Modeling and estimation for Gaussian fields indexed by graphs, application to road traffic prediction

F. Gamboa et al

IMT

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Framework

Road traffic (Mediamobile) :

- Activity: Real-time prediction of traveling time
- Aim: Understand the speed process on the road traffic network
- Observations :
  - Fixed sensors: corrupted values
  - Cars fleet: unobserved areas
  - The graph is known
- Probem: Use the spatial dependency for:
  - Spatial completion
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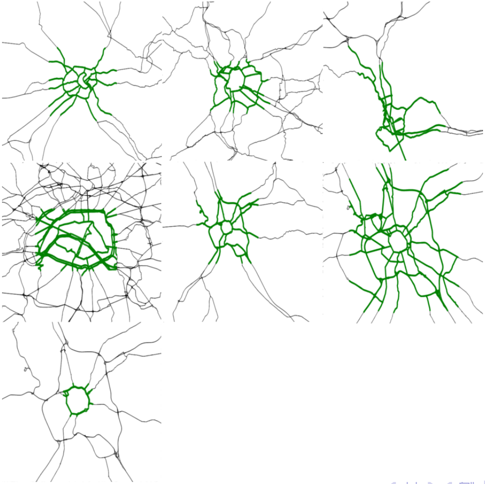
Modeling : Random process $(X_i^{(n)})_{n\in\mathbb{Z},i\in G}$



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Objective: Yield a parametric model $(\mathcal{K}_\theta)_{\theta\in\Theta}$ for covariance operators of X

Outline

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Problem

Speed of vehicles on the road network **at a fixed time**:
zero-mean Gaussian field $(X_i)_{i \in G}$ indexed by the vertices of a graph.

Aim: Chose a model for covariance operators

Modeling constraints

- Adaptability to physical modeling
- Compatibility with classical cases (time series, $\mathbb{Z}^d$, homogeneous tree...)
- Extension of classical tools from time series (spectral representation, Whittle's estimation...)

$\Rightarrow$ Define covariance operators from a spectral construction

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Graph

Model: Zero-mean Gaussian field $(X_i)_{i \in G}$ indexed by the vertices G of a graph **G**.

Definition (Unoriented weigthed graph)

$\mathbf{G} = (G, W)$:

- *G set of vertices (countable)*
- $W \in [-1, 1]^{G \times G}$ *Weighted adjacency operator (symmetric)*

Neighbors: $i \sim j$ if $W_{ij} \neq 0$
Degree of a vertex: $D_i = \sharp \{j, i \sim j\}$.

Assumption (H_0)

- $D := \sup_{i \in G} D_i < +\infty$, **G** *has bouded degree*
- $\forall i \in G, \sum_{j \in G} |W_{ij}| \leq 1$ *even renormalizing*

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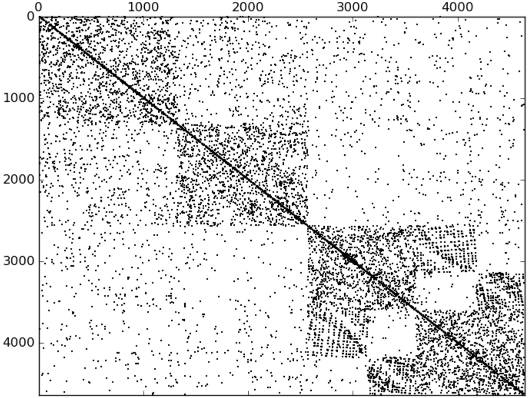
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Modeling for covariance operators

Models for covariance operators (of the speed field)

$$\mathcal{K}(f) = f(W)$$

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$$\mathcal{K}(f) = f(W)$$

W acts on $l^2(G)$:

$$\forall u \in l^2(G), \forall i \in G, (Wu)_i := \sum_{j \in G} W_{ij} u_j .$$

Modeling for covariance operators

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$$\mathcal{K}(f) = f(W)$$

Under H_0

W is a bounded Hilbertian self-adjoint operator in $B_G := l^2(G) \to l^2(G)$:

$$\|W\|_{2,op} \leq 1.$$

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Definition (Identity resolution)

$\mathcal{M}$ σ-algebra $E : \mathcal{M} \to B_G$ such that $\forall \omega, \omega' \in \mathcal{M}$,

1. *$E(\omega)$self-adjoints projectors.*
2. $E(\emptyset) = 0, E(\Omega) = I$
3. $E(\omega \cap \omega') = E(\omega)E(\omega')$
4. *Si $\omega \cap \omega' = \emptyset$, alors $E(\omega \cup \omega') = E(\omega) + E(\omega')$*

