Mathematical analysis of a model which combines total variation and wavelet for image restoration*

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Abstract

We give a mathematical analysis of a model previously introduced for image restoration. This model combines wavelet approaches and total variation approaches in a natural way. We prove the existence of a solution to the model. Then we show a way to approximate it in such a way that the approximation can be computed (this is done by penalization). We also show a simple experiment which illustrates why the model seems promising.

1 Introduction

We address in this paper the issue of image restoration. More precisely, we assume an image $v \in L^2(\mathbb{T})$ ($\mathbb{T}$ is the torus in dimension 2) is obtained from the observation of a landscape, modeled by a function $u \in L^2(\mathbb{T})$, using a measurement tool. The degradation of this measurement tool is itself modeled by

$$v = H(u) + b$$

where H is a linear and continuous operator from $L^2(\mathbb{T})$ into itself and b is a Gaussian noise. For instance, this includes the case of a convolution with a kernel $h \in L^1(\mathbb{T})$

This inverse problem has been widely studied and until now mostly two kinds of approach have been developed and opposed: the wavelet type approach and the variational ones. Among variational approaches, those based on the minimization of the total variation, as introduced in [20], are often considered as being the most efficient (see [2, 4, 9, 14, 17, 18]). On the other hand, wavelet soft-thresholding methods was introduced by Donoho and Johnstone and are studied and extended in several papers (see [6, 7, 12, 13, 19, 21]). They somewhat are an extension of Fourier based methods such as Wiener one [1]. Note that recently some attempts have been made to combine both approaches [3, 5, 8, 21].

This paper complements a previous one (see [16]). It contains mathematical proves of some theorems (in Section 2) stated there and a simple and comprehensive experiment (in Section 3).

In [16], we exposed a natural way to combine wavelet types and variational methods. We say "natural" since it appears that both kinds of approach can be expressed in a unique framework. This framework is then used to combine them.

This model simply consists in looking for a solution of

$$\begin{array}{rl} \text{Minimize,} & \int_{\mathbb{T}} |\nabla w| \\ \text{under the constraint} & (H(w) - v) \in \mathcal{N}_{\mathcal{D},\tau} \end{array} \quad (1)$$

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where $\int_{\mathbb{T}} |\nabla w|$ is the total variation and

$$\mathcal{N}_{\mathcal{D},\tau} = \{w \in L^2(\mathbb{T}), \forall \Psi \in \mathcal{D}, |\langle w, \Psi \rangle| \leq \tau\}$$

for a dictionary $\mathcal{D} = \{\Psi_l\}_{l \in I}$ of functions of $L^2(\mathbb{T})$ (typically wavelets).

Note that we abuse of the notation $\int_{\mathbb{T}} |\nabla w|$ since mathematically ∇w (or more precisely the weak derivatives of w) is a Radon measure. The total variation should therefore be understood as the total mass of this Radon measure: $|Dw|(\mathbb{T})$. However, when w is continuously differentiable $|Dw|(\mathbb{T}) = \int_{\mathbb{T}} |\nabla w|$. One can refer to [10] for a description of the total variation as well as for theorems used in this paper. We would like to recall three important properties of the total variation:

- it is lower semicontinuous
- On the space $BV(\mathbb{T}) = \{w \in L^1(\mathbb{T}), \int_{\mathbb{T}} |\nabla w| < \infty\}$,

$$\int_{\mathbb{T}} |\nabla w| + \int_{\mathbb{T}} |w|$$

  is a norm and this normed space satisfies a compactness property.
- Poincaré inequality holds in $BV(\mathbb{T})$.

Again, if the reader is not familiar with these notions he can refer to [10] where they are properly stated and proved.

In the following, $\|w\|_p$ refers to the norm in $L^p(\mathbb{T})$: $\left(\int_{\mathbb{T}} |w|^p\right)^{\frac{1}{p}}$.

2 Mathematical analysis of the model

In this section, we are going to show that the proposed model has a solution which can be approximatively computed.

