$$

(including ∞) are regular, i.e. it is of Fuchs type. Further, the condition $\operatorname{trace} \mathbf{A}(h) \equiv (\det \mathbf{A}(h))'$ implies that the characteristic exponents of (2) at h_0 and h_1 are $\{0, 1\}$. In the formulation of our main result below, we assume for definiteness that $h_0 < h_1$. A similar result holds if $h_0 > h_1$. Clearly if $h_0 < h_1$, and the function $\mathbf{I}(h)$ is analytic in a neighborhood of $h = h_0$, then it also possesses an analytic continuation in the complex domain $\mathbb{C} \backslash [h_1, \infty)$.

Definition 1. The real vector space of functions V is said to be Chebyshev in the complex domain $\mathcal{D} \subset \mathbb{C}$ provided that every function $I \in V \backslash \{0\}$ has at most $\dim V - 1$ zeros in $\mathcal{D}$. V is said to be Chebyshev with accuracy k in $\mathcal{D}$ if any function $I \in V \backslash \{0\}$ has at most $k + \dim V - 1$ zeros in $\mathcal{D}$.

Definition 2. Let $I(h)$, $h \in \mathbb{C}$ be a function, locally analytic in a neighborhood of ∞, and $s \in \mathbb{R}$. We shall write $I(h) \lesssim h^s$, provided that for every sector S centered at ∞ there exists a non-zero constant C_S such that $|I(h)| \leqslant C_S |h|^s$ for all sufficiently big $|h|$, $h \in S$.

For systems (2) satisfying (H1) and (H2), the characteristic exponents at infinity are $-\lambda$ and $-\mu$ where $\lambda' = 1/\lambda$ and $\mu' = 1/\mu$ are the eigenvalues of the constant matrix $\mathbf{A}'$. According to (H2), $\lambda + \mu = 2$. Let us denote $\lambda^* = 2$ if λ is integer and $\lambda^* = \max(|\lambda - 1|, 1 - |\lambda - 1|)$ otherwise.

Table 1
Examples of systems for integrals $\mathbf{I} = (I_1, I_2)$ and Hamiltonian functions H which satisfy hypotheses (H1)–(H3)

No.	$H, \mathbf{I}$	Σ	$\mathbf{A}$	$\det \mathbf{A}$, $\operatorname{tr} \mathbf{A}$
1	$H = y^2 + x^2 - x^3$ $\mathbf{I} = (\int_{H=h} y\,dx, \int_{H=h} xy\,dx)$	$(0, \frac{4}{27})$	$\begin{pmatrix} \frac{6}{5}h & -\frac{4}{15} \\ \frac{4}{35}h & \frac{6}{7}h - \frac{16}{105} \end{pmatrix}$	$\frac{36}{35}h^2 - \frac{16}{105}h$ $\frac{72}{35}h - \frac{16}{105}$
2	$H = y^2 + x^2 - xy^2$ $\mathbf{I} = (\int_{H=h} y\,dx, \int_{H=h} xy\,dx)$	$(0, 1)$	$\begin{pmatrix} \frac{4}{3}h & -\frac{4}{3} \\ \frac{4}{15}h & \frac{4}{5}h - \frac{16}{15} \end{pmatrix}$	$\frac{16}{15}h^2 - \frac{16}{15}h$ $\frac{32}{15}h - \frac{16}{15}$
3	$H = \frac{1}{2}y^2 + \frac{1}{2}x^2 - \frac{1}{3}x^3 + xy^2$ $\mathbf{I} = (\int_{H=h} y\,dx, \int_{H=h} x^2y\,dx)$	$(0, \frac{1}{6})$	$\begin{pmatrix} \frac{3}{2}h & -\frac{1}{2} \\ \frac{3}{16}h & \frac{3}{4}h - \frac{3}{16} \end{pmatrix}$	$\frac{9}{8}h^2 - \frac{3}{16}h$ $\frac{9}{4}h - \frac{3}{16}$
4	$H = y^2 + x^2 + x^4$ $\mathbf{I} = (\int_{H=h} y\,dx, \int_{H=h} x^2y\,dx)$	$(0, \infty)$	$\begin{pmatrix} \frac{4}{3}h & -\frac{2}{3} \\ -\frac{2}{15}h & \frac{4}{5}h + \frac{4}{15} \end{pmatrix}$	$\frac{16}{15}h^2 + \frac{4}{15}h$ $\frac{32}{15}h + \frac{4}{15}$
5	$H = y^2 + x^2 - x^4$ $\mathbf{I} = (\int_{H=h} y\,dx, \int_{H=h} x^2y\,dx)$	$(0, \frac{1}{4})$	$\begin{pmatrix} \frac{4}{3}h & -\frac{2}{3} \\ \frac{2}{15}h & \frac{4}{5}h - \frac{4}{15} \end{pmatrix}$	$\frac{16}{15}h^2 - \frac{4}{15}h$ $\frac{32}{15}h - \frac{4}{15}$
6	$H = y^2 + x^2 + x^2y^2$ $\mathbf{I} = (\int_{H=h} y\,dx, \int_{H=h} x^2y\,dx)$	$(0, \infty)$	$\begin{pmatrix} 2h & -2 \\ -\frac{2}{3}h & \frac{2}{3}h + \frac{4}{3} \end{pmatrix}$	$\frac{4}{3}h^2 + \frac{4}{3}h$ $\frac{8}{3}h + \frac{4}{3}$
7	$H = y^2 + x^2 - x^2y^2$ $\mathbf{I} = (\int_{H=h} y\,dx, \int_{H=h} x^2y\,dx)$	$(0, 1)$	$\begin{pmatrix} 2h & -2 \\ \frac{2}{3}h & \frac{2}{3}h - \frac{4}{3} \end{pmatrix}$	$\frac{4}{3}h^2 - \frac{4}{3}h$ $\frac{8}{3}h - \frac{4}{3}$
8	$H = x^{-3}(y^2 - 2x^2 + x)$ $\mathbf{I} = (\int_{H=h} x^{-3}y\,dx, \int_{H=h} x^{-4}y\,dx)$	$(-1, 0)$	$\begin{pmatrix} \frac{4}{3}h & \frac{4}{3} \\ \frac{4}{15}h & \frac{4}{5}h + \frac{16}{15} \end{pmatrix}$	$\frac{16}{15}h^2 + \frac{16}{15}h$ $\frac{32}{15}h + \frac{16}{15}$