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Spectral decomposition

$$\exists E, \mathcal{M}, W = \int_{\mathcal{M}} \lambda \mathrm{d}E(\lambda)$$

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Definition

Local measures

$$\forall i, j \in G, \forall \omega \in \mathcal{M}, \mu_{ij}(\omega) = E_{ij}(\omega).$$

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Caracterized by:

$$\forall i, j \in G, \forall k \in \mathbb{Z}, \left(W^k\right)_{ij} = \int_{\mathrm{Sp}(W)} \lambda^k \mathrm{d}\mu_{ij}(\lambda).$$

Models for covariance operators, spectral density

Definition (Construction of the covariance operators)

Let g be an positive function, analytic over $\mathrm{Sp}(W)$,

$$\mathcal{K}(g) = \int_{\mathrm{Sp}(W)} g(\lambda)\mathrm{d}E(\lambda),$$

- *g polynomial:* $MA_q^{(W)}$
- $\frac{1}{g}$ *polynomial:* $AR_p^{(W)}$ ...

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Remarks:

- $\mathcal{K}(g) = g(W)$
- Dependency in W
- Analogy with $\mathbb{Z}$

$G = \mathbb{Z}$: compatibility with time series

Adjacency operator

$$W_{ij} = \frac{1}{2}\mathbf{1}_{|i-j|=1}.$$

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$$\forall i,j \in G, \forall k \in \mathbb{Z}, \left(W^k\right)_{ij} = \frac{1}{\pi}\int_{[-1,1]} \lambda^k \frac{T_{|j-i|}(\lambda)}{\sqrt{1-\lambda^2}}\mathrm{d}\lambda.$$

T_k : $k^{\text{ième}}$ Chebychev polynomials

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Spectral density

$$f(t) = g(\cos(t))$$

$$\mathcal{K}(g)_{ij} = \frac{1}{2\pi}\int_{[-\pi,\pi]} f(t)\cos\left((j-i)t\right)\mathrm{d}t := \left(\mathcal{T}(f)\right)_{ij}$$

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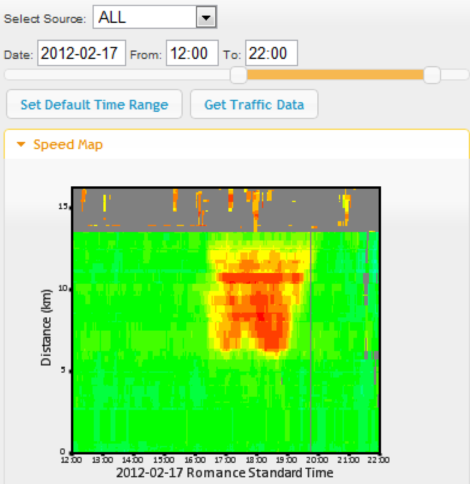

Ideas

Framework: Parametric model of covariances operators

$$\mathcal{K}(f_\theta) = f_\theta(W).$$

Aim: Parametric estimation

Remark: Spectral density $\sim$ Asymptotic eigendistribution of the covariance operators

Computational issues

- $\log \det$ Term of the log-likelihood
- Γ^{-1} term of the log-likelihood

Other important ideas

- Trace measure
- Tappered periodogram

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Problem:

- $\Theta \subset \mathbb{R}$ compact
- $(f_\theta)_{\theta \in \Theta}$ parametric family of spectral densities associated to $\mathcal{K}(f_\theta) = f_\theta(W)$
- Asymptotic on $(\mathbf{G}_n)_{n \in \mathbb{N}}$ sequence of nested subgraphs inducted by $\mathbf{G}$
  Example $G = \mathbb{Z}$: $G_n = [1, n]$.
- $\theta_0 \in \mathring{\Theta}$, $\mathbf{X} \sim \mathcal{N}(0, \mathcal{K}(f_{\theta_0}))$
- We observe the restriction X_n of $\mathbf{X}$ to $\mathbf{G}_n$, cov : $\mathcal{K}_n(f_\theta)$
- $m_n = \sharp G_n$

Aim: Estimate θ_0 with a maximum likelihood method:

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Whittle's approximation for $\mathbb{Z}$, log det

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$$\bar{L}_n(\theta) := -\frac{n}{2}\left(\log(2\pi)+\frac{1}{2\pi}\int_{[0,2\pi]}\log\left(f_\theta\right)\mathrm{d}t+\frac{1}{n}X_n^T\left(\mathcal{T}_n(f_\theta)\right)^{-1}X_n\right)$$

