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C. R. Acad. Sci. Paris, Ser. I 347 (2009) 565–570

Numerical Analysis

Average performance of the approximation in a dictionary using an ℓ_0 objective

François Malgouyres [a], Mila Nikolova [b]

[a] *Université Paris 13, CNRS UMR 7539 LAGA, 99, avenue J.B. Clément, 93430 Villetaneuse, France*
[b] *CMLA, ENS Cachan, CNRS, PRES UniverSud, 61, avenue President Wilson, 94230 Cachan, France*

Received 31 October 2008; accepted after revision 16 February 2009

Available online 5 April 2009

Presented by Yves Meyer

Abstract

We consider the minimization of the number of non-zero coefficients (the ℓ_0 "norm") of the representation of a data set in a general dictionary under a fidelity constraint. This (nonconvex) optimization problem leads to the sparsest approximation. The average performance of the model consists in the probability (on the data) to obtain a K-sparse solution—involving at most K nonzero components—from data uniformly distributed on a domain. These probabilities are expressed in terms of the parameters of the model and the accuracy of the approximation. We comment the obtained formulas and give a simulation. ***To cite this article: F. Malgouyres, M. Nikolova, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***
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Résumé

Performance moyenne de l'approximation la plus parcimonieuse avec un dictionnaire. Nous étudions la minimisation du nombre de coefficients non-nuls (la «norme» ℓ_0) de la représentation d'un ensemble de données dans un dictionnaire arbitraire sous une contrainte de fidélité. Ce problème d'optimisation (non-convexe) mène naturellement aux représentations les plus parcimonieuses. La performance moyenne du modèle est décrite par la probablilité que les données mènent à une solution K-parcimonieuse – contenant pas plus de K composantes non-nulles – en supposant que les données sont uniformément distribuées sur un domaine. Ces probabilités s'expriment en fonction des paramètres du modèle et de la précision de l'approximation. Nous commentons les formules obtenues et fournissons une illustration. ***Pour citer cet article : F. Malgouyres, M. Nikolova, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***
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Version française abrégée

Etant donné un dictionnaire $A = \{a_1, \ldots, a_M\} \in \mathbb{R}^{N\times M}$, $\operatorname{rank}(A) = N$, nous considérons l'approximation la plus parcimonieuse $u^* \in \mathbb{R}^M$ des données observées $d \in \mathbb{R}^N$ telle que $Au^* \approx d$. Celle-ci est une solution du problème d'optimisation sous contrainte $(\mathcal{P}_d)$, *cf.* (2), où # dénote la cardinalité, $\|.\|$ est une norme et $\tau > 0$ un paramètre. Pour chaque $d \in \mathbb{R}^N$, la contrainte dans (2) est non vide et u^* vérifie (3).

E-mail addresses: malgouy@math.univ-paris13.fr (F. Malgouyres), nikolova@cmla.ens-cachan.fr (M. Nikolova).

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doi:10.1016/j.crma.2009.02.026

$(\mathcal{P}_d)$ n'est qu'une réecriture de la *meilleure approximation à K termes* (BKTA), *cf.* [3], qui résoud (4). Sous des hypothèses sur le support de la distributon des données, BKTA obéit à des bornes de la forme (5), où u_K^* est une solution de (4) pour $K \to \infty$. L'évaluation des performances fournie par ces bornes est gouvernée par les pires données d, même si c'est une situation invraisemblable. En outre, les résultats sont vagues quand $M > N$ car les performances sont déduites *via* des approximations de (4).

Afin d'évaluer les performances de (2), nous proposons la méthodologie dite *Performance Moyenne en Approximation* : elle consiste à estimatimer $\mathbb{P}(\text{val}(\mathcal{P}_d) \leqslant K)$, $\forall K \in \{0, \ldots, N\}$ en fonction de τ, $\|.\|$ et A, où la probabilité porte sur la variable aléatoire d dont la loi est connue. Par exemple, on peut modéliser des données parcimonieuses comme uniformément distribuées dans une boule ℓ_1. Plus $\mathbb{P}(\text{val}(\mathcal{P}_d) \leqslant K)$ est grande pour K petit, meilleur est le modèle. Cette note résume les résultats obtenus dans [6]. Nous espérons qu'elle ouvre une nouvelle voie pour des recherches futures.