Take $s \geqslant \lambda^*$ and consider the real vector space of functions

$$V_s = \{I(h) = P(h)I_1(h) + Q(h)I_2(h):\ P, Q \in \mathbb{R}[h], I(h) \lesssim h^s\},$$

where $\mathbf{I} = (I_1(h), I_2(h))^\top$ is a non-trivial solution of (2), holomorphic in a neighborhood of $h = h_0$. As $\lambda, \mu \notin \{0, 1, 2\}$, the vector function $\mathbf{I}(h)$ is uniquely determined, up to multiplication by a constant, and $I_1(h_0) = I_2(h_0) = 0$ (see Proposition 1). Clearly, V_s is invariant under linear transformations in (2) and affine changes of the argument h. The restriction $s \geqslant \lambda^*$ is taken to guarantee that V_s is not empty.

Recall that $h_0 < h_1$ are the roots of $\det A(h) = 0$. Our main result in Section 2 is the following.

Theorem 1. *Assume that conditions* (H1)–(H3) *hold. If* $\lambda \notin \mathbb{Z}$, *then* V_s *is a Chebyshev vector space with accuracy* $1 + [\lambda^*]$ *in the complex domain* $\mathcal{D} = \mathbb{C} \setminus [h_1, \infty)$. *If* $\lambda \in \mathbb{Z}$, *then* V_s *coincides with the space of real polynomials of degree at most* $[s]$ *which vanish at* h_0 *and* h_1.

As an application of Theorem 1 let us consider a polynomial perturbation of a planar Hamiltonian system

$$\dot{x} = H_y + \varepsilon f(x, y),$$

$$\dot{y} = -H_x + \varepsilon g(x, y), \tag{3}$$

where ε is a small parameter, the degree of the polynomials f, g does not exceed n and H is some of the Hamiltonians from Table 1. Define the function

$$h \to I(h) = \oint_{H=h} [g(x, y)\, dx - f(x, y)\, dy], \quad h \in \Sigma. \tag{4}$$

As is well known, if $I(h) \not\equiv 0$ in Σ, then the number of limit cycles in (3) bifurcating for small ε from the periodic orbits of the unperturbed Hamiltonian system is bounded by the number of isolated zeros of $I(h)$ in Σ. Define the linear space $\mathcal{V}_n$ of integrals given by (4) for $\deg f, g \leqslant n$. Denote by h_1 the non-zero critical value of the Hamiltonian and by $\mathcal{D}$ the complex plane cut along the part of the real axis between h_1 and ∞ not containing the other critical value $h_0 = 0$. Then applying Theorem 1, we obtain the following results.

Theorem 2. *For each of systems* 1–5 *in Table* 1, *the linear space of integrals* $\mathcal{V}_n$ *is Chebyshev with accuracy one in* $\mathcal{D}$. *In particular*, $\mathcal{V}_n$ *is Chebyshev in* Σ.

Theorem 3. *For systems* 6 *and* 7 *in Table* 1, *the linear space of integrals* $\mathcal{V}_n$ *is Chebyshev with accuracy one in* $\mathcal{D}$, *if* $n \leqslant 6$, *and with accuracy* $[\frac{n+1}{4}]$, *if* $n \geqslant 7$. *In particular*, $\mathcal{V}_n$ *is Chebyshev in* Σ, *if* $n \leqslant 6$, *and Chebyshev with accuracy* $[\frac{n-3}{4}]$, *if* $n \geqslant 7$.

Roughly speaking, Theorems 2 and 3 imply that, for systems 1–7 from Table 1, the number of limit cycles in (3) born out of periodic orbits under small polynomial perturbations which are transversal to the integrable directions, is less than the dimension of the linear space of these perturbations (with certain accuracy if $n \geqslant 7$ in cases 6 and 7). Clearly, a bound obtained by establishing the Chebyshev property, is always the optimal one.

Case 8 from Table 1 is non-Hamiltonian one and requires slightly different approach. See the end of the paper for results about it.

Let us recall that Theorem 2 in case 1 was proved earlier by Petrov [8]. Some less general (or a little bit different) results concerning cases 3–5 can be found in [2,6,7,9].

2. The Chebyshev property

We intend first to obtain a normal form for the matrices satisfying (H1) and (H2). For this purpose, we perform in (2) a linear transformation bringing $\mathbf{A}'$ to a diagonal form and then translate the critical value h_0 to the origin. The matrix in (2) takes the form

$$\mathbf{A}(h) = \begin{pmatrix} \dfrac{2h-h_1}{2\lambda} & \dfrac{\omega h_1}{2\lambda} \\ \dfrac{h_1}{2\mu\omega} & \dfrac{2h-h_1}{2\mu} \end{pmatrix}, \tag{5}$$

where h_1 is the non-zero critical value and ω is a free parameter. This is the normal form we will use in this section. In applications, another normal form takes place. To obtain it, we apply additional linear transformation in (5) $(I_1, I_2) \to (I_1, I_1/\omega + I_2)$ bringing $\mathbf{A}(h)$ to

$$\mathbf{A}(h) = \begin{pmatrix} \dfrac{h}{\lambda} & \dfrac{\omega h_1}{2\lambda} \\ \dfrac{(\lambda-\mu)h}{\lambda\mu\omega} & \dfrac{h}{\mu} - \dfrac{h_1}{\lambda\mu} \end{pmatrix}. \tag{6}$$

Evidently, Eqs. (5) and (6) present three-parameter families of matrices which can be reduced to two-parameter ones by moving h_1 to 1. We note that all the examples in Table 1 are taken in the normal form (6), with $\frac{1}{2} \leqslant \lambda < \mu \leqslant \frac{3}{2}$.

