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PETROV MODULES AND ZEROS OF ABELIAN INTEGRALS

BY

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ABSTRACT. – We prove that the Petrov module $\mathcal{P}_f$ associated to an arbitrary semiweighted homogeneous polynomial $f \in \mathbf{C}[x, y]$ is free and finitely generated. We compute its generators and use this to obtain a lower bound for the maximal number of zeros of complete Abelian integrals. © Elsevier, Paris

1. Statement of the results

Let $f \in \mathbf{C}[x, y]$ be a polynomial and consider the quotient vector space $\mathcal{P}_f$ of polynomial one-forms $\omega = Pdx+Qdy$, modulo one-forms $dA+Bdf$ where A, B are polynomials. $\mathcal{P}_f$ is a module over the ring of polynomials $\mathbf{C}[t]$, under the multiplication $R(t) \cdot \omega = R(f)\omega$.

Recall that a function $f : \mathbf{C}^2 \to \mathbf{C}$ is called weighted homogeneous (wh) of weighted degree d and type $\mathbf{w} = (w_x, w_y)$, $w_x = weight(x)$, $w_y = weight(y)$ if

$$f(z^{w_x}x, z^{w_y}y) = z^d f(x, y), \quad \forall z \in \mathbf{C}^*. \tag{1}$$

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We shall also suppose that w_x, $w_y \leq d/2$. By analogy to the case of an isolated singularity of a germ of an analytic function [2], we give the following

DEFINITION 1. – *A polynomial $f \in \mathbf{C}[x,y]$ is called semiweighted homogeneous (swh) of weighted degree $wdeg(f) = d$ and type* $\mathbf{w}$ *if it can be written as $f = \sum_{i=0}^{d} f_i$, where f_i are wh-polynomial of weighted degree i and type* $\mathbf{w}$, *and the polynomial $f_d(x,y)$ has an isolated critical point at the origin.*

Note that according to this definition a wh-polynomial with non-isolated critical point is not semiweighted homogeneous. We define the weighted degree of a one-form $\omega = Pdx + Qdy$ as $wdeg(\omega) = \max\{wdeg(P) + w_x, wdeg(Q) + w_y\}$.

THEOREM 1.1. – *Let $f \in \mathbf{C}[x,y]$ be a swh-polynomial. The $\mathbf{C}[t]$ module $\mathcal{P}_f$ is free and finitely generated by μ one-forms $\omega_1, \omega_2, \ldots, \omega_\mu$, where $\mu = (d - w_x)(d - w_y)/w_x w_y$. Each one-form ω_i can be defined by the condition*

$$d\omega_i = g_i dx \wedge dy$$

where $g_1, g_2, \ldots, g_\mu$ is a monomial basis of the quotient ring $\mathbf{C}[x,y]/\langle f_x, f_y\rangle$. For every polynomial one-form ω there exist polynomials $a_k(t)$ of degree at most $(wdeg(\omega) - wdeg(\omega_k))/wdeg(f)$ such that in $\mathcal{P}_f$ holds $\omega = \sum_{k=1}^{\mu} a_k(t)\omega_k$.

The number $\mu = \dim \mathbf{C}[x,y]/\langle f_x, f_y\rangle$ is the global Milnor number of f, and it equals the sum of "local" Milnor numbers associated to the isolated critical points of f. The module $\mathcal{P}_f$ appeared first in a paper by PETROV [9] where the above result was announced in the case $f(x,y) = y^2 + P(x)$, where $P(x)$ is a degree $d \geq 2$ polynomial. Indeed f is a swh polynomial of degree d and type $w_x = 1$, $w_y = d/2$. The Milnor number of f is $d-1$ and a monomial basis of $\mathbf{C}[x,y]/\langle f_x, f_y\rangle$ is given by $\{1, X, \ldots, X^{d-2}\}$. As $x^k dx \wedge dy = -d(yx^k dx)$ then $\{ydx, xydx, \ldots x^{d-2}ydx\}$ is a "monomial" basis of $\mathcal{P}_f$. Of course here this can be also checked by direct combinatorial computations.