Theorem 1 *Let $v \in L^2(\mathbb{T})$ and H be a linear operator continuous from $L^2(\mathbb{T})$ into itself. Let $\mathcal{D} \subset L^2(\mathbb{T})$ and $\tau > 0$. Assume that $BV(\mathbb{T}) \cap \{w \in L^2(\mathbb{T}), H(w) - v \in \mathcal{N}_{\mathcal{D},\tau}\} \neq \emptyset$ and that there exists $C > 0$ such that for any $w \in \{w \in L^2(\mathbb{T}), H(w) - v \in \mathcal{N}_{\mathcal{D},\tau}\}$, $|\int_{\mathbb{T}} w| \leq C$. Then (1) admits a solution $u_\infty \in BV(\mathbb{T}) \cap \{w \in L^2(\mathbb{T}), H(w) - v \in \mathcal{N}_{\mathcal{D},\tau}\}$.*

Proof. Since $BV(\mathbb{T}) \cap \{w \in L^2(\mathbb{T}), H(w) - v \in \mathcal{N}_{\mathcal{D},\tau}\} \neq \emptyset$, we can build a sequence $(u_n)_{n \in \mathbb{N}}$ such that

$$\forall n \in \mathbb{N}, u_n \in \{w \in L^2(\mathbb{T}), H(w) - v \in \mathcal{N}_{\mathcal{D},\tau}\}$$
$$\text{and } \lim_{n \to \infty} \int_{\mathbb{T}} |\nabla u_n| = \inf_{w \in \{w \in L^2(\mathbb{T}), H(w) - v \in \mathcal{N}_{\mathcal{D},\tau}\}} \int_{\mathbb{T}} |\nabla w|$$

Since $\left(\int_{\mathbb{T}} |\nabla u_n|\right)_{n \in \mathbb{N}}$ converges and is positive, there exists $C' \in \mathbb{R}$ such that for any $n \in \mathbb{N}$

$$0 \leq \int_{\mathbb{T}} |\nabla u_n| \leq C'$$

Moreover, since there exists $C > 0$ such that for any $w \in \{w \in L^2(\mathbb{T}), H(w) - v \in \mathcal{N}_{\mathcal{D},\tau}\}$, $|\int_{\mathbb{T}} w| \leq C$, Poincaré's inequality (see [10], pp189) ensures us that there exists $C'' > 0$ such that

$$\|u_n\|_2 \leq C''$$

Moreover, since $\mathbb{T}$ is compact, we are sure that $\|u_n\|_1$ is also bounded. Therefore, because of compactness of $BV(\mathbb{T})$ (see [10], pp 176) and $L^2(\mathbb{T})$, there exists a sub-sequence $(u_{f(n)})_{n \in \mathbb{N}}$ of $(u_n)_n \in \mathbb{N}$ (here f is an increasing function going from $\mathbb{N}$ into itself) which converges in $L^1(\mathbb{T})$ and converges weakly in $L^2(\mathbb{T})$ to a function $u_\infty \in L^2(\mathbb{T})$ (and therefore in $L^1(\mathbb{T})$).

Let us show that u_∞ is a solution to (1).

Since H is continuous from L^2 into itself, there exists H^* linear and continuous from $L^2(\mathbb{T})$ into itself such that for any w_1 and w_2 in $L^2(\mathbb{T})$

$$\langle H(w_1), w_2\rangle = \langle w_1, H^*(w_2)\rangle.$$

Therefore, for any $\Psi \in \mathcal{D}$,

$$\begin{aligned}
|\langle H(u_\infty) - v, \Psi\rangle| &= |\langle H(u_\infty), \Psi\rangle - \langle v, \Psi\rangle| \\
&= |\langle u_\infty, H^*(\Psi)\rangle - \langle v, \Psi\rangle| \\
&= |\lim_{n\to\infty} \langle u_{f(n)}, H^*(\Psi)\rangle - \langle v, \Psi\rangle| \\
&= \lim_{n\to\infty} |\langle H(u_{f(n)}) - v, \Psi\rangle| \leq \tau
\end{aligned}$$

So $u_\infty \in \{w \in L^2(\mathbb{T}), H(w) - v \in \mathcal{N}_{\mathcal{D},\tau}\}$. Moreover, since $(u_{f(n)})_{n\in\mathbb{N}}$ converges in $L^1(\mathbb{T})$ to u, lower semi-continuity in $BV(\mathbb{T})$ (see [10], pp 172) ensures us that

$$\int_{\mathbb{T}} |\nabla u_\infty| \leq \liminf_{n\to\infty} \int_{\mathbb{T}} |\nabla u_{f(n)}| = \inf_{w\in\{w\in L^2(\mathbb{T}), H(w)-v\in\mathcal{N}_{\mathcal{D},\tau}\}} \int_{\mathbb{T}} |\nabla w|$$