Whittle's approximation for $\mathbb{Z}$, log det

$$\frac{1}{n}\log\det\left(\mathcal{T}_n(f_\theta)\right) \to \frac{1}{2\pi}\int_{[0,2\pi]}\log\left(f_\theta\right)\mathrm{d}t.$$

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Aim: Extension to the graph case

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Let $\delta_n = \sharp \delta G_n$.
Example $G = \mathbb{Z}$: $\delta_n = 2$

log det approximation for a graph

Assumption (Existence of the trace measure)

H_1 : $\exists \mu, \frac{1}{m_n} \sum_{g \in G_n} \mu_{gg} \to \mu$

Assumption (Edge effects)

H_2 : $\delta_n = o(m_n)$

Whittle's approximation for G, log det

$$\frac{1}{m_n} \log\det\left(\mathcal{K}_n(f_\theta)\right) \to \int_{Sp(W)} \log\left(f_\theta\right) \mathrm{d}\mu(t)$$

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$$\forall A \in M_{m_n}(\mathbb{R}), \mathbf{b_n}(A) = \frac{1}{\delta_n} \sum_{i,j \in G_n} |A_{ij}|$$

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Lemma (Asymptotic homomorphism)

$$\mathbf{b_n}\Big(\mathcal{K}_n(f)\mathcal{K}_n(g) - \mathcal{K}_n(fg)\Big) \leq \frac{1}{2}\alpha(f)\alpha(g),$$

where, if $f = \sum_k f_k x^k$,

$$\alpha(f) = \sum_k |f_k| \, (k+1)$$

Consistency

Let $\theta_n, \bar{\theta}_n, \tilde{\theta}_n$ resp. arg max of

$$L_n(\theta) := -\frac{1}{2}\left(m_n \log(2\pi) + \log\det\left(\mathcal{K}_n(f_\theta)\right) + X_n^T \left(\mathcal{K}_n(f_\theta)\right)^{-1} X_n\right)$$

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Assumption (H_3)

- $\theta \to f_\theta$ *injective*
- $\forall \lambda \in Sp(W), \theta \to f_\theta(\lambda)$ *continuous.*
- $\forall \theta \in \Theta, \alpha(\log(f_\theta)) \leq \rho < +\infty$

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Theorem (Consistancy of the Whittle's estimate)

The estimators θ_n,$\bar{\theta}_n$, $\tilde{\theta}_n$ *converge* $P_{f_{\theta_0}}$*-a.s. to the true value* θ_0.

Asymptotic normality and efficiency

We need:

$$\frac{1}{\sqrt{m_n}} \mathbb{E}\left[L_n'(\theta_0)\right] \to 0$$

Problem: Not true in general !!!

- $\mathbb{Z}^d$: X. Guyon, R. Dahlhaus
- Order: $\frac{\delta_n}{m_n}$

Solution: Extension of the tappered periodogram $\mathcal{Q}$.

Framework:

- **Strong** assumptions on the symmetries of the graph
- Contruction of $\mathcal{Q}$
- AR_L
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$$-2L_n^{(u)}(\theta) := m_n \log(2\pi) + m_n \int \log(f_\theta(x))\mathrm{d}\mu(x) + X_n^T \left(\mathcal{Q}_n(\frac{1}{f_\theta}) \right) X_n.$$

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$$\theta_n^{(u)} = \arg\max L_n^{(u)}.$$

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Theorem (Asymptotic normality)

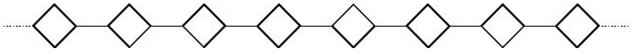
For $\theta_0 \in \mathring{\Theta}$, in the AR_L or MA_L cases, and under assumptions on the graph and the family of spectral densities,$\theta_n^{(u)}$ converges to θ_0, and is asymptotically normal and efficient:

$$\sqrt{m_n}(\theta_n^{(u)} - \theta_0) \to \mathcal{N}\left(0, \left(\frac{1}{2}\int \frac{(f'_{\theta_0})^2}{f_{\theta_0}^2}\mathrm{d}\mu\right)^{-1}\right).$$