The proof of Theorem 1.1 is based on its hand on the following

THEOREM 1.2. – *Let $f \in \mathbf{C}[x,y]$ be a polynomial with only isolated critical points, and suppose that for every $t \in \mathbf{C}$ the fibre $f^{-1}(t) \subset \mathbf{C}^2$*

is connected. Every polynomial one-form ω on $\mathbf{C}^2$ satisfies the following condition

$$(*) \qquad \forall t \in \mathbf{C},\ \omega|_{f^{-1}(t)} = 0 \, in \, H^1(f^{-1}(t)) \Leftrightarrow \omega = 0 \, in \, \mathcal{P}_f.$$

Note that he above theorem holds under fairly week assumptions on f. For example any good polynomial [8] has isolated critical points and connected fibres. Recall that any tame [3] polynomial is good, any swh polynomial is tame, and any nice or Morse-plus polynomial ([6], [7], [14]) is swh. In the case when f is a degree d polynomial with $(d-1)^2$ distinct critical points Theorem 1.2 is proved by Il'yashenko [6].

2. Proofs

Let $g(y) = y^d + \cdots$ be a degree d polynomial. Consider the global Milnor fibration

$$\mathbf{C} \xrightarrow{g} \{\mathbf{C} - \Sigma\}$$

where $\Sigma = \{t_1, t_2, \ldots t_{d-1}\}$ is the set of the critical values of g, and each fibre $g^{-1}(t)$ consists of d distinct points $y_1(t), y_2(t), \ldots y_d(t)$. The associate (co)homology Milnor bundle is a holomorphic vector bundle with fibre the vector space $\widetilde{H}^0(g^{-1}(t))(\widetilde{H}_0(g^{-1}(t)))$ of reduced (co)homologies. Let

$$\delta(t) = y_j(t) - y_i(t) \in \widetilde{H}_0(g^{-1}(t), \mathbf{Z})$$

be a locally constant (with respect to the Gauss-Manin connection) multivalued section of the homology Milnor bundle.

Lemma 2.1. – *Let $s(t)$ be a holomorphic section of the cohomology Milnor bundle of the polynomial $g(y)$ such that for any locally constant section $\delta(t) \in \widetilde{H}_0(g^{-1}(t))$ holds*

(i) *in any sector on $\mathbf{C}$ with a vertex at ∞ the function $\langle s(t), \delta(t)\rangle$ grows at most as a polynomial*

(ii) *in any sector on $\mathbf{C}$ with a vertex at $t_i \in \Sigma$ the function $\langle s(t), \delta(t)\rangle$ is bounded. Then $s(t)$ is induced by the function $\sum_{k=1}^{d-1} A_k(t) y^k$, $A_k(t) \in \mathbf{C}[t]$*

$$\langle s(t), \delta(t)\rangle = \sum_{k=1}^{d-1} A_k(t)(y_j^k(t) - y_i^k(t)).$$

Proof. – Any functions $h(y)$ defines a geometric section of the cohomology Milnor bundle by the formula $\langle h, \delta \rangle = h(y_j) - h(y_i)$. As the polynomials $y, y^2, \ldots, y^{d-1}$ form a global basis of geometric sections of $\widetilde{H}^0(g^{-1}(t))$, then $s(t) = \sum_{k=1}^{d-1} A_k(t) y^k$ for some holomorphic functions $A_k(t)$. The conditions (i), (ii) imply that $A_k(t)$ are meromorphic on $\mathbf{CP}^1$ so they are rational functions. Suppose that some coefficient $A_k(t)$ has a pole at $t = t_r \in \Sigma$. Then there exists a non-zero section $\widetilde{s} = \sum_{k=1}^{d-1} c_k y^k$, $c_k = \text{const}$, of the cohomoly Milnor bundle which vanishes of order at least one at t_r:

$$(2)\quad |\langle \widetilde{s}(t), y_j(t) - y_i(t) \rangle| = \left| \sum_{k=1}^{d-1} c_k (y_j^k - y_i^k) \right| \leq 0(|t - t_r|), t \to t_r, \forall i, j.$$

Clearly the degree $d-1$ polynomial $\sum_{k=1}^{d-1} c_k y^k$ takes the same values at the d (not necessarily distinct) roots $y_1(t_r), y_2(t_r), \ldots y_d(t_r)$ of $g(y) - t_r$. It follows that there exists at least one critical point of g, say $y_r(t_r)$, of multiplicity $m \leq d$, and which is a zero of the polynomial

$$\sum_{k=1}^{d-1} c_k y^k - \sum_{k=1}^{d-1} c_k y_r^k(t_r)$$

of multiplicity $m' < m$. Finally, if $y'(t), y''(t)$ are two distincts roots of $g(y) - t$ which tend $y_r(t_r)$ as $t \to t_r$, then

$$|y'(t) - y_r(t_r)| = O(|t - t_r|^{1/m}), |y''(t) - y_r(t_r)| = O(|t - t_r|^{1/m}),$$
$$|y'(t) - y''(t_r)| = O(|t - t_r|^{1/m})$$

so

$$|\langle \widetilde{s}(t), y'(t) - y''(t) \rangle| = O(|t - t_r|^{m'/m}), t \to t_r$$

which contradicts to (2).