Prior to proving Theorem 1, we need some preparation. Without any loss of generality, we may use the normal form (5), with $h_1 = 1$. Hence, we will consider $t = (h - h_0)/(h_1 - h_0)$ as the argument and will assume throughout this section that (2) is rewritten as a system $\mathbf{I}(t) = \mathbf{A}(t)\mathbf{I}'(t)$ for $\mathbf{I}(t) = (x(t), y(t))^\top \equiv (I_1(h), I_2(h))^\top$, with

$$\mathbf{A}(t) = \begin{pmatrix} \dfrac{2t-1}{2\lambda} & \dfrac{\omega}{2\lambda} \\ \dfrac{1}{2\mu\omega} & \dfrac{2t-1}{2\mu} \end{pmatrix}. \tag{7}$$

Proposition 1. *The functions $x(t) = I_1(h)$ and $y(t) = I_2(h)$ satisfy equations*

$$t(t-1)x'' = \lambda(\lambda-1)x, \tag{8}$$

$$t(t-1)y'' = \mu(\mu-1)y. \tag{9}$$

Proof. The most easy proof is a straightforward calculation which we have left to the reader (cf. [9]). $\square$

Proposition 2. *Let $\lambda \neq 0, 1$ and $x(t)$ be a non-trivial solution of* (8) *which is analytic in a neighborhood of $t = 0$ (or $t = 1$). Then $x(t) \neq 0$ for $t < 0$ (respectively, for $t > 1$). In particular, if $\lambda \in \mathbb{Z}$, then $x(t)$ is a special kind of ultra-spherical polynomial and has all of its zeros in the interval $[0, 1]$.*

Proof. The assertion is well known for λ integer. In this case $x(t)$ is a kind of ultra-spherical (Gegenbauer) polynomial [1] of degree λ if $\lambda \geqslant 2$ and of degree $1 - \lambda$ if $\lambda \leqslant -1$. Although the result might be known for λ which is not integer too, we will for completeness give the proof for this case. Let $x(t)$ be analytic near $t = 0$ (the other case is similar). Take the function

$$z(t) = \frac{t^2 - t}{2 - \lambda} x' + \frac{1 - \lambda t}{2 - \lambda} x.$$

Then $z' = tx' - \lambda x$ and x', z' together satisfy a system

$$(t^2 - t)x'' = (\lambda - 1)(tx' - z'),$$

$$(t^2 - t)z'' = (\lambda - 1)(tx' - tz').$$

As $x'(0) \neq 0$ and $z'(0) = 0$, the ratio $w = z'/x'$ is an analytical function in a neighborhood of $t = 0$ satisfying the Riccati equation

$$\frac{t^2 - t}{\lambda - 1} w'(t) = w^2(t) - 2tw(t) + t$$

and $w(0) = 0$. Consider in the (t, w)-plane the zero isocline given by the hyperbola $w^2 - 2tw + t = 0$. It goes through the origin and has a vertical asymptote at that point. It is easy to conclude that for $t < 0$, the graphic of w is placed inside the left branch of the hyperbola and either $w(t) > 0$ or $w(t) < 0$ for all $t < 0$, depending on whether $w'(0)$ is negative or positive. Therefore $x'(t)$ and $z'(t)$ do not change signs for $t < 0$. As $x(0) = 0$, the assertion follows.

Proposition 3. *Let $\lambda < 1$ and $x(t)$ be a non-trivial solution of* (8) *which is analytic in a neighborhood of $t = 0$. If $\lambda \notin \mathbb{Z}$, then $x(t)$ has at most $1 + [\lambda^*]$ zeros in the complex domain $D = \mathbb{C} \backslash [1, \infty)$.*

Proof. Consider the analytic continuation of $x(t)$ in the complex domain $D = \mathbb{C} \backslash [1, \infty)$. We shall count the zeros of $x(t)$ in D by making use of the argument principle. Let R be a big enough constant and r a small enough constant. Denote by $\tilde{D}$ the set obtained by removing the small disc $\{|t - 1| < r\}$ from $D \cap \{|t| < R\}$. To estimate the number of the zeros of $x(t)$ in $\tilde{D}$, we shall evaluate the increment $\Delta_{\partial \tilde{D}} \mathrm{Arg}\, x(t)$ of the argument of the function $x(t)$ along the boundary of $\tilde{D}$, traversed in a positive direction. Then, according to the argument principle, we have that the number of the zeros of $x(t)$ in $\tilde{D}$ equals

$$\frac{\Delta_{\partial\tilde{D}} \operatorname{Arg} x(t)}{2\pi}.$$

The monodromy group of the equation in (8) is reducible if and only if $\lambda \in \mathbb{Z}$ [5, Theorem 4.3.2]. Therefore, if $\lambda \notin \mathbb{Z}$, then in a neighborhood of $t = 1$ we have

$$x(t) = \xi(t) \log(t-1) + \eta(t),$$

where $\xi(t)$, $\eta(t)$ are analytic in a neighborhood of $t = 1$, $\xi(t)$ is a non-trivial solution of (8), $\xi(1) = 0$. Moreover, a local analysis shows that $\lim_{t \to 1^-} x(t) = \eta(1) = \text{const} \neq 0$. Therefore the increase of the argument of $x(t)$, when running the boundary of $\{|t-1| < r\}$, is close to zero. Along the half-line $(1, \infty)$ the imaginary part of $x(t)$ equals $\pi\xi(t)$ which does not vanish, by Proposition 2. Finally, if $|t|$ is sufficiently big then we have

$$|x(t)| \leqslant c|t|^{\lambda} \quad \text{if } \lambda > \tfrac{1}{2}, \qquad |x(t)| \leqslant c|t|^{1-\lambda} \text{ if } \lambda < \tfrac{1}{2},$$

$$|x(t)| \leqslant c|t^{\frac{1}{2}} \log t| \quad \text{if } \lambda = \tfrac{1}{2},$$

where c is a non-zero constant. The increase of the argument of $x(t)$, when running the boundary of $\{|t| < R\}$ is close to $2\pi\lambda^*$. Summing up the above information, we obtain that the increase of the argument of $x(t)$, when running the boundary of D, is at most $2\pi + 2\pi\lambda^*$. We conclude that $x(t)$ has at most $1 + [\lambda^*]$ zeros in D which completes the proof of Proposition 3.

We also need a more detailed information about the structure of the linear space V_s and an explicit formula for $\dim V_s$. The only interesting case is when λ and μ are not integer.

