LJUBOMIR GAVRILOV
MOHAMMED OUAZZANI-JAMIL
RÉGIS CABOZ

Bifurcation diagrams and Fomenko’s surgery on Liouville tori of
the Kolossoff potential $U = \rho + (1/\rho) - k \cos \phi$


<http://www.numdam.org/item?id=ASENS_1993_4_26_5_545_0>
BIFURCATION DIAGRAMS AND FOMENKO'S SURGERY ON LIOUVILLE TORI OF THE KOLOSSOFF POTENTIAL $U = \rho + \frac{1}{\rho} - k \cos \varphi$

BY LJUBOMIR GAVRILOV, MOHAMMED OUAZZANI-JAMIL AND REGIS CABOZ

ABSTRACT. — By making use of the rich algebraic structure of the problem and Fomenko's theory of surgery on (bifurcations of) Liouville tori, we give a complete description of the topology and bifurcations of the invariant level sets of the Kolossoff system corresponding to the integrable potential $U = \rho + \frac{1}{\rho} - k \cos \varphi$.

I. Introduction

Consider the motion of a particle of unit mass on the plane $(x, y)$ in a potential field

$$U = a \rho + \frac{b}{\rho} + c \cos \varphi + d \sin \varphi, \quad a, b, c, d \in \mathbb{R}$$

where $x = \rho \cos \varphi, y = \rho \sin \varphi$. Without loss of generality one may suppose (after a rotation and $\mathbb{R}$-linear change of $\rho$ and $U$) that

$$U(x, y) = \pm \rho \pm \frac{1}{\rho} - k \cos \varphi, k \in \mathbb{R}$$

The corresponding Hamiltonian function is:

$$H = \frac{1}{2} (p_x^2 + p_y^2) + U(x, y)$$

and the energy level sets $\{ H = h \} \subset \mathbb{R}^4$ are compact if $U = \rho + \frac{1}{\rho} - k \cos \varphi$. The Hamiltonian system

$$\begin{align*}
  x' &= \frac{dH}{dp_x}, \\
  p_x' &= -\frac{dH}{dx} \\
  y' &= \frac{dH}{dp_y}, \\
  p_y' &= -\frac{dH}{dy}
\end{align*}$$

(1)
4 \text{critica sectio}

is integrable and the second integral of motion reads:

\[ F = -(k^2 + y^2)p_x^2 + 2y(x-k)p_xp_y - p_y^2(x-k)^2 - \frac{2k(x-k)(kx-1)}{\sqrt{x^2 + y^2}} \]

The integrability of the system \( (1) \) was discovered by Kolossoff [8] who used it to linearize the celebrated Kovalevskaya top.

In the present paper we give a complete description of the topology of the level sets
\[ A_h = \{(x,y,p_x,p_y) \in \mathbb{R}^4 : H = h, F = f \} \subset \mathbb{R}^4. \]

For doing that we find first the bifurcation diagram \( B \) of the problem \( (1) \), \textit{i.e.} the set of critical values of the energy-momentum mapping
\[ (x,y,p_x,p_y) \rightarrow (F,H). \]

It turns out (like in Hénon-Heiles system [5], Gorjatchev-Tchaplygin [4] and Kovalevskaya top [9], [10]) that \( B \) is exactly the discriminant locus of a certain polynomial whose coefficients are functions in \( f, h, k \). The latter is closely related to the algebraic structure of the complexified system \( (1) \). This structure is studied in section 2 where we prove that the complexified generic level set \( \{ H = h, F = f \} \) is an affine part of an Abelian variety (Theorem 1). Contrary to the most of the known examples [1], the Hamiltonian flows corresponding to \( H \) and \( F \) do not linearize on this Abelian variety. Thus the system \( (1) \) is not algebraically completely integrable in the sense of Adler and van Moerbeke [1].

For non-critical values of \( F \) and \( H \) the level set \( A_h \) is, according to Liouville theorem, a finite union of two-dimensional tori. Their number is related to the number of ovals of an associated genus two Riemann surface and could be calculated by making use of the results of chapter 2 (see Theorem 2 of section 3). At last, in section 4, we describe the structure of singular level sets \( A_h \). According to Fomenko’s theory of surgery on (bifurcations of) Liouville tori they turn out to be homeomorphic to a finite list of two-dimensional complexes. To “guess” exactly which bifurcation takes place we use once again the reach algebraic structure of the problem. Namely, each bifurcation of Liouville tori is related to a bifurcation of ovals on a Riemann surface (the last being easily studied). Thus we find all generic bifurcations of Liouville tori as \( f \) and \( h \) pass through the bifurcation diagram \( B \) (Theorem 3 and Theorem 4 of section 4).
II. Algebraic structure

Denote by $A_c$ the complex affine algebraic variety:

$$A_c = \{(x, y, p_x, p_y, z) \in \mathbb{C}^5 : H = h, \ F = f, \ x^2 + y^2 = z^2, \ z \neq 0 \} \subset \mathbb{C}^5,$$

where

$$H(x, y, p_x, p_y, z) = \frac{1}{2}(p_x^2 + p_y^2) + z + \frac{1}{z} - \frac{k}{x},$$

$$F(x, y, p_x, p_y, z) = -(k^2 + y^2)p_x^2 + 2y(x - k)p_xp_y - p_y^2(x - k)^2 - \frac{2k(x - k)(kx - 1)}{z}.$$

The variety $A_c$ is invariant under the (complex) flow of the (complexified) system (1). Consider also the polynomial

$$\varphi(u) = -2(u^3 - hu^2 + (1-k^2)u - f/2)$$

and the corresponding hyperelliptic curve

(3) \hspace{1cm} K: \{w^2 = (u^2 - k^2)\varphi(u)\}.

Remark. – $K$ is precisely the curve used by Kovalevskaya [11] to integrate the Kovalevskaya top.

Theorem 1. – If the polynomial $(u^2 - k^2)\varphi(u)$ has no double roots then the affine algebraic variety $A_c$ is a smooth complex manifold which is biholomorphically equivalent to the complex manifold $\mathbb{A}_c \setminus \mathcal{D}$, where $\mathbb{A}_c$ is a complex algebraic torus (Abelian variety) and $\mathcal{D}$ is a divisor. $\mathbb{A}_c$ is a two-sheeted unramified covering of the Jacobi variety $\text{Jac}(K)$ of the genus algebraic two curve $K$. The trajectories of the Hamiltonian flow generated by $H$ on $A_c$ are straight lines on which, however, the motion is non-linear. The trajectories of the Hamiltonian flows generated by $H + sF, s \neq 0$ on $A_c$ are not straight lines.