Proof of Theorem 1.2. – Fix a constant $x_0 \in \mathbf{C}$ and for every $t \in \mathbf{C}$ let $\{y_1(t), y_2(t), \ldots y_d(t)\}$ be the unordered set of roots of the polynomial $g(y) - t$, where $g(y) = f(x_0, y)$. Let ω be a polynomial one-form on $\mathbf{C}^2$ satisfying the condition $(*)$. For any $P = (x, y) \in \mathbf{C}^2$ define, following Il'yashenko [6], the multivalued function

$$F_\omega(P) = \int_{P_i}^{P} \omega$$

where $P_i = P_i(t) = (x_0, y_i(t))$, $t = f(x_0, y_i(t))$, and the path of integration is taken along an arc contained in the connected affine algebraic curve $f^{-1}(t)$. The function $F_\omega(P)$ does not depend on the path of integration but it is determined only up to an addition of

$$\int_{P_i}^{P_j} \omega$$

where the path of integration is contained again in $f^{-1}(t)$. It is easy to check, following for example YAKOVENKO [14], that $F_\omega(P)$ grows at infinity no faster than some polynomial in x, y, $P = (x, y)$, and that

$$\int_{P_i(t)}^{P_j(t)} \omega$$

grows at infinity no faster than some polynomial in t. Let s be a section of the cohomology Milnor bundle of the polynomial in one variable $g(y)$ defined by the formula

$$\langle s(t), P_j(t) - P_i(t)\rangle = \int_{P_i(t)}^{P_j(t)} \omega.$$

As s is obviously holomorphic and satisfies the condition (i), (ii) of Lemma 2.1, then it is induced by the polynomial function $\sum\limits_{k=1}^{d-1} A_k(t)y^k$ and hence

$$\int_{P_i(t)}^{P_j(t)} \omega = \int_{P_i(t)}^{P_j(t)} d\left(\sum_{k=1}^{d-1} A_k(f)y^k\right)$$

Replacing eventually ω by $\omega - d\left(\sum_{k=1}^{d-1} A_k(f)y^k\right)$ we may suppose without loss of generality that the function $F_\omega(P)$ is single-valued. As it grows at infinity as a polynomial then it has a removable singularity along the infinite line of the projectivized complex plane $\mathbf{C}^2$, so $A(x,y) = F_\omega(x,y)$ is a polynomial in (x,y). Let $\omega = Pdx + Qdy$, where P, $Q \in \mathbf{C}[x,y]$ and derive A along the vector field $f_y \frac{\partial}{\partial x} - f_x \frac{\partial}{\partial y}$ tangent to $f^{-1}(t)$. We obtain

$$A_x f_y - A_y f_x = P f_y - Q f_x \Leftrightarrow (P - A_x) f_y = (Q - A_y) f_x.$$

As f_x and f_y have no common factors then there exists a polynomial $B \in \mathbf{C}[x,y]$ such that $P - A_x = B f_x$, $Q - A_y = B f_y$, so $\omega = dA + Bdf$. $\square$

Choose a monomial basis $g_1, g_2, \ldots, g_\mu$ of representative classes of the quotient ring $\mathbf{C}[x,y]/\langle (f_d)_x, (f_d)_y\rangle$, where f_d is the highest order weight homogeneous part of the swh polynomial f. They form also a basis for $\mathbf{C}[x,y]/\langle f_x, f_y\rangle$. Suppose that the one-forms $\omega_1, \omega_2, \ldots, \omega_\mu$ defined by (1) are chosen to be monomial. Let $\gamma_1(t), \gamma_2(t), \ldots, \gamma_\mu(t)$ be be a continuous family of cycles which form a basis of $H_1(f^{-1}(t), \mathbf{Z})$ for any non-critical value $t \in \mathbf{C}$. Then the Wronskian function