Proposition 4. *Let $s \geqslant \lambda^*$ and λ, μ be not integer. Then*

$$\dim V_s = \begin{cases} 2s-1 & \text{if } \lambda-\mu \text{ and } s-\frac{1}{2} \text{ are integer,} \\ [s-\lambda]+[s-\mu]+2 & \text{otherwise.} \end{cases}$$

Proof. Without loss of generality, we can use the coordinates in which $\mathbf{A}$ takes form (7) and $\mathbf{I}(h) = (x(t), y(t))^{\top}$. To reduce the number of cases, let us assume that $\lambda > \mu$ (when $\lambda < \mu$, the analysis is similar).

We begin our analysis with the case when $s \geqslant \lambda$. Assume first that $\lambda - \mu$ is not integer. Then one can take any solution of (2) near infinity in the form

$$\mathbf{I} = \begin{pmatrix} x \\ y \end{pmatrix} = a \begin{pmatrix} t^{\lambda} - \frac{\lambda}{2} t^{\lambda-1} + \cdots \\ \alpha t^{\lambda-1} + \cdots \end{pmatrix} + b \begin{pmatrix} \beta t^{\mu-1} + \cdots \\ t^{\mu} - \frac{\mu}{2} t^{\mu-1} + \cdots \end{pmatrix},$$

where

$$\alpha = \frac{\lambda}{2\omega(\mu - \lambda + 1)}, \qquad \beta = \frac{\mu\omega}{2(\lambda - \mu + 1)}.$$

Since $\mathbf{I}$ is analytic in a neighborhood of zero, the constants a and b are both non-zero. Indeed, if $ab = 0$ then $\mathbf{I}$ defines a one-dimensional subspace in the space of all solutions, which is invariant under the monodromy group of (2), and hence of (8) and (9). This is however impossible, as the latter groups are irreducible for $\lambda, \mu \notin \mathbb{Z}$.

Given $s \geqslant \lambda$, then the function $I(h)$ in the definition of V_s contains monomials of the form $t^k x$, $0 \leqslant k \leqslant K$, $t^l y$, $0 \leqslant l \leqslant L$, where $K \leqslant \min(s - \lambda, s - \mu + 1)$, $L \leqslant \min(s - \lambda + 1, s - \mu)$. Using that $\lambda + \mu = 2$ and $\lambda > \mu$, one obtains $K \leqslant s - \lambda + \min(0, 2\lambda - 1) = s - \lambda$. Similarly, $L \leqslant s - \mu + \min(0, 2\mu - 1) = s - \lambda + 1$ if $\lambda - \mu > 1$ and $L \leqslant s - \mu$ otherwise.

Among these monomials, other special combinations may be involved in V_s if $\lambda - \mu > 1$. Define the functions $z_1 = ty - \alpha_1 x$, $z_m = tz_{m-1} - \alpha_m x$, $m \geqslant 2$, where $\alpha_1 = \alpha$ and the constant α_m is determined so that the coefficient at t^λ in z_m is zero. Denote $M = [s - \mu] - K - 1$. Clearly, then $t^{K+1} z_m \in V_s$ for $1 \leqslant m \leqslant M$. Moreover, any combination $t^{K+1}(P(t)x + Q(t)y)$ which belongs to V_s is a linear combination of the "monomials" $t^{K+1} z_m$.

Thus, $\dim V_s = K + L + 2$ for $|\lambda - \mu| < 1$ and $\dim V_s = K + L + M + 2$ otherwise, which yields $\dim V_s = [s - \lambda] + [s - \mu] + 2$ in both cases.

Assume now that $\lambda - \mu$ is integer but λ and μ are not. If $\lambda - \mu > 1$, then one can take any solution of (2) near infinity in the form

$$\mathbf{I} = a\begin{pmatrix} t^\lambda - \frac{\lambda}{2} t^{\lambda - 1} + \cdots \\ \alpha t^{\lambda - 1} + \cdots \end{pmatrix} + (a\gamma \log t + b)\begin{pmatrix} \beta t^{\mu - 1} + \cdots \\ t^\mu - \frac{\mu}{2} t^{\mu - 1} + \cdots \end{pmatrix}, \quad \gamma \neq 0.$$

As in the previous case, this yields $K = [s - \lambda]$, $L = [s - \lambda + 1] = K + 1$ and $M = [s - \mu] - K - 1$ if $s - \frac{1}{2}$ is not integer, $M = s - \mu - K - 2$ if $s - \frac{1}{2}$ is integer. In the first case we obtain the same result as above, and in the second case $\dim V_s = K + L + M + 2 = 2s - 1$.

Finally, if $\lambda = \frac{3}{2}$, $\mu = \frac{1}{2}$, we have respectively

$$\mathbf{I} = a\begin{pmatrix} t^{\frac{3}{2}} - \frac{3}{4} t^{\frac{1}{2}} - \frac{9}{64} t^{-\frac{1}{2}} + \cdots \\ \frac{3}{8\omega} t^{-\frac{1}{2}} + \cdots \end{pmatrix} + \left(-\frac{3a}{4\omega} \log t + b\right)\begin{pmatrix} \frac{\omega}{8} t^{-\frac{1}{2}} + \cdots \\ t^{\frac{1}{2}} - \frac{1}{4} t^{-\frac{1}{2}} + \cdots \end{pmatrix}.$$

Clearly $K = [s - \frac{3}{2}]$, $L = [s - \frac{1}{2}]$ if $s - \frac{1}{2}$ is not integer and $L = K$ otherwise. Since no other combinations are involved in V_s in this case, the result follows immediately.

In the case when $\lambda > s \geqslant \lambda^*$, the analysis is simpler. We use the same formulas for $\mathbf{I}$ as above. One has either (a) $\lambda > s \geqslant \lambda - 1$ and $\lambda - \mu \geqslant 1$, or (b) $\lambda > s \geqslant \mu$ and $\lambda - \mu < 1$. If $s = \mu = \frac{1}{2}$, then V_s is empty. In all other cases, $y \in V_s$. In case (b), V_s contains no other functions. In case (a), if $s \geqslant \mu + 1$ and $\lambda - \mu$ is not integer, then also $z_m \in V_s$

for $1 \leqslant m \leqslant [s-\mu]$. The same is true if $\lambda-\mu$ is integer but $s-\frac{1}{2}$ is not. Finally, if both $\lambda-\mu$ and $s-\frac{1}{2}$ are integer, and $s \geqslant \mu+2$, then V_s contains the functions z_m, $1 \leqslant m \leqslant [s-\mu]-1$. Clearly, in all the cases above we obtain a formula for $\dim V_s$ as asserted.