Theorem 1 will be proved later in this section. We recall that the Hamilton-Jacobi equation corresponding to (1) separates in the following $(\lambda, \mu)$ coordinates (see [8] for details):

$$x = \frac{\lambda \mu}{k} + k$$

$$y = \frac{1}{k} \sqrt{(\lambda^2 - k^2)(\mu^2 - k^2)}.$$
The canonical variables \((p^1, p_\mu, \lambda, \mu)\) on \(T^* \mathbb{R}^2\) are given by

\[
\begin{align*}
  p_\lambda &= \frac{(\lambda^2 - k^2) \mu p_\mu - (\mu^2 - k^2) \lambda p_\mu}{k(\lambda^2 - \mu^2)}, \\
  p_\mu &= \frac{\sqrt{(\lambda^2 - k^2)(\lambda^2 - \mu^2)}(\lambda p_\lambda - \mu p_\mu)}{k(\lambda^2 - \mu^2)}
\end{align*}
\]

In these new variables the integrals of motion take the form

\[
H = \frac{(\lambda^2 - k^2) p_\lambda^2 - (\mu^2 - k^2) p_\mu^2 + 2(1 - k^2)(\lambda - \mu) + 2(\lambda^3 - \mu^3)}{2(\lambda^2 - \mu^2)},
\]

\[
F = -\mu^2 (\lambda^2 - k^2) p_\lambda^2 + \lambda^2 (\mu^2 - k^2) p_\mu^2 - 2\lambda\mu (\lambda \mu + k^2 + k^2 - 1)(\lambda - \mu)
\]

and hence on each level set \(A_\varepsilon\) holds

\[
(6) \quad p_\lambda = \frac{\varphi(\lambda)}{\sqrt{\lambda^2 - k^2}}, \quad p_\mu = \frac{\varphi(\mu)}{\sqrt{\mu^2 - k^2}}.
\]

For a further use we note also the relation

\[
(7) \quad F = p_\mu^2 (\mu^2 - k^2) + 2\mu^3 - 2\mu^2 H + 2\mu(1 - k^2).
\]

Denote by \(d/dt_s\) the time derivative along the Hamiltonian flow of the function \(H_s = H + s F\). By making use of the equations

\[
\frac{d\lambda}{dt_s} = \frac{\partial H}{\partial p_\lambda}, \quad \frac{d\mu}{dt_s} = \frac{\partial H}{\partial p_\mu}
\]

and (6) one obtains

\[
\begin{align*}
  \frac{d\lambda}{\sqrt{\varphi(\lambda)(\lambda^2 - k^2)}} + \frac{d\mu}{\sqrt{\varphi(\mu)(\mu^2 - k^2)}} &= -2 s dt_s \\
  \lambda^2 \frac{d\lambda}{\sqrt{\varphi(\lambda)(\lambda^2 - k^2)}} + \mu^2 \frac{d\mu}{\sqrt{\varphi(\mu)(\mu^2 - k^2)}} &= dt_s
\end{align*}
\]

The system (8) can be also written in the following equivalent form

\[
\begin{align*}
  \frac{d\lambda}{\sqrt{\varphi(\lambda)(\lambda^2 - k^2)}} + \frac{d\mu}{\sqrt{\varphi(\mu)(\mu^2 - k^2)}} &= -2 s dt_s \\
  \frac{\lambda d\lambda}{\sqrt{\varphi(\lambda)(\lambda^2 - k^2)}} + \frac{\mu d\mu}{\sqrt{\varphi(\mu)(\mu^2 - k^2)}} &= \frac{1 - 2 s \lambda \mu}{\lambda + \mu} dt_s
\end{align*}
\]

The flow of Kolossoff system (1) corresponds to \(s = 0\), and obviously \(t_s |_{s=0} = t\). The system (9) implies, roughly speaking, that our initial system linearizes on an Jacobian.
variety after using a "new time"

\[ dt = \frac{dt}{\lambda + \mu}. \]  (10)

The time \( \tau \) will play an important role and it is exactly the "Kovalevskaya time" (see [8] for details).

Define now the Abel-Jacobi map

\[ \zeta: S^2 K \to \text{Jac}(K): (P_1, P_2) \mapsto \left( \int_{P_m} \omega_1 + \int_{P_n} \omega_1, \int_{P_m} \omega_2 + \int_{P_n} \omega_2 \right) \]

where

\[ \omega_1 = \frac{du}{\sqrt{\varphi(u)(u^2 - k^2)}}, \quad \omega_2 = \frac{u \, du}{\sqrt{\varphi(u)(u^2 - k^2)}}. \]

\( P_1, P_2, P_4 \in K \) is the "infinite" point on \( K \) and \( S^2 K \) is the second symmetric product of \( K \).

Solving the Jacobi inversion problem (9), we obtain the explicit solutions of our initial problem (1) [2]. Thus \( x, y, p_x, p_y, z = \sqrt{x^2 + y^2} \) can be expressed in terms of genus two theta functions living on the \( \text{Jac}(K) \). These functions however are not single-valued as it can be seen from (4). Indeed to each point on the symmetric product \( S^2 K \) of the curve \( K \) (which is birational to \( \text{Jac}(K) \) according to Jacobi theorem) correspond two values of \( (x, y, p_x, p_y) \). On the other hand these functions do not have branch points on \( \text{Jac}(K) \) and hence they are root functions (Wurzelfunktionen [14]) on \( \text{Jac}(K) \).