$$W(t) = \det\left(\int_{\gamma_i(t)} \omega_j\right)$$

is single-valued and hence a polynomial in t. It is known that the general fibres $f^{-1}(t)$ and $f_d^{-1}(t)$ are equivalent up to an isotopy [5]. Denote by $\{\gamma_i^d(t)\}_i$ the image of $\{\gamma_i(t)\}_i$ in $H_1(f_d^{-1}(t), \mathbf{Z})$ under this isotopy and define also the polynomial function

$$W_d(t) = \det\left(\int_{\gamma_i^d(t)} \omega_j\right).$$

Define at last the discriminant function of f by the formula

$$\Delta(t) = (t-t_1)^{\mu_1}(t-t_1)^{\mu_2}\cdots(t-t_1)^{\mu_s} \tag{3}$$

where μ_i is the sum of local Milnor numbers of the critical points of f associated to its critical value t_i. We have $\mu = \sum_i \mu_i$. To prove Theorem 1.1 we need the following

LEMME 2.2. – *There exists a non-zero constant c such that $W_d(t) = ct^\mu$ and $W(t) = c\Delta(t)$.*

Proof. – The covariant derivative $d\omega_j/dt$ of ω_j coincides with the Gel'fand-Leray form of $g_j dx \wedge dy$

$$\frac{d\omega_j}{dt} = g_j \frac{dx \wedge dy}{df}.$$

It is well known [2] that

$$\det\left(\int_{\gamma_i^d(t)} g_j \frac{dx \wedge dy}{df}\right) = c = \text{const} \neq 0$$

which combined with

$$\int_{\gamma_i^d(t)} \omega_j = t^{\frac{\deg(\omega_j)}{d}} \int_{\gamma_i^d(t)} \omega_j,$$
$$\frac{d}{dt}\int_{\gamma_i^d(t)} \omega_j = \int_{\gamma_i^d(t)} \frac{d\omega_j}{dt} = t^{\frac{\deg(\omega_j)}{d}-1} \int_{\gamma_i^d(1)} \frac{d\omega_j}{dt}$$

gives $W_d(t) = ct^{\mu}$.

To prove that $W(t) = c\Delta(t)$ we use that an isotopy which connects $f^{-1}(t)$ to $f_d^{-1}(t)$ can be chosen in the following way [5]. The change of variables

$$x \to xt^{w_x/d},\ y \to yt^{w_y/d} \tag{4}$$

transforms the fibre $f^{-1}(t)$ to $\{(x,y) \in \mathbf{C}^2 : f(xt^{w_x/d},\ yt^{w_y/d}) - t = 0\}$ and the fibre $f_d^{-1}(t)$ to $f_d^{-1}(1)$. When $t \to \infty$, the fibre $\{(x,y) \in \mathbf{C}^2 : f(xt^{w_x/d},\ yt^{w_y/d}) - t =\}$ goes over $f_d^{-1}(1)$. Taking into consideration that the one-forms ω_k are monomial, we conclude that

$$W(t) = W_d(1)t^{\mu}(1 + 0(1/t)) = W_d(t)(1 + 0(1/t)).$$

This shows that $W(t)$ is a degree μ polynomial with leading term ct^{μ}. On the other hand $W(t)$ vanishes at the critical values t_i of f. If f has μ distinct critical points we are done. If not, we may use the following trick.

Consider a deformation $f_{a,b,t}(x,y) = f(x,y) + ax + by - t$ of $f(x,y)$. The discriminant of $f_{a,b,t}$ is the algebraic set $\Sigma_{a,b,t}$ of (a,b,t) such that 0 is a critical value of $f_{a,b,t}$. $\Sigma_{a,b,t}$ is an irreducible surface in $\mathbf{C}^3$ as it may be parameterized

$$a = -f_x(x,y), b = -f_y(x,y), t = f(x,y) + ax + by.$$

Thus $\Sigma_{a,b,t} = \{(a,b,t) : \Delta(a,b,t) = 0\}$ for some irreducible polynomial $\Delta(a,b,t)$ which is called the discriminant polynomial of $f_{a,b,t}$. As the Milnor number of $f(x,y) + ax + by$ equals the Milnor number of $f(x,y)$, then Δ is of degree μ in t for any a, b so we may normalize, $\Delta(a,b,t) = t^{\mu} + \cdots$ This agrees with the definition (3) and we have $\Delta(0,0,t) = \Delta(t)$. Let $\gamma_1(t,a,b), \gamma_2(t,a,b), \ldots, \gamma_{\mu}(t,a,b)$ be a continuous family of cycles which form a basis of $H_1\big(\{f_{a,b,t}(x,y)=0\},\mathbf{Z}\big)$ for any $(a,b,t)\not\in\Sigma_{a,b,t}$ and consider the Wronskian