Proof of Theorem 1. For integer λ, μ the assertion is obvious since I_1 and I_2 are different ultra-spherical polynomials which have no common zeros except the simple ones at h_0 and h_1.

Assume below that $\lambda, \mu \notin \mathbb{Z}$ and let $\lambda > \mu$ (for definiteness). Suppose as before that the matrix $\mathbf{A}$ takes form (7), and let $I(t) = P(t)x(t) + Q(t)y(t) \in V_s$, where $(x(t) = I_1(h), y(t) = I_2(h))$ is the holomorphic solution of (2) vanishing at the origin. When $P(t) \equiv 0$, the assertion is evident. When $P \not\equiv 0$, we use again the argument principle to count the zeros of $I(t)$ in $D = \mathbb{C} \backslash [1, \infty)$. Consider in D the meromorphic function

$$F(t) = P(t)\frac{x(t)}{y(t)} + Q(t).$$

Below we calculate the increase of its argument when running the boundary of D. The local structure of the solutions of (8), (9) in a neighborhood of $t=1$ implies that $\lim_{t\to 1} x(t) \neq 0$, $\lim_{t\to 1} y(t) \neq 0$. Therefore the increase of the argument of $F(t)$, when running the boundary of $\{|t-1|<r\}$ is close to zero. As $x(t), y(t)$ are real-analytic on $(-\infty, 1)$, then along the half-line $(1, \infty)$

$$\operatorname{Im} F(t) = P(t) \operatorname{Im} \frac{x(t)}{y(t)} = P(t) \frac{\det \begin{pmatrix} \overline{y(t)} & y(t) \\ \overline{x(t)} & x(t) \end{pmatrix}}{2i|y(t)|^2}.$$

As $(\overline{x(t)}, \overline{y(t)})^\top$ is the analytic continuation of $(x(t), y(t))^\top$ along a loop contained in D, and the monodromy group of (9) is not reducible for $\mu \notin \mathbb{Z}$, then the solutions $(\overline{x(t)}, \overline{y(t)})^\top$ and $(x(t), y(t))^\top$ are linearly independent. This together with $y(t) \neq 0$ for $h \in (1, \infty)$ (Proposition 2) shows that the imaginary part of $F(t)$ has at most $\deg P$ zeros on $(1, \infty)$. Suppose finally that $|t|$ is sufficiently big. As $|y(t)| \geqslant c|t|^{\lambda^*}$ then $F(t) \lesssim t^{s-\lambda^*}$. Summing up the above information, we obtain that the increase of the argument of $F(t)$, when running the boundary of D is at most $2\pi(1 + \deg P + s - \lambda^*)$. Moreover, in the exceptional case when $\lambda^* = \frac{1}{2}$, one has $F(t) \sim ct^{s-\frac{1}{2}}/\log t$ for large $|t|$, which yields a stronger result: the increase of the argument of F on $|t| = R$ is strictly less than $2\pi(s-\frac{1}{2})$. Therefore the total increase of the argument in this case is $< 2\pi(1 + \deg P + s - \frac{1}{2})$. This fact is useful only if $s - \frac{1}{2} \in \mathbb{N}$ but we need it below. One can deduce from the preceding proof of Proposition 4 that $\deg P = [s-\lambda]$ if $\lambda - \mu \leqslant 1$, $\deg P = [s-\mu-2]$ if $\lambda-\mu > 1$ and $s-\frac{1}{2}$ are both integers, and $\deg P = [s-\mu-1]$ otherwise. (If $\deg P < 0$, one takes $P \equiv 0$.) On one hand, $\lambda^* = \mu$ if

$\lambda - \mu \leqslant 1$ and $\lambda^* = \lambda - 1$ otherwise. Therefore, by Proposition 4, the difference between the number of zeros and poles in D of the meromorphic function $F(t) = I(t)/y(t)$ is bounded by $\dim V_s - 1$. By Proposition 3, this yields that $I(t)$ has at most $[\lambda^*] + \dim V_s$ zeros in D. Theorem 1 is proved.

3. The applications

In this section we prove Theorems 2 and 3. Before that, let us point out that some but not everything included in Table 1 is an evident fact. However, since the procedure of deriving the related Fuchsian systems is more or less known [10], we are not going to discuss in more detail how all these systems were obtained.

Given i, j non-negative integers, denote $I_{ij}(h) = \int\int_{H<h} x^i y^j \, dx \, dy$. Then

$$\mathscr{V}_n = \left\{ I(h) = \sum_{0 \leqslant i+j \leqslant n-1} c_{ij} I_{ij}(h) \right\}. \tag{10}$$

Lemma 1. *Let* $\mathbf{I} = (I_1, I_2)$ *be as in Table* 1. *Then for* $n \geqslant 3$ *one can express the function* $I(h)$ *from* (10) *in the form* $I(h) = \alpha(h) I_1(h) + \beta(h) I_2(h)$ *where* $\alpha(h)$ *and* $\beta(h)$ *are polynomials of degrees as follows*:

(i) $\deg \alpha = [\frac{n-1}{2}]$, $\deg \beta = [\frac{n-2}{2}]$ *in cases* 1 *and* 2;
(ii) $\deg \alpha = [\frac{n-1}{3}]$, $\deg \beta = [\frac{n-3}{3}]$ *in case* 3;
(iii) $\deg \alpha = [\frac{n-1}{2}]$, $\deg \beta = [\frac{n-3}{2}]$ *in cases* 4 *and* 5;
(iv) $\deg \alpha = \deg \beta = [\frac{n-3}{2}]$ *in cases* 6 *and* 7.

Moreover, the coefficients in $\alpha(h)$ *and* $\beta(h)$ *may take arbitrary values, except in case* (iv). *The dimension of the vector space* $\mathscr{V}_n$ *in case* (iv) *equals* $[\frac{n-1}{2}] + [\frac{n-1}{4}] + 1$.