Consider the Abelian variety \( \tilde{A}_c = \mathbb{C}^2 / \mathbb{Z} \{ e_1, e_2, e_3, 2e_4 \} \) where

\[ \text{Jac}(K) = \mathbb{C}^2 / \mathbb{Z} \{ e_1, e_2, e_3, e_4 \}. \]

If the basis \( (e_1, e_2, e_3, e_4) \) of the period lattice is chosen in a proper way then the function \( x, y, p_x, p_y, z \) become single-valued on \( \tilde{A}_c \). Let us fix such a basis. The natural projection

\[ \pi: \tilde{A}_c \to \text{Jac}(K) \]  (11)

corresponds to the involution

\[ (x, y, p_x, p_y, z) \to (x, -y, p_x, -p_y, z) \]  (12)

on \( \tilde{A}_c \). Consider the mapping

\[ i: \mathbb{C}^2 \to \mathbb{C}P^7: (x, y, z, p_x, p_y) \to [f_0, f_1, \ldots, f_7] \]
where

\[
\begin{align*}
  f_0 &= 1 \\
  f_1 &= x \\
  f_2 &= y \\
  f_3 &= z \\
  f_4 &= xp_y - yp_x \\
  f_5 &= f_4^2 \\
  f_6 &= f_5(f_k - kp_y) \\
  f_7 &= (p_x^2 - p_y^2)y - 2p_xp_yx - 2f_2f_3.
\end{align*}
\]

(13)

**Lemma 1.** - The functions \( f_0, i=0,1, \ldots, 7 \) considered as single-valued meromorphic functions on \( \mathbb{A}_c \) provide a smooth embedding of \( \mathbb{A}_c \) into \( \mathbb{C}P^7 \).

**Proof of theorem 1 assuming the above lemma.** - As the functions \( f_0, f_1, \ldots, f_7 \) provide an embedding of \( \mathbb{A}_c \) into \( \mathbb{C}P^7 \) (Lemma 1) then the closure \( i(\mathbb{A}_c) \) of \( i(\mathbb{A}_c) \) in \( \mathbb{C}P^7 \) is biholomorphically equivalent to \( \mathbb{A}_c \). Consider the divisors \( D_\infty \) and \( D'_\infty \) defined by

\[ (\lambda \mu)_\infty = 2(\zeta(P_\infty) + \zeta(K)) = 2D_\infty \]

and

\[ (\zeta)_\infty = (\lambda + \mu)_\infty = D'_\infty. \]

Obviousy \( D'_\infty \sim 2D_\infty \). It is easily seen that \( \mathbb{A}_c \) is biholomorphically equivalent to \( i(\mathbb{A}_c) \setminus (D_\infty \cup D'_\infty) \). Indeed \( i \) is a biholomorphic mapping between some neighbourhood \( V_{A_c} \) of \( \mathbb{A}_c \) in \( \mathbb{C}^4 \setminus \{z \neq 0\} \) and \( i(V_{A_c}) \subset \mathbb{C}P^7 \). To check that it suffice to note that if \( (x, y, p_x, p_y, z) \in \mathbb{A}_c \) then

\[ \det \left( \frac{\partial (f_1, f_3, f_5, f_6, f_7)}{\partial (x, y, p_x, p_y, z)} \right) = kyz \]

and

\[ \det \left( \frac{\partial (f_1, f_3, f_5, f_6, f_7)}{\partial (x, y, p_x, p_y, z)} \right) = -4p_x(p_x^2y + p_y^3 - p_y^3 - p_y^2) \]

and hence rank \( i = 5 \) (otherwise the equality \( y = p_x = 0 \) implies \( \text{disc}((k^2 - u^2) \varphi(u)) = 0 \)).

As \( j(\mathbb{A}_c) = \mathbb{A}_c \setminus D_\infty \) is a smooth complex manifold, it is concluded that \( \mathbb{A}_c \) is also a smooth complex manifold. \( \triangle \)

**Proof of Lemma 1.** - For an arbitrary divisor \( D \subset \mathbb{A}_c \) we denote

\[ L(D) = \{ f \text{ meromorphic on } \mathbb{A}_c, (f) \geq D \} \]

As \( \zeta(K) \) defines \( (1, 1) \) polarization on Jac(K) then \( D_\infty = \pi^{-1}z(\zeta(K)) \) defines \( (1, 2) \) polarization on \( \mathbb{A}_c \). Thus \( D_\infty \) defines \( (2, 4) \) polarization on \( \mathbb{A}_c \) and \( \dim L(2D_\infty) = 2 \times 4 = 8 \), [7].

To prove lemma 1, it is enough to check that the functions \( f_0, f_1, \ldots, f_7 \) provide a basis of \( L(2D_\infty) \). First of all let us note that \( f_7 \) blow up only along \( D_\infty \). Indeed in \( \lambda, \mu \)
coordinates we have

\[ f_1 = 1 \]
\[ f_1 = \frac{\lambda \mu}{k} + k \]
\[ f_2 = \frac{1}{k} \sqrt{\lambda^2 - k^2}(k^2 - \mu^2) \]
\[ f_3 = \lambda + \mu \]
\[ f_4 = \frac{1}{(\lambda - \mu)} \left\{ \sqrt{\lambda^2 - k^2} \sqrt{\phi(\lambda)} - \sqrt{(\lambda^2 - k^2)} \sqrt{-\phi(\mu)} \right\} \]
\[ f_5 = f_4 \]
\[ f_6 = \frac{1}{(\lambda - \mu)} \left\{ \mu \sqrt{\lambda^2 - k^2} \sqrt{-\phi(\mu)} - \lambda \sqrt{\phi(\lambda)} - \sqrt{(\lambda^2 - k^2)} \sqrt{-\phi(\mu)} \right\} \]
\[ f_7 = \frac{1}{k(\lambda - \mu)} \left\{ 2(\lambda \mu - k^2) \sqrt{\phi(\lambda)} \sqrt{-\phi(\mu)} - \sqrt{(\lambda^2 - k^2)}(k^2 - \mu^2)(\phi(\lambda) + \phi(\mu)) \right\} - 2f_2f_5. \]

To prove that \( f_i \in \mathcal{D}(2 \mathcal{D}_\infty) \) we shall find, following [1], the asymptotic expansions of \( x, y, z \) as functions of the time \( \tau \) in a neighbourhood of a generic point \( \tau^0 \in \mathcal{D}_\infty \). Formulae (4) imply that \( \lambda + \mu = \sqrt{x^2 + y^2} \) and hence the changing of time in the system (1) is equivalent to multiplying each equation by \( z \). According to (9) and (4) the variables \( x, y, z \) are meromorphic in \( \tau \) and the corresponding Laurent series are:

\[
\begin{align*}
\sum_{j=0}^{\infty} x_j \tau^{j-2}, & \quad \sum_{j=0}^{\infty} p_{x_j} \tau^{j-1} \\
\sum_{j=0}^{\infty} y_j \tau^{j-2}, & \quad \sum_{j=0}^{\infty} p_{y_j} \tau^{j-1} \\
\sum_{j=0}^{\infty} z_j \tau^{j-2} & \quad \sum_{j=0}^{\infty} p_{z_j} \tau^{j-1}
\end{align*}
\]  