$$W(a,b,t)=\det\left(\int_{\gamma_i}\omega_j\right).$$

The function $W(a,b,t)$ is a polynomial in a,b,t [2] which vanishes along $\Sigma_{a,b,t}$ and hence it factorizes $W(a,b,t)=c(a,b,t)\Delta(a,b,t)$ where $c(a,b,t)$ is a polynomial. As before we check that the degree of $W(a,b,t)$ in t is μ so $c(a,b,t)$ does not depend on t. It remains to replace $a=0$, $b=0$.

Proof of Theorem 1.1. – Let $\omega'_1,\omega'_2,\dots,\omega'_\mu$ be polynomial one forms. As in the proof of Lemma 2.2 we may show that

$$\det\left(\int_{\gamma_i(t)}\omega'_j\right)=c(t)\Delta(t)$$

where $c(t)$ is a polynomial depending on ω'_j. Let ω be a fixed polynomial one form. The Kramer formulae together with Lemma 2.2 show that the linear system

$$\int_{\gamma_i(t)}\omega=\sum_{k=1}^{\mu}a_k(t)\int_{\gamma_i(t)}\omega_k,\ i=1,2,\dots,\mu$$

can be solved with respect to $a_k(t)$ and $a_k(t)$ are polynomials. Changing the variables x, y as in (4) and using [5] we conclude that $\deg(a_k(t))\leq |wdeg(\omega)-wdeg(\omega_k))/wdeg(f)|$. We note at last that the polynomial one-form

$$\omega-\sum_{k=1}^{\mu}a_k(t)\omega_k$$

satisfies condition $(*)$ and according to Theorem 1.2 is equal to zero in $\mathcal{P}_f$ □

3. Zeros of Abelian integrals

Let $f\in\mathbf{R}[x,y]$ be a real polynomial and $\delta(t)\subset f^{-1}(t)\subset\mathbf{R}^2$ be a continuous family of ovals defined for $t\in K$, where K is a compact real segment. For every real one-form ω on $\mathbf{R}^2$ denote by $N_K(f,\omega)$ the number of the zeros of the complete Abelian integral

$$I(t) = \int_{\delta(t)} \omega$$

on the interval K. The problem of finding the number

$$N_K(f,n) = \sup_{\deg(\omega)\leq n} N_K(f,\omega)$$

was stated first by ARNOLD (*see* for example [1], [7]) in relation with the second part of the 16th Hilbert problem. A solution of the problem is known only in the case $f(x,y) = y^2 + P(x)$ where $P(x)$ is a real polynomial of degree at most four with only real critical values (*see* PETROV [9], [10] for the case $deg\,(P(x)) = 3$, [11] for the case $deg\,(P(x)) = 4$ with a symmetry, and [12] for the generic case $deg\,(P(x)) = 4$). It was recently proved [7] that for generic fixed f the number $N_K(f,n)$ has at most an exponential growth as $n \to \infty$.

More generally, for any real vector space V of real one-forms on $\mathbf{R}^2$ denote

$$N_K(f,V) = \sup_{\omega\in V} N_K(f,\omega).$$

The image of V under the natural projection $V \to \mathcal{P}_f$ is again a real vector space which we denote by V_f.

Following [4] we say that the real vector space V satisfies the condition $(\star)$ if and only if for every polynomial one-form $\omega \in V$

$$(\star) \qquad \int_{\delta(t)} \omega \equiv 0 \Leftrightarrow \omega = 0 \text{ in } \mathcal{P}_f.$$

We have the following obvious

PROPOSITION 3.1. – *If V satisfies condition $(\star)$ then*

$$N_K(f,V) \geq \dim_{\mathbf{R}} V_f - 1.$$

An important case when the condition $(\star)$ is satisfied is given by

PROPOSITION 3.2. – *Let $\delta(t) \subset f^{-1}(t) \subset \mathbf{R}^2$ be a continuous family of ovals surrounding a single elliptic critical point of f. If $f \in \mathbf{R}^2[x,y]$ is a* *swh Morse polynomial with distinct critical values, then the space of all real polynomial one-forms satisfies* $(\star)$