Principaux résultats

Pour $\theta > 0$ et toute fonction $f : \mathbb{R}^N \to \mathbb{R}$, les sous-ensembles de niveau θ sont définis par (6). Pour tout $J \subset \{1, \ldots, M\}$, le sous espace $\mathcal{A}_J$ est défini dans (7), son complément orthogonal est désigné par $\mathcal{A}_J^\perp$ et la projection orthogonale sur $\mathcal{A}_J^\perp$ par $P_{\mathcal{A}_J^\perp}$. Le sous ensemble $\mathcal{A}_J^\tau$ est défini dans (8). Pour tout $K \leqslant N$, nous posons (*cf.* [6] pour les détails) $\mathcal{J}(K)$ par (9), avec la convention que $\mathcal{J}(0) = \{\emptyset\}$. On a alors :

Théorème 0.1. *Pour toute matrice $A \in \mathbb{R}^{N\times M}$,* $\text{rank}(A) = N$ *et pour toute norme $\|.\|$, on a*

$$K \in \{0, \ldots, N\} \quad \Rightarrow \quad \left\{d \in \mathbb{R}^N, \text{val}(\mathcal{P}_d \leqslant K)\right\} = \bigcup_{J \in \mathcal{J}(K)} \mathcal{A}_J^\tau.$$

Pour tout $n \in \mathbb{N}$, la measure de Lebesgue de $C \subset \mathbb{R}^n$ est notée par $\mathbb{L}^n(C)$. Nous définissons les constantes C_J comme dans (10) où f_d est une norme telle que d est uniforme sur $B_{f_d}(\theta)$. Pour tout $K = 0, \ldots, N$, les constantes $\overline{C}_K$ sont définies dans (11). A l'aide de ces notations, nous présentons le résultat principal de [6].

Théorème 0.2. *Soit f_d et $\|.\|$ deux normes, et $A \in \mathbb{R}^{N\times M}$ avec* $\text{rank}(A) = N$. *Pour tout $\theta > 0$, soit d une variable aléatoire uniformément distribuée sur $B_{f_d}(\theta)$. Alors, $\forall K \in \{0, \ldots, N\}$*

$$\mathbb{P}\big(\text{val}(\mathcal{P}_d) \leqslant K\big) = \overline{C}_K (\tau/\theta)^{N-K} + o\big((\tau/\theta)^{N-K}\big) \quad \textit{pour } \tau/\theta \to 0. \tag{1}$$

Commentaires sur les résultats

(a) *Une limitation.* D'après le Théorème 0.2, la précision de $\mathbb{P}(\text{val}(\mathcal{P}_d) \leqslant K)$ augmente lorsque $\tau/\theta \to 0$. Notons que la fonction $o()$ dépend de tous les ingrédients du modèle (f_d, A, $\|.\|$, K et N). Afin d'améliorer les bornes pour des τ/θ raisonnables, il nous faut améliorer les mesures des ensembles $\mathcal{A}_J^\tau$ et de leurs intersection. Dans des cas très généraux, on se heurte à des problèmes ouverts en mathématiques fondamentales, *cf.* [7].

(b) *Amélioration cruciale en approximation.* Si l'on enrichit le dictionnaire avec un *nouvel* élement $a_{M+1} \in \mathbb{R}^N$ (non collinéaire avec aucune de colonnes de A), alors la valeur de $\overline{C}_K$ dans (11) augmente et en conséquence la probabilité dans (1) augmente. *Donc le modèle est améloré.* Cependant, cela ne changera rien dans l'évaluation utilisant (5).

(c) *Le rôle de* $\|.\|$. Dans (2) et (4), le choix de $\|.\|$ est d'une importance cruciale. Les équations (10), (11) et (12) suggèrent d'optimiser une somme pondérée des mesures de $P_{\mathcal{A}_J^\perp}(B_{\|.\|}(1))$.

(d) *Approximation versus compressed-sensing (CS).* Les hypothèses typiques en CS imposent des matrices A telles que les sous espaces vectoriels dans $\mathcal{J}(K)$ sont presque orthogonaux quand K est assez petit. Alors $\mathbb{L}^N(\mathcal{A}_J^\tau \cap \mathcal{A}_{J'}^\tau \cap B_{f_d}(\theta))$, pour $(J, J') \in (\mathcal{J}(K))^2$ diminue, mais de toutes façons ces termes sont asymptotiquement négligeables. l'impact d'une telle hypothése sur les $\overline{C}_K$ est une queston ouverte.