(14)

(here \( \tau \) stays for \( \tau - \tau_0 \)). After substituting the above series in the Kolossoff system (1) one obtains a recurrent system of linear equations for the coefficients \( x_j, y_j, z_j \). The general solution (14) depends effectively upon three free parameters \( \alpha, \gamma, \delta \):

\[
\begin{align*}
x & = \frac{\alpha}{\tau^2} + \frac{(k \beta^3 - 4 \gamma \alpha)}{4 \beta} + \frac{\delta \tau}{\beta} + \ldots \\
y & = \frac{\beta}{\tau^2} - \frac{(k \alpha \beta + 4 \gamma)}{4} - \frac{\alpha \delta}{\beta} + \frac{\delta \tau}{\beta} + \ldots \\
z & = \frac{-2 + 2 \gamma}{\tau^2} + \frac{\delta}{\beta} + \ldots
\end{align*}
\]  

(15)
where $\alpha^2 + \beta^2 = 4$ (for details about the general procedure of finding the series (15) we refer the reader to [1] or [6, 15]). After substituting (15) in (14), we obtain

\[
\begin{align*}
    f_0 &= 1 \\
    f_1 &= \frac{\alpha}{t^2} + \ldots \\
    f_2 &= \frac{\beta}{t^2} + \ldots \\
    f_3 &= -\frac{2}{t^2} + \ldots \\
    f_4 &= \frac{k \beta}{t} + \ldots \\
    f_5 &= \frac{k^2 \beta^2}{t^2} + \ldots \\
    f_6 &= \frac{12 \delta}{t^2} + \ldots \\
    f_7 &= -2 \frac{(k \alpha \beta + 6 \gamma)}{t^2} + \ldots
\end{align*}
\]

(16)

The complex constants $\alpha$ (or $\beta$ such that $\alpha^2 + \beta^2 = 4$), $\gamma$, $\delta$ parametrize the pole divisor $\mathcal{D}_\omega$. Indeed substituting (15) in \{ $H = h$, $F = f$, $z^2 = x^2 + y^2$ {\} we obtain the genus three curve

\[
\begin{align*}
    \gamma &= \frac{2 k \beta - k \alpha \beta}{16}, \\
    \delta^2 &= \frac{\beta}{72} (k^3 \alpha \beta^3 + 8 k^2 \gamma \beta^2 - 2 k (1 + k^2) \alpha \beta - 32 k^2 \gamma - 2 f \beta), \\
    \alpha^2 + \beta^2 &= 4
\end{align*}
\]

(17)

$\mathcal{D}_\omega$ is a double unramified covering of the genus two curve

\[
\delta^2 = \frac{(\alpha^2 - 4)}{144} (k^3 \alpha^3 + 2 h k^2 \alpha^2 + 4 k (1 - k^2) \alpha + 4 f)
\]

(18)

and obviously this curve (18) coincides with (3) after making the substitution

$$
\alpha \rightarrow \frac{2 u}{k}, \quad \delta \rightarrow \frac{w}{3 k}.
$$

Equations (16) and (18) imply that $f_0, f_1, \ldots, f_7$ are linearly independent on $\mathcal{A}_c$ which completes the proof of lemma 1. \triangle
III. Topology of Regular Level Sets

In this section we shall describe the topological type of $A_k$ for all generic constants $f$, $h$, $k \in \mathbb{R}$. The system (1) will be considered as a real system of differential equations.

According to Theorem 1 $A_k$ is a smooth real manifold if the polynomial $(k^2 - u^2) \varphi (u)$ has no double roots. Define the bifurcation set

$$B = \{(f, h, k) \in \mathbb{R}^3 : \text{disc}((u^2 - k^2) \varphi (u)) = 0 \}. $$

It is clear that the topological type of $A_k$ may change only as $(f, h, k)$ passes through $B$. Thus in each connected component of the set $\mathbb{R}^3 \setminus B$ the level set $A_k$ has the same topological type. Note that the bifurcation set $B \subset \mathbb{R}^3 \{f, h, k\}$ is invariant under the involution

$$(f, h, k) \rightarrow (f, h, -k)$$

and the topological type of the level set $A_k$ is one and the same at the points $(f, h, k)$ and $(f, h, -k)$. Thus it is enough to consider $k \geq 0$.

**Theorem 2.** — The set $\mathbb{R}^3 \setminus B \cap \{k \geq 0\}$ consists of 12 connected components. The sections of these components with the plane $\{k=\text{const.}\}$ are shown on figure 1. If $(f, h, k) \in \mathbb{R}^3 \setminus B$ the level set $A_k$ is (diffeomorphic to) a torus, to a disjoint union of two tori, or it is the empty set as it is shown in table I.

**Remark.** — The notation $2T$ in table I means a disjoint union of 2 two-dimensional tori.

**Proof of Theorem 2.** — The complex conjugation

$$(x, y, z, p_x, p_y) \rightarrow (\bar{x}, \bar{y}, \bar{z}, \bar{p}_x, \bar{p}_y)$$

acts as an antiholomorphic involution on $A_C$. The set of its fixed points is the real part $\Re(A_C)$ of $A_C$ and $A_k = \Re(A_C) \cap \{z > 0\}$. Consider also the natural antiholomorphic involution $\tau$ of the Kovalevskaya curve (3) given in $(w, u)$ coordinates by:

$$\tau: \ (w, u) \rightarrow (\bar{w}, \bar{u}).$$

It induces an antiholomorphic involution on the symmetric product $S^2 K$ and hence on $\text{Jac}(K)$ and $\Lambda_C$. Formulae (4), (5), (6) imply that this involution coincides with the complex conjugation (20) on $A_C$. The upshot is that in order to describe $A_k$ it is enough to study the projection

$$\pi: \ A_C \rightarrow \text{Jac}(K)$$

and the pair $(K, \tau)$.

**Remark.** — The pair $(K, \tau)$ where $K$ is a Riemann surface and $\tau$ is an antiholomorphic involution on $K$ is called Klein surface. For the theory of Klein surfaces we refer the reader to [12].