Proof. – Let $D \subset \mathbf{C}$ be a disc containing the critical values $t_1, t_2, \dots, t_\mu$ of f, and let $t_0 \in \partial D$. Any system of mutually non-intersecting paths $s_1, s_2, \dots, s_\mu$ starting from t_0 and ending at $t_1, t_2, \dots, t_\mu$, respectively, and numbered in the order they start from t_0 defines a distinguished basis of vanishing cycles $\gamma_1(t_0), \gamma_2(t_0), \dots, \gamma_\mu(t_0)$ of $H_1(f^{-1}(t_0), \mathbf{Z})$. Namely, if $\gamma_1(t), \gamma_2(t), \dots, \gamma_\mu(t)$, $t \in D \backslash \{\cup_i t_i\}$, are the corresponding continuous families of cycles, then each cycle $\gamma_i(t)$ vanishes along the path s_i as t tends to the critical value t_i (*see* [2] for a detailed definition). As in the "local" case we associate to the distinguished basis of vanishing cycles its Dynkin diagram. Recall this is a graph, and that each vertex of the graph corresponds to a vanishing cycle γ_i. Two distinct vertices corresponding to γ_i and γ_j are joined by k edges (k dotted edges) if the intersection number $(\gamma_i \cdot \gamma_j)$ is k (respectively $-k$). It is easy to see that the Dynkin diagram of f coincides with the Dynkin diagram of its highest weight-homogeneous part f_d [5] and hence the diagram is connected [2].

Suppose now that $\delta(t) \subset f^{-1}(t) \subset \mathbf{R}^2$ is a continuous family of ovals surrounding a single elliptic critical point of f. Then $\delta(t)$ vanishes along a suitable path and it can be included into a basis of vanishing cycles as above. If $\delta(t) = \delta_i(t)$ for some i, and $(\gamma_i \cdot \gamma_j) \neq 0$, then $\int_{\gamma_i(t)} \omega \equiv 0$. Indeed, consider a loop $l_p \in \pi_i(D \backslash \{\cup_i t_i; t_0\})$, generated by the path s_j and which makes one turn around t_j anticlokwise (*fig.* 1). The Picard-Lefschetz formula [2] implies that the analytic continuation of the complete Abelian integral $\int_{\gamma_i(t)} \omega$ defined for t in a neighborhood of t_0, along the loop l_j is the integral

$$\int_{\gamma_i(t)} \omega - (\gamma_i \cdot \gamma_j) \int_{\gamma_i(t)} \omega.$$

It follows $\int_{\gamma_i(t)} \omega \equiv 0$. As the Dynkin diagram of f is connected then, proceeding by induction, we conclude that $\int_{\gamma_i(t)} \omega \equiv 0$, for $i = 1, 2, \dots, \mu$.

Remark. – The condition that $\delta(t)$ surrounds a critical point of f is necessary for the conclusion of Proposition 3.2 to hold true. Indeed, if $\delta(t)$ is a family of ovals homologous to zero on the compactified curve $f^{-1}(t)$, and ω is a differential of second kind, then $\int_{\delta(t)} \omega \equiv 0$. At the same time ω may be not equal to zero in $H^1_{DR}(f^{-1}(t), \mathbf{C})$. The

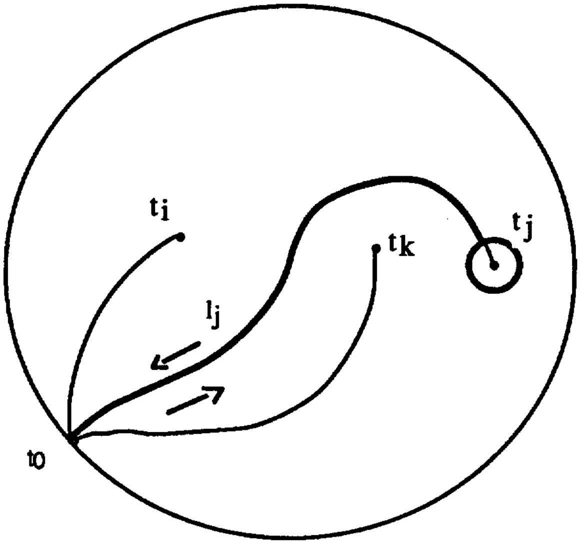

Fig. 1. – A system of μ paths defining a basis of vanishing cycles.

condition that the critical values of f are all distinct is necessary too, as it may be seen from [11].

