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Remark on the Number of Critical Points of the Period

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1. STATEMENT OF THE RESULT

Consider the following Hamiltonian system on the plane

$$\begin{cases} \mathring{x} = y \\ \mathring{y} = \partial V(x)/\partial x, \qquad \text{overdot} = d/dt, \end{cases} \tag{1.1}$$

where $V(x)$ is an arbitrary polynomial of degree four. All topologically different phase portraits of (1.1) are given in Fig. 1. Let $\{\gamma(p)\}_{p \in \Delta}$ be a continuous family of periodic solutions, parameterized by $p = \frac{1}{2}y^2 - V(x)$, and defined on a maximal open interval $\Delta \subset \mathbb{R}$. The period function $T(p) = \int_0^T dt = \int_{\gamma(p)} dx/y$ assigns to the periodic solution $\gamma(p)$ its minimum period. The system (1.1) possesses 0, 1, 2, or 3 period functions (see Fig. 1).

THEOREM (Chow and Sanders [1]). *The period functions of* (1.1) *can have at most three critical points* (*including the multiplicities*).

In the present paper we improve the above result.

THEOREM 1. *The period functions of* (1.1) *can have at most one simple critical point. More precisely, if a period function has a critical point, then the phase portrait of the system* (1.1) *is topologically equivalent to Fig.* 1d.

To prove Theorem 1 we use well-known methods of algebraic geometry. Namely, the period function $T(p)$ satisfies a second order Picard–Fuchs equation and $x(p) = T'(p)/T(p)$ satisfies a Riccati equation. Instead of this equation we consider the equivalent polynomial autonomous system on the plane $\mathbb{R}^2\{x, p\}$ and study its global phase portrait. As it was noted in [2, Sect. 5], the phase curve $x(p) = T'(p)/T(p)$ possesses the following

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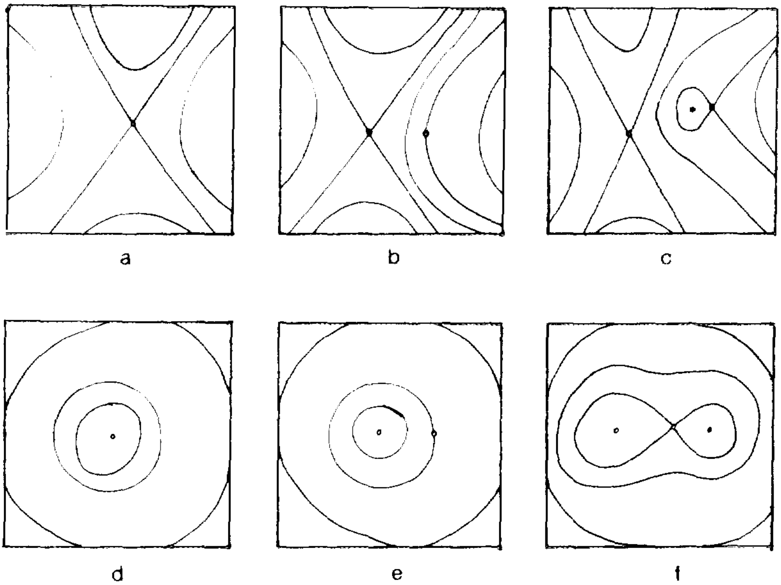

FIG. 1. Phase portraits of system (1.1).

fundamental property. Suppose that for $p = p_0$ the periodic solution $\gamma(p)$ vanishes. Then $\lim_{p \to p_0} T'(p)/T(p) = x_0 \neq \pm\infty$, the equilibrium point (p_0, x_0) is a saddle and $(p = t,\ x = T'(t)/T(t))$ is a separatrix solution in a neighbourhood of (p_0, x_0). At last we use the fact that for all p the Abelian integral $T'(t)$ can be expressed explicitely as a linear combination of two Abelian integrals which do not vanish for $p \in \Delta$ (formula (2.8)). This is the main point of our proof as the remaining assertions can be proved along the same lines as in [1]. Nevertheless we prefer to study autonomous systems on the plane than Riccati equations because of the simple geometrical property of the phase curve $x(p) = T'(p)/T(p)$, explained above.