and u_∞ is a solution of (1). □

Note that the fact that there exists $C > 0$ such that $w \in \{w \in L^2(\mathbb{T}), H(w) - v \in \mathcal{N}_{\mathcal{D},\tau}\}$, $|\int_{\mathbb{T}} w| \leq C$ is obviously satisfied when there exists $\varphi \in \mathcal{D}$ such that $\langle u, 1_{|\mathbb{T}}\rangle = \langle H(u), \varphi\rangle$ (or there exists $(\alpha_n)_{n\in\mathbb{N}} \in l^1(\mathbb{N})$ such that this φ satisfies $\varphi = \sum_{n\in\mathbb{N}} \alpha_n \Psi_n$ for some $\Phi_n \in \mathcal{D}$). For instance when H is a convolution with a kernel h whose Fourier transform satisfies $\hat{h}(0,0) \neq 0$, putting $1_{|\mathbb{T}}$ (or $\frac{1}{\hat{h}(0,0)} 1_{|\mathbb{T}}$) in $\mathcal{D}$ suffices.

Note also that in the case of a convolution, if $\hat{h}(0,0) = 0$, we can simply replace the minimizing sequence $(u_n)_{n\in\mathbb{N}}$ by $(v_n = u_n - \int_{\mathbb{T}} u_n)_{n\in\mathbb{N}}$. It is clear that $(v_n)_{n\in\mathbb{N}}$ is still a minimizing sequence and that we can follow the proof above and show that it converges (up the extraction of a sub-sequence). The same argument applies to the proof of the next theorem. This shows that our hypotheses could be sharper.

We can unfortunately not guaranty the uniqueness of the solution to (1). Indeed neither $\{w \in L^2(\mathbb{T}), H(w) - v \in \mathcal{N}_{\mathcal{D},\tau}\}$ nor the total variation are strictly convex. However, we could state with this regard some results similar to the one given in [2] and [9].

In order to find a solution to (1), we could of course use a relaxation method, but it might converge very slowly. We could also use a projected steepest descent algorithm but, at each iteration, the projection needs a lot of calculus and can probably be compared to a matching pursuit. This leads us to define a method by penalization.

Theorem 2 *Let $v \in L^2(\mathbb{T})$ and H be a linear operator continuous from $L^2(\mathbb{T})$ into itself. Let $\mathcal{D} \subset L^2(\mathbb{T})$ be a countable set and $\tau > 0$. Assume that $BV(\mathbb{T}) \cap \{w \in L^2(\mathbb{T}), H(w) - v \in \mathcal{N}_{\mathcal{D},\tau'}\} \neq \emptyset$ for a $\tau' < \tau$ and that there exists $C > 0$ such that for any $w \in \{w \in L^2(\mathbb{T}), H(w) - v \in \mathcal{N}_{\mathcal{D},\tau}\}$, $|\int_{\mathbb{T}} w| \leq C$. Then, for any $\epsilon > 0$,*

$$E_\epsilon(w) = \int_{\mathbb{T}} |\nabla w| + \frac{1}{\epsilon} \sum_{\Psi\in\mathcal{D}} \Big(\sup\left(|\langle H(w) - v, \Psi\rangle| - \tau, 0\right)\Big)^2 \tag{2}$$

has a solution $w_\epsilon \in BV(\mathbb{T}) \cap L^2(\mathbb{T})$. Moreover, we can extract a sequence $(w_{\epsilon_n})_{n\in\mathbb{N}}$ (with $\lim_{n\to\infty} \epsilon_n = 0$) that converges in $L^1(\mathbb{T})$ and converges weakly in $L^2(\mathbb{T})$ to a function $w_0 \in BV(\mathbb{T}) \cap \{w \in L^2(\mathbb{T}), H(w) - v \in \mathcal{N}_{\mathcal{D},\tau}\}$. w_0 is solution to (1).

Proof. The proof is decomposed into the following steps:

1. There exists a $w_\epsilon \in BV(\mathbb{T}) \cap L^2(\mathbb{T})$ minimizing (2).
2. There exists $w_0 \in L^2(\mathbb{T})$ and a sub-sequence $(w_{\epsilon_n})_{n\in\mathbb{N}}$ (with $\lim_{n\to\infty} \epsilon_n = 0$) of $(w_\epsilon)_{\epsilon\in\mathbb{R}}$ such that the sub-sequence converges in $L^1(\mathbb{T})$ and converges weakly in $L^2(\mathbb{T})$ to w_0.
3. $w_0 \in \{w \in L^2(\mathbb{T}), H(w) - v \in \mathcal{N}_{\mathcal{D},\tau}\}$.