**ANNALES SCIENTIFIQUES DE L'ÉCOLE NORMALE SUPÉRIEURE**
Fig. 1. - The set \( B \cap \{ k = \text{const.} \} \) for \( k \geq 0 \).
**Table I**

Topological type of $\mathbb{A}_k$ and real roots of the polynomial $(u^2 - k^2) \varphi(u)$ for $(f, h, k) \in \mathbb{R}^2 \setminus \mathcal{B}$ (see fig. 1).

<table>
<thead>
<tr>
<th>Domain</th>
<th>Roots</th>
<th>Topological type</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$-k &lt; u_1 &lt; u_2 &lt; u_3$</td>
<td>T</td>
</tr>
<tr>
<td>1'</td>
<td>$u_1 &lt; -k &lt; u_2 &lt; u_3$</td>
<td>Ø</td>
</tr>
<tr>
<td>2</td>
<td>$-k &lt; u_1 &lt; k &lt; u_2 &lt; u_3$</td>
<td>T</td>
</tr>
<tr>
<td>2'</td>
<td>$u_1 &lt; u_2 &lt; -k &lt; u_3$</td>
<td>Ø</td>
</tr>
<tr>
<td>3</td>
<td>$u_1 &lt; -k &lt; u_2 &lt; u_3$</td>
<td>2 T</td>
</tr>
<tr>
<td>3'</td>
<td>$u_1 &lt; u_2 &lt; -k &lt; u_3$</td>
<td>Ø</td>
</tr>
<tr>
<td>4</td>
<td>$u_1 &lt; -k &lt; k$</td>
<td>Ø</td>
</tr>
<tr>
<td>4'</td>
<td>$-k &lt; u_1 &lt; -k$</td>
<td>Ø</td>
</tr>
<tr>
<td>5</td>
<td>$-k &lt; u_1 &lt; k$</td>
<td>Ø</td>
</tr>
<tr>
<td>6</td>
<td>$-k &lt; u_1 &lt; u_2 &lt; u_3$</td>
<td>Ø</td>
</tr>
<tr>
<td>7</td>
<td>$-k &lt; k &lt; u_1 &lt; u_2 &lt; u_3$</td>
<td>Ø</td>
</tr>
<tr>
<td>7'</td>
<td>$u_1 &lt; u_2 &lt; u_3 &lt; -k &lt; k$</td>
<td>Ø</td>
</tr>
</tbody>
</table>

**Definition.** A connected component of the set of fixed points of $\tau$ on $K$ is called an oval.

To determine the ovals of $K$ it suffices to study the real roots of the polynomial $(u^2 - k^2) \varphi(u)$ for different values of $f$, $h$ and $k$. These roots are shown on table I. Using the formulae (4), (5), (6) and the condition $(x, y, z, p, q, r) \in \mathbb{R}^5$ we obtain that $\mathbb{A}_h \neq \emptyset$ only if $(f, h, k)$ belongs to domain 1, 2 or 3. There we find exactly two “admissible”

<table>
<thead>
<tr>
<th>Domain</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Projection of the “admissible” ovals on $z$-plane</td>
<td>$\Delta_1 = [u_1, u_2]$</td>
<td>$\Delta_1 = [u_1, k]$</td>
<td>$\Delta_1 = [-k, k]$</td>
</tr>
<tr>
<td></td>
<td>$\Delta_1 = [k, u_3]$</td>
<td>$\Delta_2 = [u_2, u_3]$</td>
<td>$\Delta_3 = [u_2, u_3]$</td>
</tr>
</tbody>
</table>

ovals whose projections on the $z$-plane are given by the intervals $\Delta_1$ and $\Delta_3$ (see table II). The product of the “admissible” ovals in $S^2 \mathbb{K}$ [and hence in $\text{Jac}(K)$] gives a Liouville torus. Thus we proved that $\pi(\mathbb{A}_h)$ consists of a torus $T$. There are two possibilities for $\mathbb{A}_h = \pi^{-1}(T)$ (recall that $\mathbb{A}_c$ is a double covering of $\text{Jac}(K) \setminus \emptyset$ and the projection is given by the map (11)):

- $\mathbb{A}_h$ is a disjoint union of two copies of $T$;
- $\mathbb{A}_h$ is homeomorphic to a torus two times “longer” than $T$.

To determine which case arises it suffices to note that when $\lambda$ (respectively $\mu$) makes one turn around the interval $\Delta_1$ (respectively $\Delta_3$) in a complex domain then the function $y$ does not change in the first case, whereas in the second case it changes the sign [we recall that the projection $\pi$ corresponds to the involution (20)]. Thus we find that in domain 1 and 2 $\mathbb{A}_h$ is a torus and in domain 3 it is a disjoint union of two tori. △
At last we shall find the topological type of the regular energy-level surface \{ H = h \}

**Lemma 2.** — The bifurcation set \( \Sigma \) of the family of surfaces

\[
Q_{h,k} = \left\{ \frac{p_x^2 + p_y^2}{2} + \frac{1}{\rho} - k \cos \varphi = h \right\}
\]

is given by the union of two lines \( \Sigma = \{ h = 2 + k \} \cup \{ h = 2 - k \} \subset \mathbb{R}^2 \{ h, k \} \). The set \( \mathbb{R}^2 \setminus \Sigma \) consists of 4 components shown on figure 2. The topological type of \( Q_{h,k} \) in each of these domains is given in table 3.

![Figure 2](image)

**Remarks.** — We note that the three dimensional constant-energy surfaces most often met in mathematical physics and theoretical mechanics are: \( S^3 \) (the sphere), \( \mathbb{R}P^3 \) (the projective space), \( T^3 \) (the torus) and \( S^2 \times S^1 \) (the direct product), see [13] for details.

