Random moment problems

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The world in Caracas for Chichi (Doctor León!!)

Agenda

Some papers and collaborators: Random Moments

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- **F. Chang, J. Kemperman, W. Studden J. (1993) A normal limit theorem for moment sequences. AOP (The seminal paper)**

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- **H. Dette, F.G. (2007) Asymptotic properties of the algebraic moment range process. Acta Math. Hungar.**

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- H. Dette et al (2009-2012) Matricial moment case.

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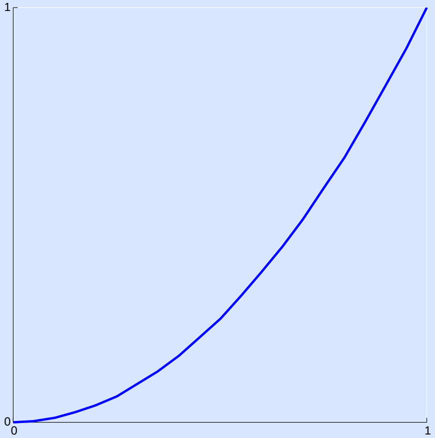
$\mathbb{M}_k =$ convex hull of the curve in $\mathbb{R}^k$, $(x^j)_{j=1,\dots,k,x\in[0,1]}$

Example: the space $\mathbb{M}_2$



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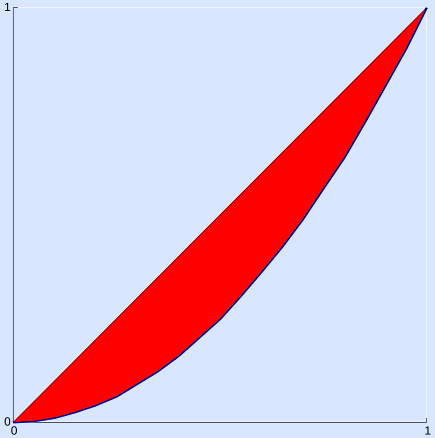

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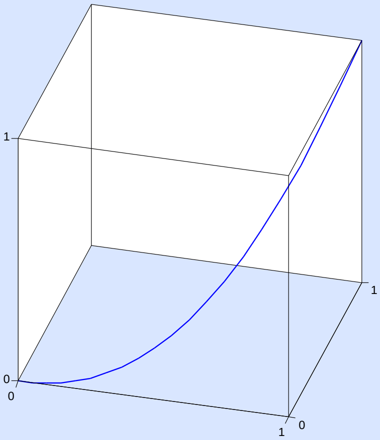

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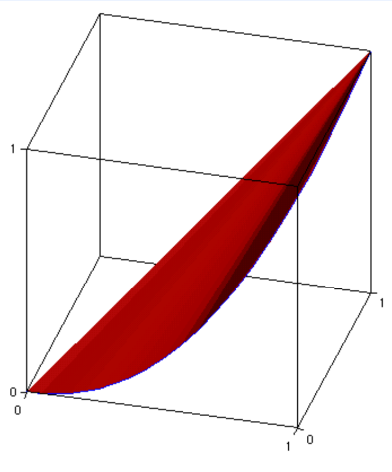

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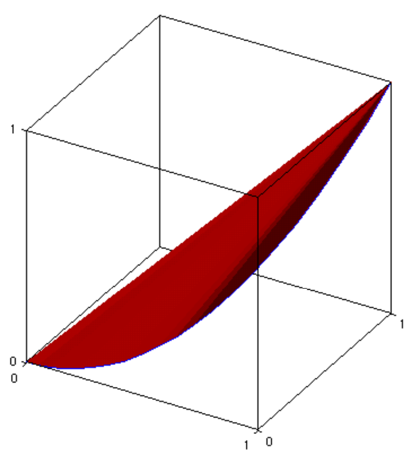