Outline

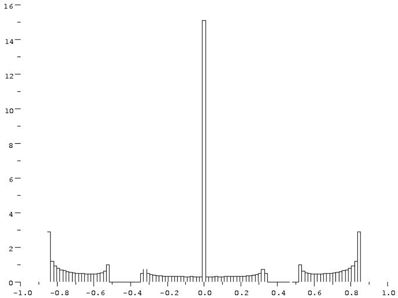
Applications

Figure: Graphe G



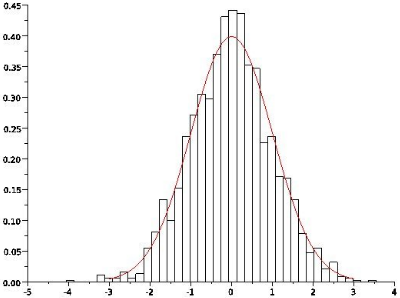
Applications

Figure: Empirical spectral measure

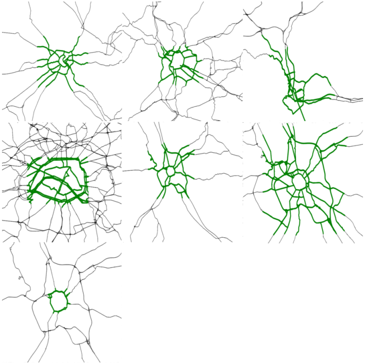


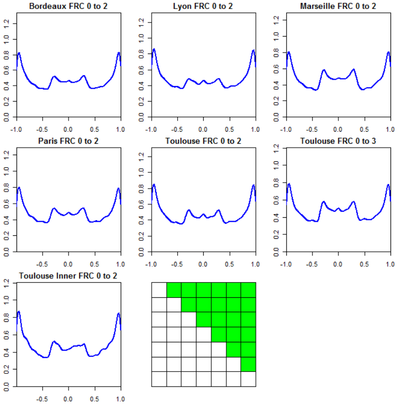
Applications

Figure: Empirical distribution of estimation error



Spectrum of the road network





Real datas

Aim: Predict missing values on *FRC* 0 in Toulouse

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Protocol:

- 10% of datas hidden to test the quality of the prediction
- Model: AR_1

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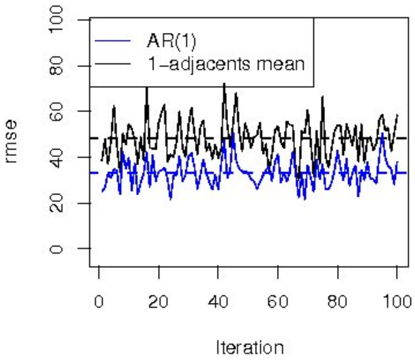
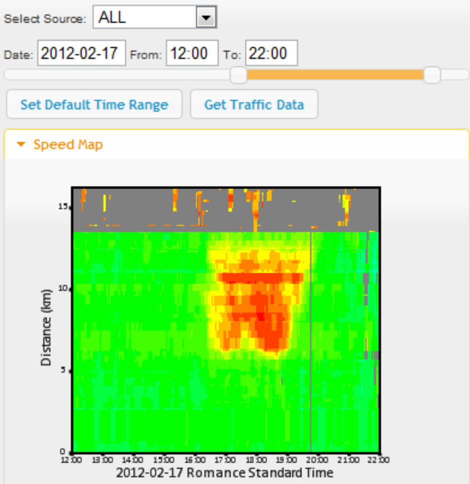
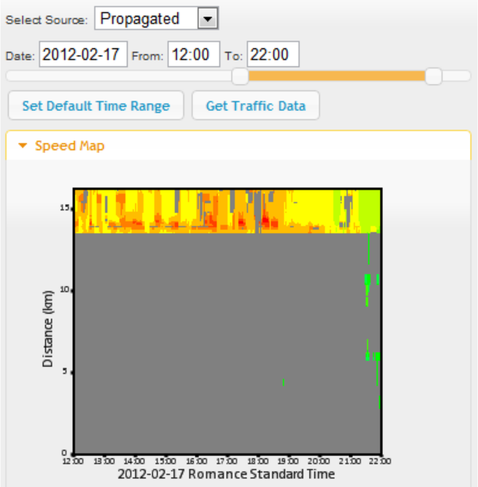


A concrete problem



A solution ?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Set Default Time Range
Get Traffic Data
Speed Map
Distance (km)
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5
10
15
12:00 13:00 14:00 15:00 16:00 17:00 18:00 19:00 20:00 21:00 22:00
2012-02-17 Romance Standard Time

Projects

In progress

- Choice/estimation of the generator
- Spectral study and modeling of the road network
- Maximum likelihood for stationary processes indexed by trees
- “Blind“ prediction

Future works ?

- Link with physicals models
- Use approximation of manifolds by graphs
- Extension of the notion of causality

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Merci !