<table>
<thead>
<tr>
<th>Domain</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Topological type</td>
<td>( \emptyset )</td>
<td>( S^3 )</td>
<td>( S^2 \times S^1 )</td>
<td>( S^3 )</td>
</tr>
</tbody>
</table>

**Proof of Lemma 2.** — The function \( H \) has exactly two critical points \( p_x = p_y = 0 \), \( y = 0 \), \( x = \pm 1 \), for \( k \neq 0 \) and a critical variety \{ \( p_x = p_y = 0 \), \( x^2 + y^2 = 1 \) \} for \( k = 0 \) with
corresponding critical values \( h = 2 \pm k (k \neq 0) \) and \( h = 2 (k = 0) \). Let us compute the topological type of \( Q_{h,k} \). If \( k = 0 \) then
\[
H = \frac{p^2 + p_j^2}{2} + \frac{(\rho - 1)^2}{\rho} + 2 \geq 2
\]
and hence for \( h < 2 \) we have \( Q_{h,k} = \emptyset \). This implies that in domain 1 \( Q_{h,k} = \emptyset \). Suppose now that \( k = 0 \). On the surface \( H = 2 + \varepsilon \) where \( \varepsilon \) is small and positive, \( \rho - 1 \) is small together with \( \varepsilon \). As
\[
\left\{ \frac{p^2 + p_j^2}{2} + \rho + \frac{1}{\rho} = 2 + \varepsilon \right\}
\]
can be written as
\[
\left\{ \frac{p^2 + p_j^2}{2} + \frac{(\rho - 1)^2}{\rho - 1 + 1} = \varepsilon \right\} \Leftrightarrow \left\{ \frac{p^2 + p_j^2}{2} + (\rho - 1)^2 - (\rho - 1)^3 + \ldots = \varepsilon \right\} \sim S^2
\]
Then \( Q_{2+k,0} \) is topologically equivalent to \( S^2 \times S^1 \) and hence in domain 3 the topological type of \( Q_{h,k} \) is \( S^2 \times S^1 \). Consider at last \( Q_{2,\varepsilon} \) for \( \varepsilon \) small and positive
\[
Q_{2,\varepsilon} = \left\{ \frac{p^2 + p_j^2}{2} + \frac{(\rho - 1)^2}{\rho} = \varepsilon \cos \varphi \right\}
\]
The set \( Q_{2,\varepsilon} \cap \{ \varphi = \text{const.} \} \) is topologically equivalent to \( S^2 \) for \( \varphi \in (-\pi/2, \pi/2) \) and to a point for \( \varphi = \pm (\pi/2) \). Hence \( Q_{2,\varepsilon} \) is topologically equivalent to \( S^3 \). This implies that in domain 2 (and 4 by a symmetry) the topological type of \( Q_{h,k} \) is \( S^3 \). \( \triangle \)

**IV. Topology of Singular Level Sets and Surgery on Liouville Tori**

In this section we shall find the topological type of the level set \( A_\mu \) for generic values \( (f, h, k) \in \mathcal{B} \) and thus we shall describe all generic bifurcations of Liouville tori (the non-generic ones are easily found by continuity). For doing that we shall use the Fomenko's classification theorem of bifurcations of (surgery on) Liouville tori [3].

In section 3 we found the topological type of level set \( A_\mu \) far from the bifurcation diagram. Suppose now that the constants \( f, h, k \) are changed in such a way, that \( (f, h, k) \) passes through the bifurcation diagram \( \mathcal{B} \). Then the topological type of \( A_\mu \) may change
and bifurcations of (surgery on) Liouville tori takes place. Consider the following three types of bifurcations (see fig. 3).

![Figure 3 - Bifurcations of two-dimensional invariant Liouville tori and the corresponding graphs.](image)

1) A (two-dimensional) torus $T^2$ is contracted to the axial circle $S^1$ and then vanishes. Denote this surgery as $T \to S^1 \to \emptyset$.

2) A torus $T$ splits into two tori by passing through the complex $S^1 \times \{ S^1 \wedge S^1 \}$ where $S^1 \wedge S^1$ is a union of two circles having exactly one common point. Denote this bifurcation as $T \to 2T$.

3) A torus $T$ becomes twice "shorter" as it spirals twice round a torus. The last complex is homeomorphic to a non-trivial section of the bundle $S^1 \wedge S^1 \to S^1$, and the corresponding bifurcation will be denoted as $T \to T$.

Following Fomenko [3] we present each of the above bifurcations by a graph shown on figure 3. An ordinary point denotes a non-singular Liouville torus. A black circle stands for a circle and a "branching" point (see fig. 3) stands for $\{ S^1 \wedge S^1 \} \times S^1$. At last asterisk denotes a set homeomorphic to a non-trivial section of the bundle $S^1 \wedge S^1 \to S^1$.

For fixed constants $h$ and $k$ let us consider the energy level surface $Q_{h,k} = \{ H = h \}$. As $f$ varies the Liouville tori contained in the level set $\{ F = f \} |_{Q_{h,k}}$ may change its topological type. Denote by $\Gamma (Q_{h,k}, F)$ the graph describing the corresponding sequence of bifurcations of Liouville tori. The main result of this section is the following
KOLOSSOFF POTENTIAL

Fig. 4. – The set $\mathcal{D}$ and the graphs $\Gamma(Q_h,k,F)$.

**THEOREM 3.** If $(h,k)$ belongs to one and the same connected component of the set

$$\mathcal{D} = \{ h \neq 2 \pm k \} \cap \left\{ h \neq \pm \left( k + \frac{1}{2k} \right) \right\} \subset \mathbb{R}^2 \{ h, k \}$$

then the graph $\Gamma(Q_h,k,F)$ is the same and it is shown on figure 4.

Theorem 3 also implies a description of all generic bifurcation of Liouville tori of our initial system (1). Namely, consider a parametrized smooth curve

$$\gamma(s): s \rightarrow (f(s), h(s), k(s)) \subset \mathbb{R}^3 \{ f, h, k \}$$

intersecting the bifurcation diagram $B$ at $s = s_0$.

**DEFINITION.** A bifurcation of Liouville tori contained in the level set

$$A_s \equiv Q_{(s), (s)} \cap \{ F = f(s) \}$$

as $s$ passes through $s_0$ is called generic, provided that $B$ is smooth in a neighbourhood of $(f(s_0), h(s_0), k(s_0))$ and $\gamma(s)$ intersects $B$ transversally.
THEOREM 4. — All generic bifurcations of Liouville tori of the system (1) are given in table IV.