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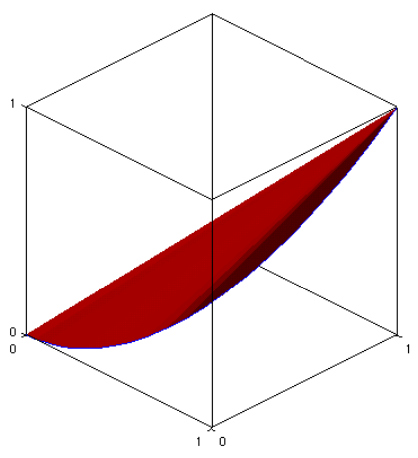

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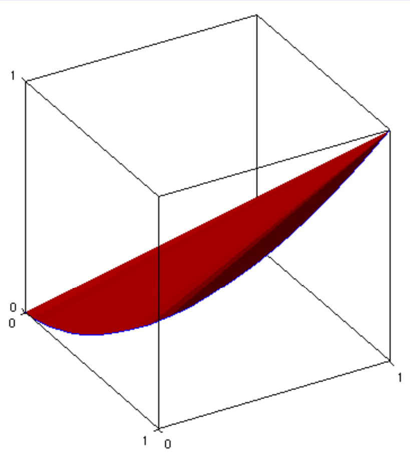

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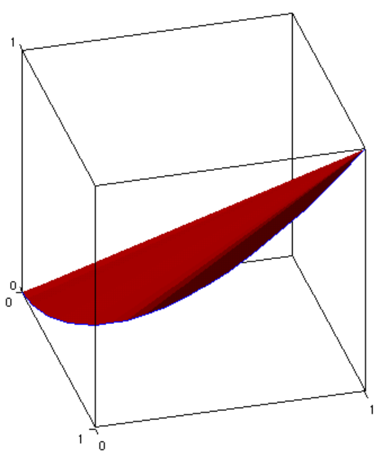

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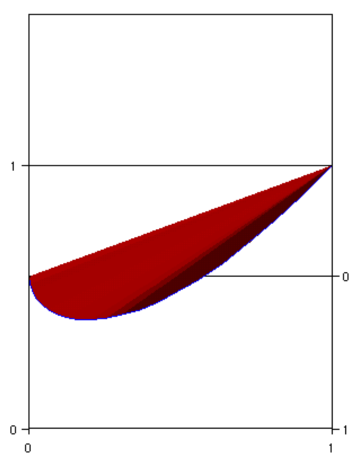

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First Stone (fixed k) see Kemperman et al. AOP 1993:

$$\lim_n Z_n^k = \overline{c}^{(k)} \text{ (a.s.)}$$

$\overline{c}^{(k)} = (\overline{c}_1, \ldots, \overline{c}_k)$ are the k first moments of the arcsine law ν

$$\nu(dx) := \frac{1}{\pi\sqrt{x(1-x)}}dx, \ (x \in [0,1])$$

$\overline{c}_1 = 0.5, \ \overline{c}_2 = 3/8, \ldots$

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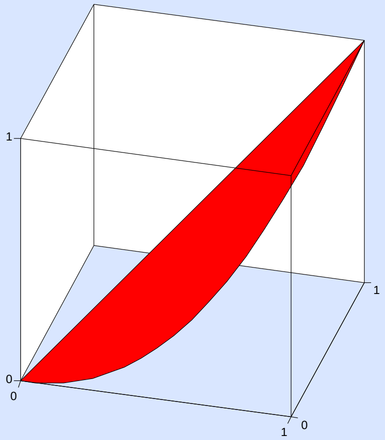

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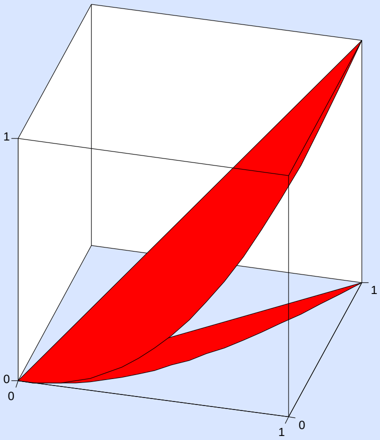