4. w_0 is a solution to (1).

Proof of 1. This proof follows the same sketch as the one of Theorem 1. First, note that since $BV(\mathbb{T}) \cap \{w \in L^2(\mathbb{T}), H(w) - v \in \mathcal{N}_{\mathcal{D},\tau}\} \neq \emptyset$, we can build a minimizing sequence $(w_\epsilon^n)_{n\in\mathbb{N}}$. Moreover, since $E_\epsilon(w_\epsilon^n)$ converges, it is bounded by a constant C'. Therefore, the total variation $\int_{\mathbb{T}} |\nabla w_\epsilon^n|$ is itself bounded by C'.

Let us show that the mean of w_ϵ^n is bounded. With that in mind, let us consider $w_{int} \in BV(\mathbb{T}) \cap \{w \in L^2(\mathbb{T}), H(w) - v \in \mathcal{N}_{\mathcal{D},\tau'}\}$ and note

$$\lambda_\epsilon^n = \sup_{\Psi\in\mathcal{D}} |\langle H(w_\epsilon^n) - H(w_{int}), \Psi\rangle|.$$

Note that either $\lambda_\epsilon^n < \tau - \tau'$, in which case $w_\epsilon^n \in \{w \in L^2(\mathbb{T}), H(w) - v \in \mathcal{N}_{\mathcal{D},\tau}\}$ and $|\int_{\mathbb{T}} w_\epsilon^n| \leq C$, or $\lambda_\epsilon^n \geq \tau - \tau'$. Note also that we have,

$$\lambda_\epsilon^n \leq \sqrt{\epsilon C'} + \tau + \tau',$$

since, for any $\Psi \in \mathcal{D}$,

$$\begin{aligned}
|\langle H(w_\epsilon^n) - H(w_{int}), \Psi\rangle| &= |\langle H(w_\epsilon^n) - v, \Psi\rangle - \langle H(w_{int}) - v, \Psi\rangle| \\
&\leq \sup(|\langle H(w_\epsilon^n) - v, \Psi\rangle| - \tau, 0) + \tau + \tau' \\
&\leq \sqrt{\epsilon C'} + \tau + \tau'.
\end{aligned}$$

In the case where $\lambda_\epsilon^n \geq \tau - \tau'$, we consider

$$w' = w_{int} + \frac{\tau - \tau'}{\lambda_\epsilon^n}(w_\epsilon^n - w_{int}). \tag{3}$$

By construction, we have

$$\begin{aligned}
|\langle H(w') - v, \Psi\rangle| &\leq |\langle H(w_{int}) - v, \Psi\rangle| + \frac{\tau - \tau'}{\lambda_\epsilon^n}|\langle H(w_\epsilon^n) - H(w_{int}), \Psi\rangle| \\
&\leq \tau' + \frac{\tau - \tau'}{\lambda_\epsilon^n}\lambda_\epsilon^n = \tau.
\end{aligned}$$

So

$$|\int_{\mathbb{T}} w'| \leq C.$$

This permits us to conclude that

$$|\int_{\mathbb{T}} w_\epsilon^n| \leq |\int_{\mathbb{T}} w_{int}| + \frac{\lambda_\epsilon^n}{\tau - \tau'}|\int_{\mathbb{T}} (w' - w_{int})| \leq C + \frac{\sqrt{\epsilon C'} + \tau + \tau'}{\tau - \tau'} 2C$$

So, there exists $C'' > 0$, such that for all $n \in \mathbb{N}$

$$|\int_{\mathbb{T}} w_\epsilon^n| \leq C''.$$

Therefore, w_ϵ^n is bounded in $BV(\mathbb{T})$, $L^2(\mathbb{T})$ and $L^1(\mathbb{T})$ (since $\mathbb{T}$ is compact). So there exists a subsequence $(w_\epsilon^{f(n)})_{n\in\mathbb{N}}$ of $(w_\epsilon^n)_{n\in\mathbb{N}}$ and a function $w_\epsilon \in L^2(\mathbb{T})$ such that $(w_\epsilon^{f(n)})_{n\in\mathbb{N}}$ converges in $L^1(\mathbb{T})$ and converges weakly in $L^2(\mathbb{T})$ to w_ϵ.