<table>
<thead>
<tr>
<th>Generic bifurcations of the level set $A_h$</th>
<th>(1 \to 2)</th>
<th>(1 \to 6)</th>
<th>(1 \to 4)</th>
<th>(2 \to 3)</th>
<th>(2 \to 5)</th>
<th>(3 \to 4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T$ (\to) $T$</td>
<td>$T$ (\to) $\emptyset$</td>
<td>$T$ (\to) $\emptyset$</td>
<td>$T$ (\to) $2T$</td>
<td>$T$ (\to) $\emptyset$</td>
<td>$T$ (\to) $\emptyset$</td>
<td>$T$ (\to) $\emptyset$</td>
</tr>
</tbody>
</table>

Before proving Theorem 3 and Theorem 4 we shall formulate Fomenko’s theorem [3] (adapted to our case).

DEFINITION. — A smooth function $F$ on a manifold $Q$ is a Bott function, provided that its critical points form nondegenerate critical smooth submanifolds. A critical submanifold of a smooth function $F$ on a manifold $Q$ is called nondegenerated, provided that the Hessian matrix $d^2F$ is nondegenerate in normal planes to the submanifold.

Now we may state the Fomenko’s classification theorem of bifurcations of two-dimensional Liouville tori.

THEOREM (Fomenko [3]). — Let $F$ be a Bott integral on a non-singular constant energy surface $Q^3$ of an integrable two-degrees of freedom Hamiltonian system. Suppose that each critical manifold of $F$ on $Q^3$ is a union of circles. Then each bifurcation of Liouville tori contained in the level set $\{F=f\}$, as $f$ varies, is a composition of the three bifurcations $T \to S^1 \to \emptyset$, $T \to 2T$, and $T \to T$ described above.

Remark. — The condition that each critical manifold of $F$ is a union of circles does not seem to be very restrictive. To our knowledge all studied integrable mechanical systems fall into this case (it may be a conjecture).

In order to apply Fomenko’s theorem we need to check that $F$ is a Bott function when restricted on an energy level surface $Q_{h,k}$.

LEMA 3. — The second integral $F$ is a Bott function on the non-singular energy level surface $Q_{h,k} = \{H=h\}$ provided that $h \neq \pm (k + (1/2)k)$.

Proof of Lemma 3. — Suppose that $Q_{h,k}$ is a non-singular compact manifold, i.e. $h \neq 2 \pm k$ (Lemma 2). If $F$ has a critical value $f$ on $Q_{h,k}$ then the corresponding level surface $A_h = \{H=h, F=f\}$ is degenerated and hence the polynomial $(u^2-k^2)\varphi(u)$ has multiple zeros. The condition $h \neq \pm (k + (1/2)k)$ means that $(u^2-k^2)\varphi(u)$ has no triple zeros on the boundary of the domains 1, 2 and 3 on figure 1, as the $h$-coordinates of the points A, B, C' are $2+k$, $k + (1/2)k$, $2-k$ for $k>0$ and $2-k$, $-k + (1/2)k$, $2+k$ for $k<0$. So let us suppose that the level set $A_h$ is degenerated and consider a degenerated connected component of it. Such a component is parametrized locally by $(\lambda, \mu)$, formulae (5), (6) and (7), at least for $\lambda \neq \mu$. If in addition $\lambda$ and $\mu$ are far from a double root of $(u^2-k^2)\varphi(u)$ then the equations (8) imply that the Hamiltonian flows of $H$ and $H+sF$ are linearly independent and hence $dH$ and $dF$ are linearly independent at such point.
Thus critical points of $F|_{Q_{k,1}}$ correspond only to $(\lambda, \mu)$ such that $\lambda$ (or $\mu$) is a double root of $(u^2 - k^2)\varphi(u)$. This is an one-dimensional analytical set and hence it is a disjoint union of circles. The last follows from the fact that the flow of $H$ on $Q_{k,1}$ has no stationary points and the critical set of $F$ on $Q_{k,1}$ is invariant under the action of this flow.

![Diagram]

Fig. 5. — Correspondence between bifurcation of roots of the polynomial $(u^2 - k^2)(u^3 - hu^2 + (1 - k^2)u - f(2))$ and bifurcations of invariant Liouville tori.

At last let us prove that the hessian matrix of $F|_{Q_{k,1}}$ is non-degenerated of the normal planes to these circles. Let $\mu = \mu_0$ be a double root of $(u^2 - k^2)\varphi(u)$. According to (7) we have

$$F|_{Q_{k,1}} = (u^2 - k^2)\mu_0^2 + 2\mu^3 - 2\mu^2 + 2\mu(1 - k^2)$$

and a critical circle of the level set $\{F|_{Q_{k,1}} = f\}$ is given by $\mu = \mu_0$, $p_\mu = 0$. The normal directions to this circle are given by derivations with respect to $\mu$ and $p_\mu$. We have

$$\frac{\partial^2 F}{\partial \mu \partial p_\mu} = \begin{pmatrix} 2(\mu_0^2 - k^2) & 0 \\ 0 & -\varphi''(\mu_0) \end{pmatrix}$$

and as $\mu_0 \neq \pm k$ then $\text{rank}(d^2(F|_{Q_{k,1}})) \geq 2$. On the other hand the Hessian $d^2(F|_{Q_{k,1}})$ is degenerated on tangent lines to the critical circle and hence $\text{rank}(d^2(F|_{Q_{k,1}})) = 2$ which completes the proof of lemma 3. △

**Proof of Theorem 3.** — Let us fix a regular energy level set $Q_{k,1}$ with a Bott integral $F$ on it, and let us consider the corresponding line $h = \text{const.}$ on figure 1 (plane $k = \text{const.}$, $h = \text{const.}$ in the space $\mathbb{R}^3 \{ f, h, k \}$). As $f$ vary the topological type of $A_h = \{Q_{k,1}\} \cap \{F = f\}$ may change. Using Theorem 2 and the Fomenko’s classification theorem we identify several possible bifurcations. For example passing from domain 3
(where $A_u \sim 2T$) to domain 2 where $(A_u \sim T)$ on figure 1 we may have the following surgeries: $2T \to T$, or composition of $T \to T$ and $T \to \emptyset$. To make the difference between the two possibilities it suffices to look at the bifurcations of roots of the polynomial $(u^2 - k^2) \phi(u)$, and more specifically the four ends of the “admissible” ovals $\Delta_1$ and $\Delta_2$. The correspondence between bifurcation of roots and tori is shown on figure 5. As the bifurcations of real roots of the polynomial $(u^2 - k^2) \phi(u)$ are easily described on table 1 then we obtain a description of the bifurcations of invariant Liouville tori of our initial system (1). By making use of figure 1 we note that if $(h, k)$ is fixed and belongs to one and the same connected component of the set

$$\mathcal{D} = \{h \neq 2 \pm k\} \cap \left\{h \neq \pm \left(\frac{k + \frac{1}{2k}}{2}\right)\right\},$$

then changing $f$ the same bifurcations of roots of the polynomial $(u^2 - k^2) \phi(u)$ take place. This implies that if $(h, k)$ belongs to one and the same connected component of the set $\mathcal{D}$ the corresponding Fomenko’s graph $\Gamma(Q_h, k, F)$ is the same and it is shown on figure 4. This completes the proof of theorem 3. △

**Definition.** — The straight line $l \subset \mathbb{R}^3 \{f, h, k\}$ is generic provided that it intersects $\mathcal{B}$ transversally.