2. The Proof

We shall use the notation and the results of [1]. One may suppose, without loss of generality, that the potential function $V(x)$ is brought in the following normal form:

$$V(x) = \frac{a}{4} \cdot x^4 + \frac{b}{2} \cdot x^2 + k \cdot x, \qquad \text{where} \quad a = \pm 1, \text{ and } b = \pm 2, 0.$$

In the cases $a=1$, $b=\pm 2$, and $a=-1$, $b=-2$, our Theorem 1 follows from [1, Sect. 4]. To this end we shall study the cases $a=-1$, $b=2$, and $a=-1$, $b=0$.

Case 1. $V(x)=-\frac{1}{4}\cdot x^4+k\cdot x$ (*see* Fig. 1d).

From [1, Formula (4.3)], we obtain the following Picard–Fuchs equation satisfied by $T=T(p)$

$$\delta\cdot T''+\delta'\cdot T'+28p\cdot T=0, \qquad \text{prime}=d/dp, \tag{2.1}$$

where $\delta=64\cdot p^3+27\cdot k^4$ is the discriminant of the polynomial $V(x)+p$. For each $k\in\mathbb{R}$ there exists an unique $p_1\leqslant 0$, such that $\delta(p_1)=0$ (see Fig. 2) and the period function $T(p)$ is defined for $p>p_1$. As $T>0$ on this interval, then the function $x(p)=T'(p)/T(p)$ takes only finite values. It satisfies the following Riccati equation

$$\delta\cdot x'+\delta'\cdot x+\delta\cdot x^2+28\cdot p=0. \tag{2.2}$$

Instead of (2.2) we consider, as in [2, Sect. 5], the equivalent autonomous system

$$\begin{cases}\mathring{x}=-\delta'\cdot x-\delta\cdot x^2-28p\\ \mathring{p}=\delta.\end{cases} \tag{2.3}$$

It has only one equilibrium point with coordinates

$$(x_0=7\cdot k^{-4/3}/36,\ p_0=-3\cdot k^{4/3}/4)$$

which is a saddle. From [1, Lemma 4.1], we obtain $\lim_{p\downarrow p_0} T'(p)/T(p)=7\cdot k^{-4/3}/4$ and hence the curve $x=T'(p)/T(p)$ is a separatrix solution of

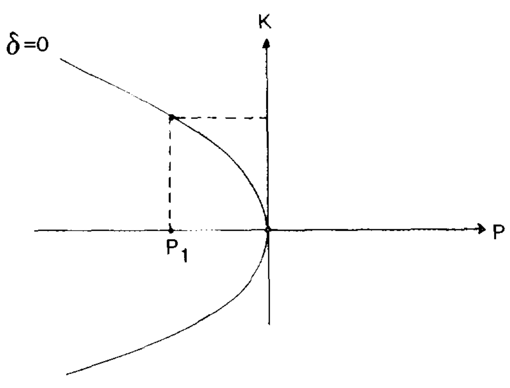

FIG. 2. The level set $64p^3+27k^4=0$.

(2.3). Suppose that this phase curve intersects the line $x=0$ at least twice. Denote the points of intersection by P_2, P_3, and put $P_1=(p_1, 0)$ (see Fig. 3a). The direction of the vector field at the points P_1, P_2, P_3, implies that there exist at least two points on the line $x=0$, and the vector field is tangent to the line at these points. In other words $-28p$ has two zeroes which is a contradiction. The same conclusion holds if $P_2=P_3=(p_2, 0)$, i.e., p_2 is a critical point of $T(p)$ of multiplicity two. Hence if $k \neq 0$, then $T(p)$ has no more than two critical points, including the multiplicities. In fact the period function has exactly one critical point. Indeed $\lim_{p\to\infty} T(p)=0$, and hence for sufficiently big positive values of p $T'(p)<0$ holds (Fig. 3b). At last suppose that $k=0$. The autonomous system corresponding to (2.2) takes the form

$$\begin{cases} \mathring{x} = -48px - 16p^2x^2 - 7 \\ \mathring{p} = 16p^2. \end{cases} \tag{2.4}$$

One easily computes that $\lim_{p\downarrow 0} T'(p)/T(p) = -\infty$, and $T'(p)<0$ for sufficiently big positive values of p. The same arguments imply that T has no critical points (see Fig. 3c).