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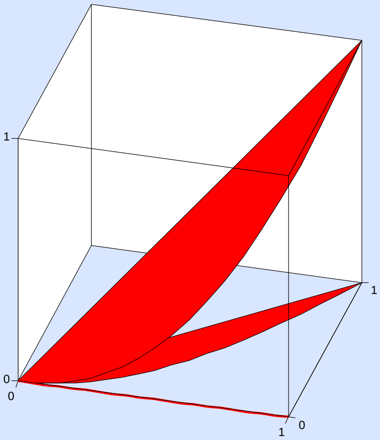

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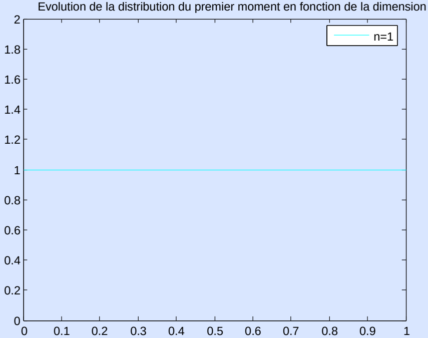

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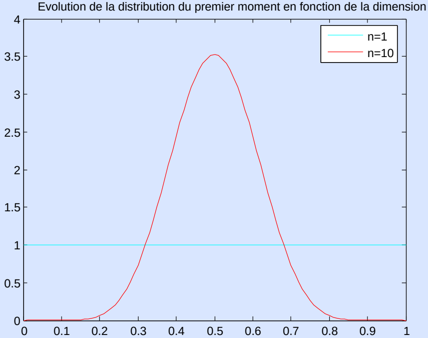

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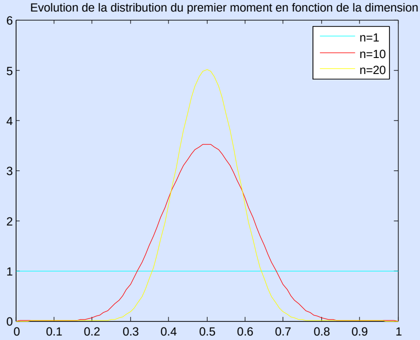

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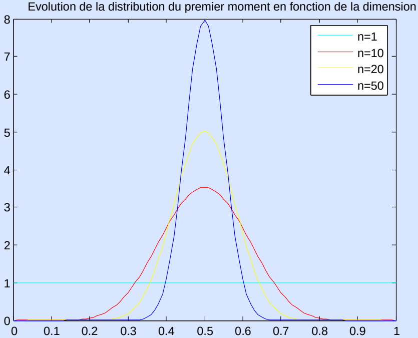

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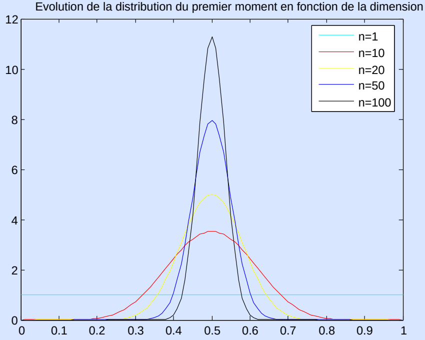

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Theorem

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In view of this result some questions

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- **Some links with other model?**

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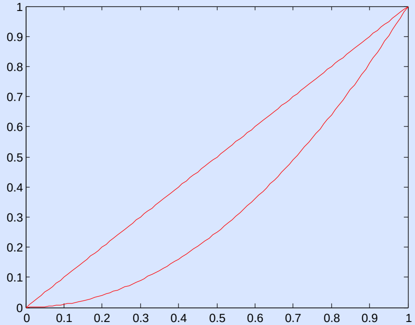
$$c_j^- := \inf_{\mu \text{ fitting } c_1, \dots, c_{j-1}} \int x^j \mu(dx), \quad \text{and the same with } \sup \text{ for } c_j^+$$