Proof. For some of the cases, the results in Lemma 1 are already known. The result in case 1 was proved by Petrov [6]. For cases 4 and 5 see [9], [7], respectively. The result for 2 follows from the considerations in [3] and [4]. The result concerning 3 is proved in [2]. Let us consider cases 6 and 7 from Table 1. By symmetry, we have $I_{ij}(h) = I_{ji}(h)$ and $I_{ij}(h) \equiv 0$ whenever i or j is an odd number. To establish the relations between the integrals $I_{ij}(h)$, we take the equation $H \equiv x^2 + y^2 + \nu x^2 y^2 = h$, $\nu = \pm 1$ and multiply both sides by the one-form $x^i y^{j+1} \, dx$. Afterwards integrate the result along the oval $H = h$ and apply Green's formula. One obtains the relation

$$(j+1) I_{i+2,j} + (j+3) I_{i,j+2} + \nu (j+3) I_{i+2,j+2} = (j+1) h I_{ij}.$$

Similarly, multiplying by $x^{i+1} y^j \, dy$ and integrating, we get another relation

$$(i+3) I_{i+2,j} + (i+1) I_{i,j+2} + \nu (i+3) I_{i+2,j+2} = (i+1) h I_{ij}.$$

Combining these equations we easily obtain

$$v(i-j)I_{i+2,j+2} = (j+1)I_{i+2,j} - (i+1)I_{i,j+2}, \quad i \neq j,$$

$$v(i+3)I_{i+2,i+2} = -(2i+4)I_{i+2,i} + (i+1)hI_{ii},$$

$$v(i+5)I_{i+4,0} = [v(i+2)h-1]I_{i+2,0} - 3I_{i,2} + hI_{i,0}. \tag{11}$$

For $i=j=0$, we get (noticing that $I_{00} = -I_1$ and $I_{20} = I_{02} = -I_2$)

$$I_{22} = \tfrac{4}{3}vI_2 - \tfrac{1}{3}vhI_1,$$

$$I_{40} = I_{04} = (-\tfrac{2}{5}h + \tfrac{4}{5}v)I_2 - \tfrac{1}{5}vhI_1.$$

Then, using (11) with i,j even, we easily prove the assertion in (iv) by induction.

It remains to calculate the dimension of the vector space $\mathscr{V}_n$. Clearly, we have $\dim \mathscr{V}_1 = 1$, $\dim \mathscr{V}_3 = 2$, $\dim \mathscr{V}_5 = 4$. By (11), the only new functions in $\mathscr{V}_{2m+1}$ (compared to $\mathscr{V}_{2m-1}$) are $I_{2m,0}$ and, if m is even, $I_{m,m}$. Hence, the integrals $I_{2k,0}$, $0 \leqslant k \leqslant m$ and $I_{2k,2k}$, $1 \leqslant k \leqslant m/2$ form a basis in $\mathscr{V}_{2m+1}$. These integrals are independent, since the leading term of $I_{2k,0}$, $k \geqslant 2$ is proportional to $h^{k-1}(vI_1 + 2I_2)$ and the leading term of $I_{2k,2k}$, $k \geqslant 1$ is proportional to $h^k I_1$. The above argument implies that $\dim \mathscr{V}_n = [\frac{n-1}{2}] + [\frac{n-1}{4}] + 1$.

Remark 1. In cases (i)–(iii) of Lemma 1, the result remains true even for $n = 1, 2$, under the convention that a polynomial $\beta(h)$ of negative degree is taken to be zero. In case (iv), one has to take $\beta(h) = 0$, $\deg \alpha = 0$ for $n = 1, 2$.

Corollary 1. *The dimension of the vector space* $\mathscr{V}_n$, $n \geqslant 1$, *related to arbitrary polynomial perturbations of degree n in* (3), *in cases* 1–7 *of Table* 1 *is as follows*:

n, *in cases* 1 *and* 2

$[\frac{2n+1}{3}]$, *in case* 3

$2[\frac{n-1}{2}] + 1$, *in cases* 4 *and* 5

$[\frac{n-1}{2}] + [\frac{n-1}{4}] + 1$, *in cases* 6 *and* 7.

Proof of Theorems 2 and 3. Let us first note that $|\lambda - \mu| \leqslant 1$ for all cases 1–7 in Table 1, which yields that $[\lambda^*] = 0$. We put

$s = \frac{n+1}{2}$ in cases 1 and 2,

$s = \frac{n}{3} + \frac{1}{2}$ in case 3,

$s = [\frac{n+1}{2}]$ in cases 4 and 5,

$s = [\frac{n-1}{2}] + \frac{1}{2}$ for $n \geqslant 3$, $s = 1$ for $n = 1, 2$ in cases 6 and 7.

It is easy to check that, with this choice of s, $\mathscr{V}_n \subset V_s$. For this purpose, one can perform the inverse transformation $(I_1, I_2) \to (I_1, I_2 - I_1/\omega)$ bringing $\mathbf{A}$ to a normal form (5) and then use the formulas for the solution given in the proof of Proposition 4. Hence, it suffices to verify in each case that $\deg \alpha + \lambda \leqslant s$, $\deg \beta + \mu \leqslant s$ (the first inequality should be strong in cases 6 and 7; see also Remark 1). Then we compare the dimensions of $\mathscr{V}_n$ and V_s (Proposition 4 and Corollary 1). One obtains that

$\dim \mathscr{V}_n = \dim V_s$ in cases 1–5, as well as in 6 and 7, provided that $n \leqslant 6$,

$\dim V_s - \dim \mathscr{V}_n = [\frac{n-3}{4}]$ in cases 6 and 7, if $n \geqslant 7$.

Thus, the results follow from Theorem 1, taking into account that $I(h)$ has always a zero at $h_0 = 0$.