Avant d'appliquer les résultats de CS [8,4] en approximation, il est important d'évaluer la différence entre les performances de $(\mathcal{P}_d)$ sous les hypothéses du CS (voir [1]) et sans restrictions supplémentaires.

Enfin, si l'on utilise $(\mathcal{P}_d)$ en approximation, l'ajout d'une nouvelle colonne a A améliore toujours la performance de $(\mathcal{P}_d)$ (voir (b)). Si $(\mathcal{P}_d)$ est utilisé en CS, la nouvelle colonne peut dégrader sa performance.

(e) *Distribution des données.* Une analyse de (10)–(11) montre que le dictionnaire A doit être construit de telle sorte que $\mathbb{L}^K(\mathcal{A}_J \cap B_{f_d}(1))/\mathbb{L}^N(B_{f_d}(1))$ est le plus grand possible en particulier quand $\#J$ est petit.

1. The ℓ_0 approximation and the evaluation of its performance

We deal with sparse approximation of observed data $d \in \mathbb{R}^N$ using a dictionary $A = \{a_1, \ldots, a_M\}$—an $N \times M$ matrix with $M \geqslant N$ and $\operatorname{rank}(A) = N$. The *sparsest* vector $u \in \mathbb{R}^M$ such that $Au \approx d$ is a solution of the constraint optimization problem $(\mathcal{P}_d)$ below:

$$(\mathcal{P}_d): \quad \begin{cases} \text{minimize}_{u\in\mathbb{R}^M} \ell_0(u), \quad \text{where } \ell_0(u) \stackrel{\text{def}}{=} \#\{1 \leqslant i \leqslant M\colon\ u_i \neq 0\}, \\ \text{under the constraint: } \|Au - d\| \leqslant \tau, \end{cases} \tag{2}$$

where # means cardinality, $\|.\|$ is a norm (e.g. the ℓ_2 norm) and $\tau > 0$ is a parameter. For any $d \in \mathbb{R}^N$, the constraint in $(\mathcal{P}_d)$ is nonempty and the minimum is reached for an u^* such that

$$\operatorname{val}(\mathcal{P}_d) \stackrel{\text{def}}{=} \ell_0(u^*) \leqslant N. \tag{3}$$

Problem $(\mathcal{P}_d)$ is simply a different way to parameterize the so called *best K-term approximation* (BKTA), defined as a solution to

$$\begin{cases} \text{minimize}_{u\in\mathbb{R}^M} \|Au - d\|, \\ \text{under the constraint: } \ell_0(u) \leqslant K, \quad \text{where } 0 \leqslant K \leqslant N. \end{cases} \tag{4}$$

The evaluation of the performances of the latter is a well developed field, see [3]. It is named "non-linear approximation" when $M = N$ and "highly non-linear approximation" when $M > N$. It is mostly developed for infinite dimensional vector spaces. Under hypotheses on the support of the data distribution, it provides bounds of the form

$$\|Au_K^* - d\| \leqslant C/K^\alpha, \quad \text{for } C > 0,\ \alpha > 0, \tag{5}$$

where u_K^* is the BKTA of d. Doing so, it gives the asymptotic behavior of $\|Au^* - d\|$ when K goes to infinity. The performance of the BKTA is governed by the worst data d, even if it is an unlikely scenario.

Unfortunately the results when $M > N$ are vagues: the bounds are typically derived from a heuristic approximation of the solution of (4) (e.g. the Orthogonal Matching Pursuit (OMP)). The resultant performances are heavily biased by the heuristic approximation. Moreover, since the BKTA is NP-hard (see [2]), another critical point is to discriminate between different heuristics. In particular, the non-linear approximation theory fails to discriminate between ℓ_1 approximation (where ℓ_0 is replaced by ℓ_1) and greedy approaches such as OMP. The latter open question is a reasonable perspective for the Average Performance in Approximation (APA).