Observe that c_j^- and c_j^+ depends only on $c_1, \dots, c_{j-1}$

Canonical moment p_2



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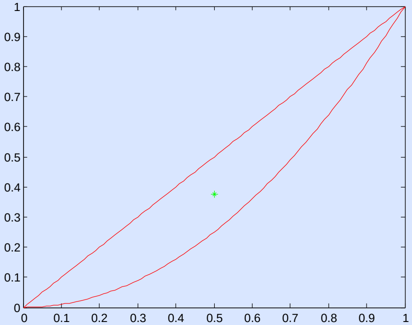

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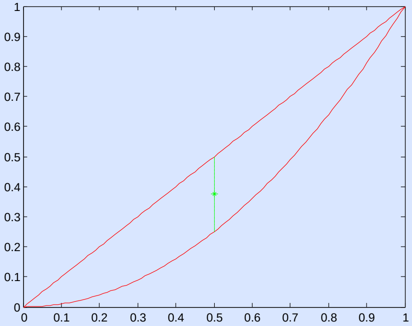

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Other asymptotics?

Large deviations for Z_n^k

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For $c \in \mathbb{M}_k$ define

$$I_k(c) := -\log[r_{k+1}(c)] - k \log 4$$

(and $I_k(c) = +\infty$ if $c \in \mathbb{R}^k$ but $c \notin \mathbb{M}_k$)
I_k is a rate function (in the terminology of large deviations)

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Not completely satisfying depends on k!!!

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- The proof relies on LD projective limit and Szegö theorem
- obvious nice Corollary linking the two Theorems:

$$r_{k+1}(c) = \exp\left(-\inf_{P \text{ fitting } c} I(P) - k \log 4\right) \quad (c \in \mathbb{M}_k).$$

Other statistic: Range process

Full evolution of the range: the range process

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- *Let* $(B_t))_{t\in[0,T]}$ *denotes a standard Brownian motion*

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Other statistic: Range process

Full evolution of the range: the range process

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Idea of the proof: $(R_t^n)_{t\in[0,T]}$ is a product of β r.vs

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- Multidimensional moment problem on the simplex: everything remains true on a polyhedral approximation of the moment space, ν is the uniform measure on the simplex (Lozada EJP)
- **More general distributions on moment spaces: Tilted distributions, generalized Dirichlet distributions (G-Lozada AOP, Barthe et al. ALEA- The non compact case: Dette et al to appear AOP)**

Another frame: trigonometric moment problem

The trigonometric moment space

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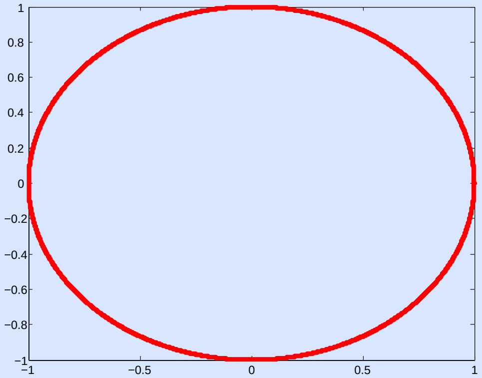
$\mathbb{M}_k^{\mathbb{T}} =$ convex hull of the curve in $\mathbb{C}^k$, $(x^j)_{j=1,\dots,k, x \in \mathbb{T}}$

Trivial example: the space $\mathbb{M}_1^{\mathbb{T}}$



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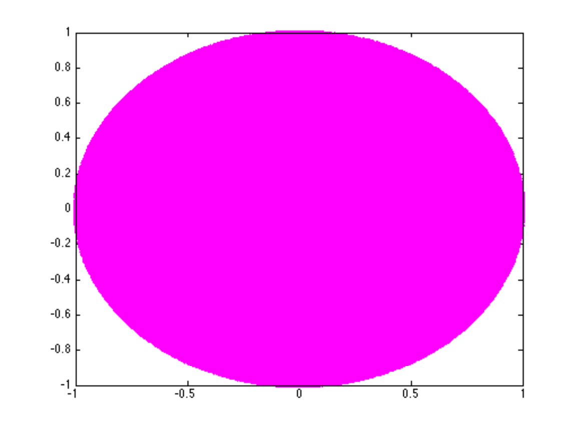