Some other examples: Let us consider in brief system 8 from Table 1. Instead of (3) and (4), we have

$$\dot{x} = H_y/M + \varepsilon f(x,y),$$

$$\dot{y} = -H_x/M + \varepsilon g(x,y), \tag{3'}$$

where $H = x^{-3}(y^2 - 2x^2 + x)$, $M(x) = x^{-4}, f, g$ are polynomials of degree at most n, and

$$I(h) = \oint_{H=h} M(x)[g(x,y)\,dx - f(x,y)\,dy], \quad h \in \Sigma = (-1,0). \tag{4'}$$

Note that in case 8, $\mathbf{I}$ is analytic in a neighborhood of $h_0 = -1$. Define by $\mathscr{V}_n$ the linear space of integrals (4′) and let $\mathscr{D} = \mathbb{C}\backslash[0,\infty)$.

Theorem 4. *For system* 8 *in Table* 1, *the linear space of integrals* $\mathscr{V}_n$ *has a dimension* $n+1$ *and is Chebyshev in* $\mathscr{D}$, *with accuracy as follows: one for* $n = 2$, *two for* $n = 1, 3$, *three for* $n = 0$ *and* $n - 3$ *for* $n \geqslant 4$.

Taking $n = 2$, we get the following result about the number of limit cycles in (3′):

Corollary 2. *For any quadratic perturbation of the reversible quadratic system* (3′), *the cyclicity of the period annulus around the center at* $(x,y) = (1,0)$ *is two.*

Proof of Theorem 4. Denote $I_{kl} = \int\int_{H<h} M(x)x^k y^l\,dx\,dy$, $I_k = \int_{H=h} M(x)x^{k-1}y\,dx$; thus $\mathbf{I} = (I_2, I_1)^\top$. By symmetry, $I_{kl} = 0$ for l odd. In the same way as above, we obtain the relations

$$I_{k,l+2} = \frac{2l+2}{2k+3l+3}(I_{k+2,l} - I_{k+1,l}), \quad k = -1, 0, \ldots, \quad l = 0, 2, \ldots,$$

$$\left(k - \frac{1}{2}\right)hI_{k+2} = (4-2k)I_{k+1} + \left(k - \frac{7}{2}\right)I_k, \quad k = 0, 1, 2, \ldots,$$

and use the first of them to get the expression

$$I(h) = \sum_{k=0}^{n} c_k I_k(h), \quad c_k \text{ independent}, \tag{12}$$

and then the second one to obtain

$$I(h) = P_0 I_1(h) + P_1(h) I_2(h), \quad \text{for } n = 0, 1, 2,$$

$$I(h) = h^{-1}[P_1(h) I_1(h) + P_2(h) I_2(h)], \quad \text{for } n = 3, 4,$$

$$I(h) = h^{3-n}[P_{n-3}(h) I_1(h) + P_{n-2}(h) I_2(h)], \quad \text{for } n \geqslant 5, \tag{13}$$

where P_k denotes a polynomial of degree k. By (12), $\dim \mathscr{V}_n = n + 1$. Given n, we choose $s = \frac{7}{4}$ if $n = 0, 1, 2$, $s = \frac{11}{4}$ if $n = 3, 4$, $s = n - \frac{5}{4}$ if $n \geqslant 5$, and consider the corresponding linear space V_s. Its dimension is $\dim V_{\frac{7}{4}} = 3$, $\dim V_{\frac{11}{4}} = 5$, $\dim V_{n-\frac{5}{4}} = 2n - 3$, respectively. For each n the function $I(h)$ in (13), multiplied by an appropriate power of h, belongs to the respective V_s. The result then follows from Theorem 1.

Our last example is concerned with the Hamiltonian

$$H = x^2 + y^2 - x^4 - a x^2 y^2 - y^4, \quad a > 2, \tag{14}$$

which comes from the cubic Hamiltonian vector field having a rotational symmetry of order 4. In complex coordinates $z = x + iy$, such a field is presented by a complex equation $\dot{z} = -iz + Az^2\bar{z} + B\bar{z}^3$, $A, B \in \mathbb{C}$, $\operatorname{Re} A = 0$. Take a polynomial perturbation in (3) which is semi-even with respect to x:

$$f(-x, y) = f(x, y), \quad g(-x, y) = -g(x, y), \quad \deg f, g \leqslant n \tag{15}$$

and consider integral (4) where $\Sigma = (\frac{1}{a+2}, \frac{1}{4})$ and the integration is along the oval $\delta(h) \subset \{H = h\}$ surrounding the center at $(\frac{1}{\sqrt{2}}, 0)$. As in Lemma 1, we can derive relations between the integrals involved in (4) and then use them to rewrite $I(h)$ in the form $I(h) = P(h) I_1(h) + Q(h) I_2(h)$ where $I_1 = \int_{\delta(h)} x^2\, dy$, $I_2 = \int_{\delta(h)} x^2 y^2\, dy$, and P, Q are polynomials with independent coefficients and degrees $[\frac{n-2}{4}]$, $[\frac{n-4}{4}]$, respectively. The related vector space $\mathscr{V}_n$ has a dimension $[\frac{n}{2}]$. The vector function $\mathbf{I} = (I_1, I_2)^\top$ satisfies a system (2) with a matrix (which is too large to fit in Table 1)

$$\mathbf{A} = \begin{pmatrix} \dfrac{4h-1}{3} & \dfrac{a-2}{3} \\ \dfrac{4h-1}{15(a+2)} & \dfrac{4h}{5} + \dfrac{a-14}{15(a+2)} \end{pmatrix}.$$

Clearly, conditions (H1)–(H3) are satisfied with $h_0 = \frac{1}{4}$, $h_1 = \frac{1}{a+2}$. Denote $\mathscr{D} = \mathbb{C}\backslash(-\infty, h_1]$. Take $s = \frac{n+1}{4}$, then evidently $\mathscr{V}_n = V_s$. Applying Theorem 1, we obtain

Theorem 5. *For any system* (3) *satisfying* (14) *and* (15), *the linear space of integrals* $\mathscr{V}_n$ *has a dimension* $[\frac{n}{2}]$. *Moreover*, $\mathscr{V}_n$ *is Chebyshev with accuracy* 1 *in* $\mathscr{D}$ *and it is Chebyshev in* Σ.

Theorem 5 is useful for estimating the number of limit cycles not surrounding the origin that are born in small semi-even polynomial perturbations of the cubic Hamiltonian vector field with a rotational symmetry of order 4.