To evaluate the performance of the sparsest approximation (i.e. $(\mathcal{P}_d)$), we propose the Average Performance in Approximation methodology which consists in estimating for every $K = 0, \ldots, N$

$$\mathbb{P}\big(\operatorname{val}(\mathcal{P}_d) \leqslant K\big) \quad \text{as a function of } \tau, \|.\| \text{ and } A,$$

where the probability is on d and d is a random variable of known distribution law. For example, sparse data are typically modeled as uniformly distributed in an ℓ_1 ball. The larger $\mathbb{P}(\operatorname{val}(\mathcal{P}_d) \leqslant K)$, for K small, the better the model. This note summarizes the results obtained in [6] and emphasizes their meaning. We hope it also provides a focus for future research.

2. Main results

For $\theta > 0$ and any real function f defined on $\mathbb{R}^N$, we denote the θ-level set of f by

$$B_f(\theta) = \big\{w \in \mathbb{R}^N,\ f(w) \leqslant \theta\big\}, \quad \theta > 0. \tag{6}$$

Let us denote, for $J \subset \{1, \ldots, M\}$,

$$\mathcal{A}_J = \operatorname{span}\{a_j\colon\ j \in J\}, \quad \text{where } a_j \text{ is the } j\text{th column of the matrix } A. \tag{7}$$

We systematically denote by $\mathcal{A}_J^\perp$ the orthogonal complement of $\mathcal{A}_J$ in $\mathbb{R}^N$ and by $P_{\mathcal{A}_J^\perp}$ the orthogonal projector onto $\mathcal{A}_J^\perp$. We set

$$\mathcal{A}_J^\tau \stackrel{\text{def}}{=} \mathcal{A}_J + B_{\|.\|}(\tau) \text{ and we have } \mathcal{A}_J^\tau = \mathcal{A}_J + P_{\mathcal{A}_J^\perp}\big(B_{\|.\|}(\tau)\big). \tag{8}$$

Geometrically, $\mathcal{A}_J^\tau$ is an infinite cylinder in $\mathbb{R}^N$: like a τ-thick coat wrapping the subspace $\mathcal{A}_J$.

For any given dimension $K \leqslant N$, we set that (see [6] for the precise definition)

$$\mathcal{J}(K) \text{ is a maximal non-redundant listing of all subspaces } \mathcal{A}_J \subset \mathbb{R}^N \text{ of dimension } K. \tag{9}$$

We always have $\#\mathcal{J}(N) = 1$ and set by convention $\mathcal{J}(0) = \{\emptyset\}$. Theorem 1 is the *key* for the results that follow. It provides an easy geometrical vision of the problem.

Theorem 1. *For any $N \times M$ matrix A with* $\operatorname{rank}(A) = N$, *any norm $\|.\|$, any $\tau > 0$ and any $K \in \{0, \ldots, N\}$, we have*

$$\big\{d \in \mathbb{R}^N, \operatorname{val}(\mathcal{P}_d \leqslant K)\big\} = \bigcup_{J \in \mathcal{J}(K)} \mathcal{A}_J^\tau.$$

For any $n \in \mathbb{N}$, the Lebesgue measure of $C \subset \mathbb{R}^n$ is denoted by $\mathbb{L}^n(C)$. Let us now define

$$C_J = \mathbb{L}^{N-K}\big(P_{\mathcal{A}_J^\perp}\big(B_{\|.\|}(1)\big)\big)\, \mathbb{L}^K\big(\mathcal{A}_J \cap B_{f_d}(1)\big), \quad \text{where } K = \dim(\mathcal{A}_J) \tag{10}$$

and f_d is a norm such that d is uniform on $\theta B_{f_d}(1)$ (typically f_d is the ℓ_1 norm for sparse data).

For any $K = 0, \ldots, N$, define the constants $\overline{C}_K$ as it follows:

$$\overline{C}_K = \frac{\sum_{J \in \mathcal{J}(K)} C_J}{\mathbb{L}^N(B_{f_d}(1))}. \tag{11}$$

Using these notations, we can state the main result of [6]:

Theorem 2. *Let f_d and $\|.\|$ be any two norms and A an $N \times M$ matrix with* $\operatorname{rank}(A) = N$. *For $\theta > 0$, consider a random variable d with uniform distribution on $B_{f_d}(\theta)$. Then, for any $K = 0, \ldots, N$*

$$\mathbb{P}\big(\operatorname{val}(\mathcal{P}_d) \leqslant K\big) = \overline{C}_K (\tau/\theta)^{N-K} + o\big((\tau/\theta)^{N-K}\big) \quad \text{as } \tau/\theta \to 0. \tag{12}$$

The proof involves three main steps:

(i) For $J \in \mathcal{J}(K)$, estimate $\mathbb{L}^N(\mathcal{A}_J^\tau \cap B_{f_d}(\theta))$. It has the form $\theta^N C_J (\tau/\theta)^{N-K} + \theta^N o((\tau/\theta)^{N-K})$.