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- Uniform measure on $\mathbb{M}_k^{\mathbb{T}}$
- Canonical coordinates:
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  - Coefficients in othogonal polynomial recursion
  - **Verblunsky coefficients, Partial autocorrelation, reflection coefficients....**

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  - Convergence of the random measure towards the uniform on $\mathbb{T}$
  - CLT for random moments (diagonal covariance)
  - **Fonctional large deviations: rate function reversed Kullback!!!**

Link with other random model: the ball

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Recall the so-called Poincaré Theorem

Theorem

Let X_n be a random vector uniformly distributed on the l_2-ball of $\mathbb{R}^n$ (or $\mathbb{C}^n$). Then,

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Revisiting this known result
→ **here there also is a canonical reparametrization giving independent coordinates** (*Stick breaking:* **Barthe et al. ALEA**)

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→ **This application is built on the Verblunsky coefficients (Barthe et al. ALEA)**

Spectral measure of random matrices

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e_1 first vector of the canonical basis.

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$$A = \Pi D \Pi^*,$$

$D := \mathbf{diag}(\lambda_i)_{i=1,\dots,N}$ $\Pi := (\pi_{ij})_{i,j=1,\dots,N}$ **unitary.**

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The spectral measure μ_A of A have moment of order k equal to $\langle e_1, A^k e_1\rangle$:

$$\mu_A(d\lambda) = \sum_k |\pi_{1k}|^2 \, \delta_{\lambda_k}(\lambda)$$

An easy nice result

Canonical moment representation + Killip et al results on tridiagonal representations give

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Assume that A has the Haar distribution on the unitary group.
Then, the canonical moment of μ_A (Verblunsky coefficients) are independent with *good beta* **distributions.**

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Assume that A has the Haar distribution on the unitary group. Then, the canonical moment of μ_A (Verblunsky coefficients) are independent with good beta distributions.

All the asymptotic results remain true!!!

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A*ssume that A has the Haar distribution on the unitary group. Then, μ_A satisfies a LDP with good rate function the reversed Kullback information with respect to the uniform.*

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YES!!!!

But up to now explicit rate functions only for β-Hermite ensemble

β-Hermite ensemble I

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- **GOE(N)**

β-Hermite ensemble I

- GOE(N) diagonal entries = independent $\mathcal{N}(0; 2/N)$

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  Eigenvectors independent and Haar distributed on $O(N)$

Δ =Vandermonde determinant

β-Hermite ensemble II

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β-Hermite ensemble III

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- **Coulomb gas model**

$$|\Delta(\lambda_1, \dots, \lambda_N)|^\beta \exp -\frac{N\beta}{4} \sum_j \lambda_j^2 .$$

β-Hermite ensemble III

- Coulomb gas model

$$|\Delta(\lambda_1, \dots, \lambda_N)|^\beta \exp -\frac{N\beta}{4} \sum_j \lambda_j^2 .$$

  - $\beta = 1$ for the GOE,
  - $\beta = 2$ for the GUE
  - $\beta = 4$ for the GSE

As $N \to \infty$, Empirical measure goes to the semicircle distribution LDP speed N^2 and rate function connected to the Voiculescu entropy.

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$(|\pi_1|^2, \ldots, |\pi_N|^2)$

- Independent of eigenvalues and
- Dir $N(\beta/2)$ distributed.

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For the β-Hermite ensemble μ_A satisfies a LDP (projective throught its moments) with rate function the reversed Kullback information with respect to the semi-circle+ function involving mass of the measure away from $[-2, 2]$.

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Coments

- Proof quite technical involving *sum rules (B. Simon et al)*
- **Contribution away from $[-2, 2]$ for the rate function involves rate function for extreme eigenvalues (see Ferral)**

End

Gracias por su atencion
Thanks for your attention
Merci
Obrigado
Danke
Grazie