Appendix. Non-oscillation and Sturm-type theorems

The classical Sturm theorem can be used to find bounds for the number of the zeros of the solutions of linear non-autonomous differential equations on a real interval. In the context of the present paper a Sturm-type non-oscillation theorem was recently proved by Petrov [9]. The proof uses of course topological arguments. It is natural to ask whether the results of the present paper could not be deduced in such a way. The answer turns out to be negative in general, and our main Theorem 1 is essentially a non-oscillation result in a complex domain. On the other hand our proofs also rely on topological arguments: the argument principle for real analytic functions in a complex domain. Therefore we may call Theorem 1 a *Sturm-type theorem in a complex domain.*

To compare these two approaches (real and complex) we give below an example in which a Sturm-type theorem in a real domain can still be proved. We shall follow closely Petrov [9]. As in the Introduction, P and Q are the polynomials from the definition of V_s and it is assumed (for definiteness) that $h_0 < h_1$.

Theorem. *Assume that conditions* (H1)–(H3) *hold and* $2\lambda \notin \mathbb{Z}$. *Then any non-trivial function in* V_s *has at most* $\deg P + \deg Q + 1$ *zeros in the interval* $(-\infty, h_0)$. *In particular, if* $|\lambda - \mu| < 1$, *then* V_s *is a Chebyshev vector space in* $(-\infty, h_0)$.

Proof. As in Section 2, it is sufficient to consider (2) as a system for $\mathbf{I}(h) = (x(t), y(t))^\top$, with $\mathbf{A}$ taken in a normal form (7). For k a non-negative integer, denote

$$\omega_{2k+1} = (k+\lambda)(k+\lambda-1), \qquad \omega_{2k+2} = (k+\mu)(k+\mu-1),$$

$$\Omega_k = \begin{pmatrix} \omega_{2k+1} & 0 \\ 0 & \omega_{2k+2} \end{pmatrix}, \qquad \mathbf{R}_k = \begin{pmatrix} \lambda+k-1 & \mu\omega \\ \lambda/\omega & \mu+k-1 \end{pmatrix}.$$

Following [9] we introduce the operator

$$\mathbf{L} = \begin{pmatrix} L & 0 \\ 0 & L \end{pmatrix}, \qquad L = t(t-1)\frac{d^2}{dt^2}.$$

By Proposition 1, we have $\mathbf{LI} = \Omega_0 \mathbf{I}$. Next, we prove that under hypotheses (H1) and (H2), the operator $\mathbf{L}$ satisfies also the following identities: $\mathbf{L}(t^k\mathbf{I}) = t^k\Omega_k\mathbf{I} - t^{k-1}\mathbf{R}_k\mathbf{I}$, $k \in \mathbb{N}$. Indeed, taking into account the form of the matrix in (7), and denoting for short $\delta = \det \mathbf{A} = t(t-1)/\lambda\mu$, we obtain

$$\begin{aligned}
\mathbf{L}(t^k\mathbf{I}) &= t(t-1)(t^k\mathbf{I})'' \\
&= t(t-1)[t^k\mathbf{I}'' + 2kt^{k-1}\mathbf{I}' + k(k-1)t^{k-2}\mathbf{I}] \\
&= [t^k\mathbf{L} + 2k\lambda\mu t^{k-1}\delta\mathbf{A}^{-1} + k(k-1)(t^k - t^{k-1})]\mathbf{I} \\
&= t^k[\Omega_0 + 2k\lambda\mu(\delta\mathbf{A}^{-1})' + k(k-1)]\mathbf{I} + t^{k-1}[2k\lambda\mu(\delta\mathbf{A}^{-1})(0) - k(k-1)]\mathbf{I} \\
&= t^k\Omega_k\mathbf{I} - t^{k-1}\mathbf{R}_k\mathbf{I}.
\end{aligned}$$

Assume that 2λ is not an integer. Then it is easy to verify that the constants ω_j are all different. This implies that there exist scalar functions of the form

$$x_k(t) = [t^k + O(t^{k-1})]x(t) + O(t^{k-1})y(t),$$

$$y_k(t) = O(t^{k-1})x(t) + [t^k + O(t^{k-1})]y(t),$$

satisfying the equations

$$Lx_k(t) = \omega_{2k+1}x_k(t), \qquad Ly_k(t) = \omega_{2k+2}y_k(t),$$

where $O(t^{k-1})$ denotes different polynomials of degree $k-1$. To verify this, we ask for a $\mathbf{I}_k(t) = (x_k(t), y_k(t))^\top$ in the form

$$\mathbf{I}_k = \sum_{j=0}^{k} \mathbf{B}_j t^j \mathbf{I},$$

where $\mathbf{B}_k$ is the unity matrix, $\mathbf{B}_j$ to be determined for $j<k$. As the operator $\mathbf{L}$ commutes with the constant matrices, we have

$$\mathbf{LI}_k = \Omega_k t^k\mathbf{I} + \sum_{j=0}^{k-1} (\mathbf{B}_j\Omega_j - \mathbf{B}_{j+1}\mathbf{R}_{j+1})t^j\mathbf{I} = \Omega_k\mathbf{I}_k$$

and the matrices $\mathbf{B}_j$, $j = k-1, k-2, \dots, 0$ are determined recursively from the equations

$$\mathbf{B}_j\Omega_j - \Omega_k\mathbf{B}_j = \mathbf{B}_{j+1}\mathbf{R}_{j+1},$$

which is possible because ω_j are all different. Therefore, there is a basis in the space V_s consisting of the eigenfunctions of the operator L. Taking into account that, by Proposition 2, x_k and y_k do not vanish for $t<0$, we apply Petrov's elimination technique [9] to prove that any function in V_s has at most $\deg P + \deg Q + 1$ isolated zeros in $(-\infty, 0)$. Especially, in the case when $|\lambda - \mu| < 1$, this means that V_s is Chebyshev in $(-\infty, 0)$.

Note that the above proof works only on the open intervals having h_0 as an endpoint and where $\det \mathbf{A}$ is positive (because the Sturm theorem applies in a backward direction here). Also note that when $|\lambda - \mu| > 1$, the above estimate, although it concerns the interval $(-\infty, h_0)$ only, is weaker than the estimate for the whole $\mathscr{D}$ obtained in Theorem 1.

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