(ii) Estimate the volume of $\bigcup_{J \in \mathcal{J}(K)} \mathcal{A}_J^\tau \cap B_{f_d}(\theta)$. An important intermediate result says that if $(J, J') \subset \{1, \ldots, N\}^2$ with $\mathcal{A}_J \neq \mathcal{A}_{J'}$ and $\dim(\mathcal{A}_J) = \dim(\mathcal{A}_{J'}) = K$, then $\mathbb{L}^N(\mathcal{A}_J^\tau \cap \mathcal{A}_{J'}^\tau \cap B_{f_d}(\theta)) = \theta^N o((\tau/\theta)^{N-K})$. It becomes negligible when $\tau/\theta \to 0$.

(iii) Divide the result of (ii) by $\theta^N \mathbb{L}^N(B_{f_d}(1))$ to obtain the sought-after probabilities.

3. Meaning of the results

(a) *A limitation.* Theorem 2 describes the behavior of $\mathbb{P}(\operatorname{val}(\mathcal{P}_d) \leqslant K)$ with increasing precision when τ/θ goes to 0. However, the $o()$ function depends on all the ingredients of the model, namely f_d, A, $\|.\|$, K and N. For most interesting scenarios, the current estimates (see [6]) are only valid for a range of values τ/θ which are too small, when compared to the values τ/θ which are met in applications. In order to provide accurate bounds for adequate τ/θ, we need better measures for the sets $\mathcal{A}_J^\tau$ and for their intersections of different orders. One can notice that, in the very general setting of $(\mathcal{P}_d)$, this comes up against fundamental mathematical problems relevant to Geometry of Banach Spaces that remain open—see [7].

(b) *Crucial improvement in Approximation.* If we enrich the dictionary with a *new* element $a_{M+1} \in \mathbb{R}^N$ (i.e. which is collinear to none of the columns of A), then A is $N \times (M+1)$ and $\mathcal{J}(K)$ contains more elements (except if $K \in \{0, N\}$). As a consequence, the value of $\overline{C}_K$ in (11) increases and then the probability in (12) increases. *Hence the model is improved.* In words, adding an element to the dictionary improves the average performance. However, the performance might not change for the worst case analysis, since we can hardly expect the new element to improve the approximation of all the worst possible data. The possibility to draw such a conclusion emphasizes the benefit of the APA over a worst-case point of view.

(c) *The role of* $\|.\|$. In $(\mathcal{P}_d)$ and in the BKTA (4), the choice of $\|.\|$ is of critical importance. Equations (10), (11) and (12) suggest to optimize a weighted sum of the measures of $P_{\mathcal{A}_J^\perp}(B_{\|.\|}(1))$. In particular, the optimal $\|.\|$ depends on the data distribution (f_d) and on the dictionary (A). We can hardly expect that there is a *universal* choice for $\|.\|$ (e.g. $\|.\| = \|.\|_2$, the Euclidean norm) such that $\overline{C}_K$ reaches its optimal value for *all* $K \in \{1, \dots, N\}$. With this regard, we anticipate that in the derivation of similar results for the ℓ_1 minimization, the corresponding term is different (see [5]), since it is governed by the Kuhn–Tucker optimality conditions.

(d) *Approximation versus compressed-sensing (CS).* In CS, it has been shown that under suitable hypotheses on the matrix A, the data d and when $\|.\|$ is the ℓ_2-norm, some of the heuristics that approximate $(\mathcal{P}_d)$ (or BKTA) provide solutions with the correct support (see [8,4,9]). These results are aggregated with methods for constructing suitable matrices A and the remark that, for these matrices A, the sparsest decomposition is unique for sparse signals. Altogether, this forms the core of CS (see [1] for a typical CS statement). In order to apply the results of [8,4] in the context of approximation it is important to assess the gap between the performance of the sparsest approximation model under the CS assumptions and without any restrictive assumptions.