The upshot is that *for each fixed k the period function has at most one critical point.*

Case 2. $V(x) = -\frac{1}{4}\cdot x^4 + x^2 + k\cdot x$ (see Figs. 1d–1f).

The period function T satisfies the following Picard–Fuchs equation (see [1, Formula (4.3)]

$$\delta B\cdot T'' + (\delta' B - \delta B')\cdot T' - \Sigma\cdot T = 0, \tag{2.5}$$

where $\Sigma = 96(1+p)^2 - 12k^2(5+21p)$, $B=-8(1+p)+9k^2$, and $\delta = 64p(1+p)^2 - 16k^2(1+9p) + 27k^4$. Note that δ is the discriminant of the

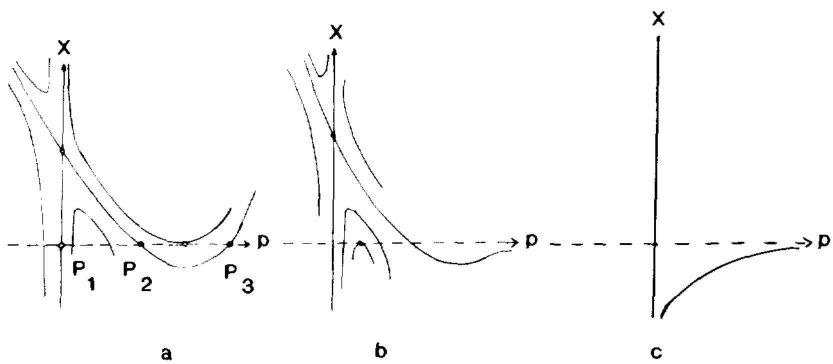

FIG. 3. Phase portrait of system (2.3).

polynomial $V(x)+p$. From (2.5) we derive the Riccati equation satisfied by $x=T'/T$

$$\delta B\cdot x'+(\delta' B-\delta B')\cdot x+\delta B\cdot x^2-\Sigma=0 \tag{2.6}$$

and the equivalent autonomous system reads

$$\begin{cases} \mathring{x}=(\delta B'-\delta' B)\cdot x-\delta B\cdot x^2+\Sigma \\ \mathring{p}=\delta B. \end{cases} \tag{2.7}$$

The curves $\delta=0$, $B=0$, and $\Sigma=0$ are given in Fig. 4. The two points of intersection of these three curves have coordinates ($p=\frac{1}{3}$, $k=\pm\frac{4}{9}\sqrt{6}$). We have three subcases

(i) $0<|k|<\frac{4}{9}\sqrt{6}$ (see Fig. 1f).

For each fixed k, there exist three points $p_1\leqslant p_2\leqslant p_3$ such that $\delta(p_i)=0$, $i=1,2,3$. Furthermore there are three one-parameter families of periodic orbits with period functions T_i, $i=1,2,3$, defined on (p_1,p_3), (p_2,p_3), and (p_3,∞), respectively. Note, however, that T_1 is equal to T_2 on (p_2,p_3), as the periodic orbits $\gamma_1(p)$, $\gamma_2(p)$ corresponding to T_1, T_2, represent homological cycles on the complex compactified curve $\{(x,y)\in\mathbb{C}^2:\frac{1}{2}y^2=V(x)+p\}$.

Consider now the phase portrait of the system (2.7) (Fig. 5): it possesses four equilibrium points with coordinates (p_i,x_i), $i=1,2,3,4$, where $B(p_4)=0$, $x_4=-\Sigma(p_4)/\delta(p_4)\,B'(p_4)$, $x_i=\Sigma(p_i)/B(p_i)\,\delta'(p_i)$, $i=1,2,3$. One easily computes (using Fig. 4) that $x_1,x_2,x_4>0$, $x_3<0$. As the