The exact results obtained for $N_K(f, V)$ in [9], [10], [11], [12] suggest, at least for a reasonable choice of f and V, the following conjecture

If V satisfies $(\star)$, *then* $N_K(f, V) = \dim_{\mathbf{R}} V_f - 1$.

The above may be also reformulated by saying that the space of complete Abelian integrals $\int_{\delta(t)} \omega$ over forms $\omega \in V$ is Chebyshev space. To this end we give three examples of computation of $\dim_{\mathbf{R}} V_f$.

1. Let f be a Morse swh polynomial of weighted degree d with distinct critical values and let V be the real vector space of polynomial one forms of weighted degree at most n. By Theorem 1.1 the projection V_f is identified to all one-forms $\sum_{k=1}^{\mu} a_k(t)\omega_k$ where $a_k(t)$ are polynomials of degree at most $[(n - wdeg\,(\omega_k))/d]$. It follows that

$$\begin{aligned}\dim V_f &= \mu + \sum_{k=1}^{\mu} \left[\frac{n - wdeg\,(\omega_k)}{d}\right] \\ &\geq \sum_{k=1}^{\mu} \frac{n - wdeg\,(\omega_k)}{d} = \mu\left(\frac{n-1}{d}\right)\end{aligned}$$

where μ is the global Milnor number of f (we used that $\sum_{k=1}^{\mu} wdeg\,(\omega_k) = \mu$). In the case where $n = d$ and f is of non-weighted degree n ($w_x = w_y = 1$) we have an exact result

$$\dim V_f = \frac{n(n-1)}{2}.$$

2. Let us put for example $f(x,y) = y^2 + P_d(x)$, where $P_d(x)$ is a real degree d polynomial with $d-1$ distinct critical values. Then f is a degree d swh polynomial, $w_x = 1$, $w_y = d/2$, $\mu(f) = d-1$, which satisfies the condition ($\star$). If V is the real vector space of polynomial one forms of weighted degree at most n then

$$\begin{aligned} N_K(f,V) \geq \dim V_f &= d-1+\sum_{k=1}^{d-1}\left\lceil \frac{n - wdeg\,(x^{k-1}ydx)}{d}\right\rceil \\ &= d-1+\sum_{k=1}^{d-1}\left\lceil \frac{n-k-d/2}{d}\right\rceil \\ &\geq d-1+\sum_{k=1}^{d-1}\left(\frac{n-k-d/2}{d}-\frac{k}{d}\right) = \frac{d-1}{d}n - \frac{d-1}{2}. \end{aligned}$$

3. Let $f(x,y) = y^2 + P_d(x)$ be as above a polynomial with distinct critical values and consider the vector space V of all real one-forms $Pdx + Qdy$ where P, Q are polynomials of (non-weighted) degree n

$$V = \{P(x,y)dx + Q(x,y)dy : \deg(P), \deg(Q) \leq n\}. \tag{5}$$

The identity

$$y^p x^q dx = \frac{p}{2}P_d'(x)\frac{x^{q+1}}{q+1}y^{p+2}dx + d\left(y^p\frac{x^{q+1}}{q+1}\right) - \frac{p}{2}\frac{x^{q+1}}{q+1}y^{p-2}df$$

shows that the one form $y^{p-2}x^{q+d}dx$ is equivalent in $\mathcal{P}_f$ to a one-form $y^pR_q(x)dx$ where $R_q(x)$ is a polynomial of degree q. Proceeding by induction we conclude that V_f is generated as a vector space by monomial one-forms

$$y^p x^q dx, q \leq d-2, p+q \leq n.$$

These one-forms are moreover $\mathbf{R}$-linearly independent in $\mathcal{P}_f$. Indeed, this holds true for

$$f^s y x^k dx, 0 \leq k \leq d-2, s \geq 0.$$

(Theorem 1.1) and any such form is equivalent in $\mathcal{P}_f$ to a $\mathbf{R}$-linear combination of one forms

$$y^{2i+1} x^j dx, i \leq s, j \leq d-2.$$

It follows that $\dim_{\mathbf{R}} V_f$ equals to the number of entire values (k, s) contained in the polygon defined by

$$k \geq 0, s \geq 0, k \leq d-2, 2s+1+k \leq n.$$

and (after some elementary computations)

$$\dim_{\mathbf{R}} V_f = \left[\frac{d-1}{2} n - \frac{(d-2)^2}{4} + \frac{1}{2} \right].$$

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