To prove Theorem 4 we note that instead of a generic smooth curve $l \subset \mathbb{R}^3 \{f, h, k\}$ it suffice to consider a generic straight line

$$\{c_1 + c_2 f + c_3 = 0, h = \text{const.}\} \subset \mathbb{R}^3 \{f, h, k\}.$$

Then Theorem 4 follows from the following

**Lemma 4.** — Let $\{c_1 + c_2 f + c_3 = 0, h = \text{const.}\}$ be a generic straight line in $\mathbb{R}^3 \{f, h, k\}$. Then $\{c_1 H + c_2 F + c_3 = 0\} \subset \mathbb{R}^4 \{x, y, p_x, p_y\}$ is a smooth surface, and $F$ is a Bott integral on it.

Indeed, instead of $H$ we may take for a Hamiltonian of (1) the function $c_1 H + c_2 F$.

The same arguments as in the proof of Theorem 3 imply the desirable result (table IV).

To the end of the paper we shall prove Lemma 4 (which generalizes Lemma 2 and Lemma 3).

Let $k = k_0$ be fixed, $(f_0, h_0, k_0) \in \mathcal{B}$ be a generic point (i.e. in a neighbourhood of it $B$ is a smooth manifold), and let $q = (x^0, y^0, p_x^0, p_y^0)$ be a point on the level set $\{H = h_0, F = f_0\}$. We shall prove that if

$$c_1 \text{ grad}(H)\big|_q + c_2 \text{ grad}(F)\big|_q = 0$$

then the straight line $\{c_1 h + c_2 f + c_3 = 0\}$ is tangent to $\mathcal{B}$ (and hence it is not generic). As the equation of a straight line tangent to $\mathcal{B}$ at the point $(f_0, h_0, k_0)$ is given by

$$\{u_0^2 - hu_0 + (1 - k^2) u_0 - f/2 = 0\} \subset \mathbb{R}^2 \{f, h\}$$

where $u_0$ is the double root of the polynomial $P(u) = (u^2 - k^2) \phi(u)$ then it is enough to prove that $c_1/c_2 = 2 u_0^2$. In $(h, u, p_x, p_y)$ coordinates defined by (4), (5) we have the identity

4e série — tome 26 — 1993 — n° 5
KOLOSSOFF POTENTIAL 563

\[
F = P_\mu^2 (\mu^2 - k^2) + 2 \mu^3 - 2 \mu^2 H + 2 \mu (1 - k^2).
\]

Then, at least far from the locus we have

\[
\begin{aligned}
\{ \lambda = \mu \} \cup \{ (\lambda^2 - k^2) (\mu^2 - k^2) = 0 \} \\
\frac{\partial F}{\partial \mu} = 2 \mu P_\mu^2 - \varphi' (\mu) - 2 \mu^2 \frac{\partial H}{\partial \mu} \\
\frac{\partial F}{\partial P_\mu} = 2 (\mu^2 - k^2) P_\mu - 2 \mu^2 \frac{\partial H}{\partial P_\mu}
\end{aligned}
\]

As grad(H) and grad(F) are colinear according to \((21)\), then

\[
\begin{aligned}
\{ 2 (\mu^2 - k^2) P_\mu = 0 \\
2 \mu P_\mu^2 - \varphi' (\mu) = 0
\end{aligned}
\]

and hence \( P_\mu = 0, \varphi' (\mu) = 0 \). Now \((6)\) implies that \( \varphi (\mu) = \varphi' (\mu) = 0 \) and hence \( \mu \) is a double root of the polynomial \((\mu^2 - k^2) \varphi (\mu)\). Suppose now that \( \lambda^0, \mu^0, P_\mu^0, P_{\mu^0} \) belongs to the locus \((22)\) and let \( (\lambda, \mu, P_\mu, P_{\mu^0}) \) tends to \( (\lambda^0, \mu^0, P_\mu^0, P_{\mu^0}) \). The vectors grad(H) and grad(F) tend to some vectors grad(H)\(^0\) and grad(F)\(^0\) and let us suppose that these vectors are colinear. Using \((6), (23)\) and \((24)\) we conclude that

\[
(\mu^2 - k^2) \varphi (\mu) \to 0 \quad \text{and} \quad 2 \mu \frac{\varphi (\mu)}{\mu^2 - k^2} - \varphi' (\mu) \to 0
\]

and hence \( \mu^0 \) is a double root of the polynomial \((\mu^2 - k^2) \varphi (\mu)\). The upshot is that if \( c_1 \text{grad}(H) + c_2 \text{grad}(F) = 0 \) then \( c_1/c_2 = 2 \mu_0^2 \), where \( \mu_0 \) is the double root of the polynomial \((\mu^2 - k^2) \varphi (\mu)\), and hence the straight line

\[
\{ c_1 h + c_2 f + c_3 = 0, k = \text{const.} \}
\]

is tangent to B. This completes the proof of Lemma 4. \( \triangle \)

Acknowledgements

Acknowledgements are due to J.-P. Codaccioni and to P. Vanhaecke for the valuable discussions.
REFERENCES


(Manuscript received February 4, 1992; revised June 11, 1992.)

L. Gavrilov (*)
M. Ouazzani-Jamil (*)
and
R. Caboz,
Laboratoire de Physique Appliquée
Université de Pau et des pays de l'Adour
64000 Pau, France
and
(*) Laboratoire de Topologie et Géométrie
URA CNRS 1408, Université Paul Sabatier
31062 Toulouse, France
(‡) Faculté des Sciences Dhar el Mehraz Fès,
Département de Physique, Fès, Morocco.