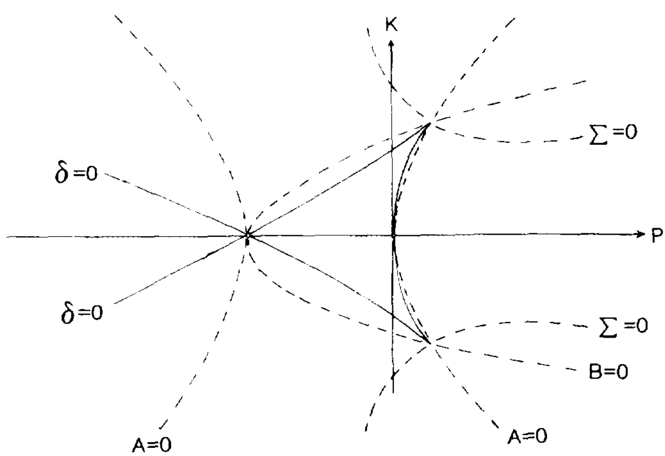

FIG. 4. Level sets of δ, Σ, A, B.

curve $x = T_1'(p)/T_1(p)$, $p_1 < p < p_3$ is a phase curve of (2.7), it passes through the points (p_4, x_4) and (p_2, x_2). Using [1, Lemma 4.1], we obtain $\lim_{p \downarrow p_1} T_1'(p)/T_1(p) = p_1$. We have also $\lim_{p \uparrow p_3} T_1 = \infty$ (see [1, p. 63]), and hence for all $p < p_3$, such that p is sufficiently close to p_3, $T_1' > 0$ holds. Now Fig. 5 implies that $\lim_{p \uparrow p_3} T_1'(p)/T_1(p) = \infty$.

Suppose that the curve $x = T_1'(p)/T_1(p)$ intersects the line $\{x = 0\}$ (see the dotted line on Fig. 5). Then it intersects this line at least twice, and hence the vector field (2.7) is tangent to $\{x = 0\}$ at some point. In other words $\Sigma(p)$ vanishes in the interval (p_1, p_3) which is a contradiction (see Fig. 4). It is concluded that T_1 and T_2 have no critical points.

Consider now the period function $T_3(p)$, $p > p_3$. We shall prove that $T_3' < 0$. Indeed [1, Formula (4.2)] implies

$$\delta \cdot T_3' = A \cdot T_3 + B \cdot \tilde{T}_3, \tag{2.8}$$

where $T_3(p) = \int_{\gamma_3(p)} dx/y$, $\tilde{T}_3(p) = \int_{\gamma_3(p)} x^2\, dx/y$, $A(p) = -16p(1+p) + 6k^2$. (the polynomials $\delta(p)$ and $B(p)$ are given after formula (2.5)). As T_3, $\tilde{T}_3 > 0$, and for all sufficiently big p, $\delta > 0$, $A < 0$, $B < 0$ hold (we use Fig. 4), then for these values of p $T_3'(p) < 0$. On the other hand Fig. 5 implies that $\lim_{p \downarrow p_3} T_3'(p)/T_3(p) = -\infty$. Hence if $T_3'(p)$ vanishes on the interval (p_3, ∞) then it vanishes at least twice, and the vector field (2.7) is tangent to $\{x = 0\}$ at some point $(0, q)$ (see Fig. 5). It means that $\Sigma(q) = 0$ and without loss of generality one may also suppose that $T_3'(q)/T_3(q) > 0$. As $A(q) < 0$, $B(q) < 0$, $\delta(q) > 0$ (see Fig. 4), Eq. (2.8) implies $T_3'(p)/T_3(p) < 0$ which is a contradiction. It is concluded that for all $p > p_3$, $T_3'(p) < 0$ holds.

(ii) $k = \pm \frac{4}{9}\sqrt{6}$ (see Fig. 1e).

Here $p_2 = p_3$ and using (2.8) we obtain as above that for $p > p_3$ $T_3' < 0$ holds. By continuity T_1 does not possess simple critical point in the interval