The typical CS hypothesis lead to special matrices A such that the vector spaces in $\mathcal{J}(K)$ are almost orthogonal when K is small enough. Then $\mathbb{L}^N(\mathcal{A}_J^\tau \cap \mathcal{A}_{J'}^\tau \cap B_{f_d}(\theta))$, for $(J, J') \in (\mathcal{J}(K))^2$ is reduced—see (ii) in the sketch of the proof of Theorem 2—but in any case these terms are asymptotically negligible when τ/θ is small. The impact of such an hypothesis on $\overline{C}_K$ is an open question, see paragraph (e). The discussion on the hypothesis "$\|.\|$ is the Euclidean norm" is in paragraph (c).

Last, if $(\mathcal{P}_d)$ is used for approximation, adding a new column to A always improves the performance of $(\mathcal{P}_d)$—see (b). However, if $(\mathcal{P}_d)$ is used for CS, the new column may degrade its performance. It is e.g. the case if the new column belongs to an element $\mathcal{J}(2)$.

(e) *Distribution of the data.* Analyzing Eqs. (10)–(11) shows that the distribution of the data (defined using f_d) occurs in coefficients $\overline{C}_K$ at first in the numerator—but restricted to a subspace of dimension K—and at second in their denominator as defined on $\mathbb{R}^N$. As a consequence, the matrix A should be designed so that the ratio $\mathbb{L}^K(\mathcal{A}_J \cap B_{f_d}(1))/\mathbb{L}^N(B_{f_d}(1))$ is as large as possible, in particular when $\#J$ is small.

4. Two examples in $\ell_p(\mathbb{R}^N)$ spaces

For any $1 \leqslant p < \infty$, the ℓ_p norm of a vector $u \in \mathbb{R}^n$, $\forall n \in \mathbb{N}$, is denoted $\|u\|_p = (\sum_{i=1}^n |u_i|^p)^{\frac{1}{p}}$. It may be useful to remind that the volume of the unit ball of $\ell_p(\mathbb{R}^n)$ reads [7]

$$V_n(p) = \big(2\Gamma(1+1/p)\big)^n / \Gamma(1+n/p), \tag{13}$$

where Γ is the usual gamma function. Remember that (beside for $p = \infty$), volumes $V_n(p)$ rapidly decay to 0 as far as n increases.

4.1. Case $\|.\| = \ell_2$, $f_d = \ell_1$ and $A = \mathrm{Id}$

In this case we write $\overline{C}_K^1$ in place of $\overline{C}_K$. Since $A = \mathrm{Id}$, for every $J \subset \{1, \dots, N\}$ we have $\dim \mathcal{A}_J = \#J$, hence

$$\mathbb{L}^{N-\#J}\big(P_{\mathcal{A}_J^\perp}\big(B_{\|.\|}(1)\big)\big) = V_{N-\#J}(2) \quad \text{and} \quad \mathbb{L}^{\#J}\big(\mathcal{A}_J \cap B_{f_d}(1)\big) = V_{\#J}(1),$$

where $V_n(p)$ is defined in (13). Using (10)–(11) we obtain that $\overline{C}_K^1 = \frac{N!}{K!(N-K)!} \frac{V_{N-K}(2) V_K(1)}{V_N(1)}$, for all $K \in \{0, \dots, N\}$.

4.2. Case $\|.\| = \ell_2$ and $f_d = \ell_2$

We label this case by writing $\overline{C}_K^{2,M}$ in place of $\overline{C}_K$, where A is of size $N \times M$, with $N \leqslant M$. Assume also that A is such that for any $J_1 \subset \{1, \dots, M\}$ with $\#J_1 < N$, we have $\dim(\mathcal{A}_{J_1}) = \#J_1$, and for any $J_2 \subset \{1, \dots, M\}$ satisfying $\#J_1 = \#J_2$ and $J_1 \neq J_2$, we have $\mathcal{A}_{J_1} \neq \mathcal{A}_{J_2}$. Hence $\#\mathcal{J}(N) = 1$ and $\#\mathcal{J}(K) = \frac{M!}{K!(M-K)!}$, for $K = 0, \dots, N-1$. Moreover, for all $J \subset \{1, \dots, M\}$ with $\#J \leqslant N$ we have $\mathbb{L}^{N-\#J}(P_{\mathcal{A}_J^\perp}(B_{\|.\|}(1))) = V_{N-\#J}(2)$ and $\mathbb{L}^{\#J}(\mathcal{A}_J \cap B_{f_d}(1)) = V_{\#J}(2)$. Using (10)–(11) yet again, $\overline{C}_K^{2,M} = \#\mathcal{J}(K) \frac{V_{N-K}(2) V_K(2)}{V_N(2)}$, for all $K \in \{0, \dots, N\}$. Observe that $\overline{C}_K^{2,M}$ depends on A only via M.