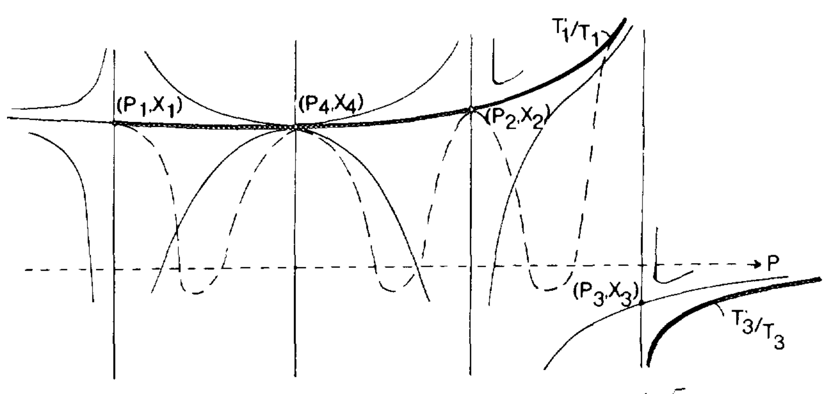

FIG. 5. Phase portrait of system (2.7) for $0 < |k| < \frac{4}{9}\sqrt{6}$.

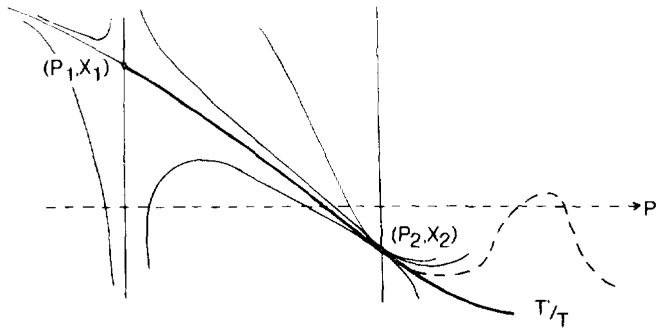

FIG. 6. Phase portrait of system (2.7) for $|k| > \frac{4}{9}\sqrt{6}$.

(p_1, p_2). If $T_1(p)$ has a double critical point then (2.5) implies that $\Sigma(p)$ vanishes in the interval (p_1, p_2) which is a contradiction.

(iii) $k = 0$ (see Fig. 1e).

Here $p_1 = p_2$. By continuity T_2 and T_3 do not possess simple critical points in the intervals (p_2, p_3) and (p_3, ∞), respectively. Suppose that T_2 (or T_3) has a double critical point. Then (2.5) implies that $\Sigma(p)$ vanishes in the interval (p_2, p_3) or (p_3, ∞) which is a contradiction. Hence T_2 and T_3 have no critical points.

(iv) $|k| > \frac{4}{9}\sqrt{6}$ (see Fig. 1d).

We have only one period function $T(p)$ defined on the interval (p_1, ∞), where $\delta(p_1) = 0$. The system (2.7) possesses two equilibrium points with coordinates (p_i, x_i), $i = 1, 2$, where $B(p_2) = 0$, $x_1 = \Sigma(p_1)/B(p_1)\,\delta'(p_1) > 0$, $x_2 = -\Sigma(p_2)/\delta(p_2)\,B'(p_2) < 0$. One easily checks that (p_1, x_1) is a saddle, and (p_2, x_2) is a node. The phase portrait of (2.7) is given on Fig. 6. Since [1, Lemma 4.1], implies that $\lim_{p \downarrow p_1} T'(p)/T(p) = x_1$, the curve $x = T'(p)/T(p)$ is a separatrix solution of the system (2.7). If T has more than one critical point in the interval (p_1, p_2), then the same arguments as in (i) show that $\Sigma(p)$ has at least three zeroes in (p_1, p_2) which is a contradiction. It is concluded that $T(p)$ has exactly one critical point in this interval. If $p > p_2$, then $A(p) < 0$, $B(p) < 0$, $\delta > 0$, and (2.8) implies $T' = (A \cdot T + B \cdot \int_{\gamma(p)} x^2\, dx/y)/\delta < 0$. Thus we have proved that the period function $T(p)$ has exactly one critical point in the interval (p_1, ∞). It completes the proof of Theorem 1.

ACKNOWLEDGMENTS

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References

1. S. N. Chow and J. A. Sanders, On the number of critical points of the period, *J. Differential Equations* **64** (1986), 51–66.
2. L. Gavrilov and E. Horozov, Limit cycles and zeroes of Abelian integrals satisfying third order Picard–Fuchs equations, *in* "Lecture Notes in Mathematics," Vol. 1455, pp. 160–186, Springer-Verlag, New York, 1990.