$$\frac{\mathbb{P}\left(\mathrm{val}(\mathcal{P}_{d,\beta}^{2,M}) \leq K\right)}{\mathbb{P}\left(\mathrm{val}(\mathcal{P}_{d}^{1}) \leq K\right)}$$

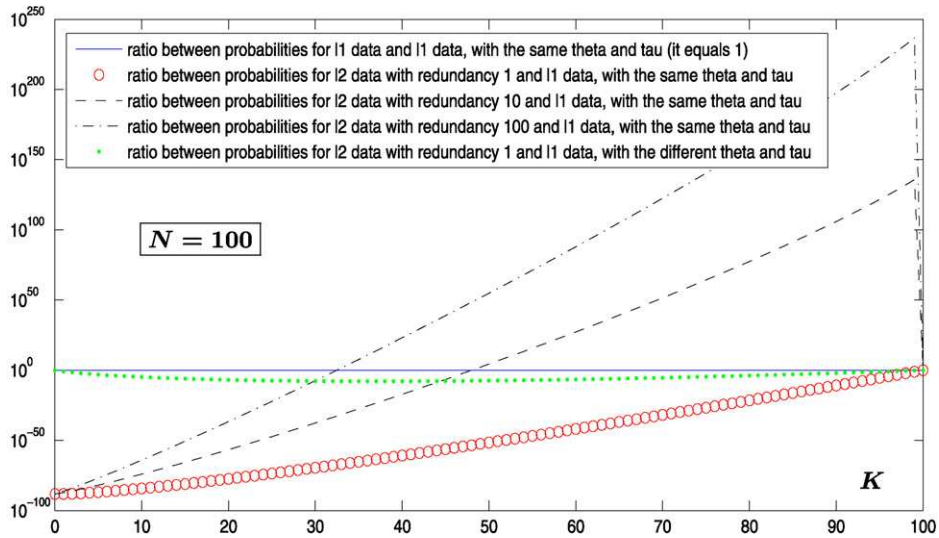

Fig. 1. All curves correspond to various values of M and β where τ/θ is small.

4.3. Simulation in problem comparison

By way of illustration, we consider the following question: How much redundancy (i.e. $\frac{M}{N}$) is needed to capture data uniformly distributed on $B_{\|.\|_2}(\beta\theta)$, for $\beta > 0$, as accurately as data uniformly distributed on $B_{\|.\|_1}(\theta)$ without redundancy?

Denoting $(\mathcal{P}_d^1)$ the problem described in Section 4.1 and $(\mathcal{P}_{d,\beta}^{2,M})$ the problem of Section 4.2, for given M and β, we have

$$\frac{\mathbb{P}(\mathrm{val}(\mathcal{P}_{d,\beta}^{2,M}) \leqslant K)}{\mathbb{P}(\mathrm{val}(\mathcal{P}_{d}^{1}) \leqslant K)} = \frac{\overline{C}_K^{2,M}}{\overline{C}_K^1 \beta^{N-K}} + o(1), \quad \text{as } \tau/\theta \to 0.$$

We display on Fig. 1 $K \to \overline{C}_K^{2,M}\beta^{N-K}/\overline{C}_K^1$ for $N = 10^2$ and: (1) $M \in \{10^2, 10^3, 10^4\}$ with $\beta = 1$; (2) $M = 10^2$ with $\beta = (\frac{V_N(1)}{V_N(2)})^{1/N}$. Experimentally, redundancy does not help when τ/θ and K are small. However, the performances for $B_{\|.\|_2}(\beta\theta)$ seem very close to those for $B_{\|.\|_1}(\theta)$. Notice that $0 < c < \beta = (\frac{V_N(1)}{V_N(2)})^{1/N} \leqslant 1$ where c is a universal constant, see [7].

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