A reasoning similar to the one of the proof of Theorem 1 permits to conclude that

$$|\langle H(w_\epsilon) - v, \Psi\rangle| = \lim_{n\to\infty} |\langle H(w_\epsilon^{f(n)}) - v, \Psi\rangle|$$

so (up to a sub-sequence)

$$\sum_{\Psi\in\mathcal{D}}\left(\sup\left(|\langle H(w_\epsilon)-v,\Psi\rangle|-\tau,0\right)\right)^2=\lim_{n\to\infty}\sum_{\Psi\in\mathcal{D}}\left(\sup\left(|\langle H(w_\epsilon^{f(n)})-v,\Psi\rangle|-\tau,0\right)\right)^2$$

and

$$\int_{\mathbb{T}}|\nabla w_\epsilon|\leq\liminf_{n\to\infty}\int_{\mathbb{T}}|\nabla w_\epsilon^{f(n)}|.$$

Therefore, $w_\epsilon\in BV(\mathbb{T})\cap L^2(\mathbb{T})$ and minimizes $E_\epsilon(w)$.

Proof of 2. This proof is in fact almost identical to the previous one. Indeed, note first that taking notations defined above, for any $\epsilon>0$

$$E_\epsilon(w_\epsilon)\leq E_\epsilon(w_{int})=\int_{\mathbb{T}}|\nabla w_{int}|=C'$$

We can define

$$\lambda_\epsilon=\sup_{\Psi\in\mathcal{D}}|\langle H(w_\epsilon)-H(w_{int}),\Psi\rangle|.$$

and once again, we have

$$\lambda_\epsilon\leq\sqrt{\epsilon C'}+\tau+\tau'.$$

Then, we can build w' using the analogue of (3) and use it to prove that there exists $C''>0$ such that for any $\epsilon>0$

$$|\int_{\mathbb{T}}w_\epsilon|\leq C''$$

We can therefore extract a sub-sequence $(w_{\epsilon_n})_{n\in\mathbb{N}}$ (with $\lim_{n\to\infty}\epsilon_n=0$) and there exists $w_0\in L^2(\mathbb{T})\cap L^1(\mathbb{T})$ such that $(w_{\epsilon_n})_{n\in\mathbb{N}}$ converges in $L^1(\mathbb{T})$ and converges weakly in $L^2(\mathbb{T})$ to w_0.

Proof of 3. First, note that because of the weak $L^2(\mathbb{T})$ convergence

$$|\langle H(w_0)-v,\Psi\rangle|=\lim_{n\to\infty}|\langle H(w_{\epsilon_n})-v,\Psi\rangle|$$

Let us now assume $|\langle H(w_0)-v,\Psi\rangle|>\tau$ for a given $\Psi\in\mathcal{D}$ and let τ_0 be such that $|\langle H(w_0)-v,\Psi\rangle|>\tau_0>\tau$. We know that there exists $N>0$ such that for all $n>N$

$$|\langle H(w_{\epsilon_n})-v,\Psi\rangle|>\tau_0.$$

Therefore,

$$E_{\epsilon_n}(w_{\epsilon_n})\geq\frac{1}{\epsilon_n}(\tau_0-\tau)^2.$$

and therefore grows to infinity with n. This contradict the fact that

$$E_{\epsilon_n}(w_{\epsilon_n})\leq E_{\epsilon_n}(w_{int})=\int_{\mathbb{T}}|\nabla w_{int}|.$$

So for any $\Psi\in\mathcal{D}$

$$|\langle H(w_0)-v,\Psi\rangle|\leq\tau.$$

Proof of 4. We remark first that for any $w\in\{w\in L^2(\mathbb{T}),H(w)-v\in\mathcal{N}_{\mathcal{D},\tau}\}$

$$\int_{\mathbb{T}}|\nabla w_{\epsilon_n}|\leq E_{\epsilon_n}(w_{\epsilon_n})\leq E_{\epsilon_n}(w)=\int_{\mathbb{T}}|\nabla w|.$$

Moreover, lower semi-continuity in $BV(\mathbb{T})$ once again ensures us that

$$\int_{\mathbb{T}}|\nabla w_0|\leq\liminf_{n\to\infty}\int_{\mathbb{T}}|\nabla w_{\epsilon_n}|.$$

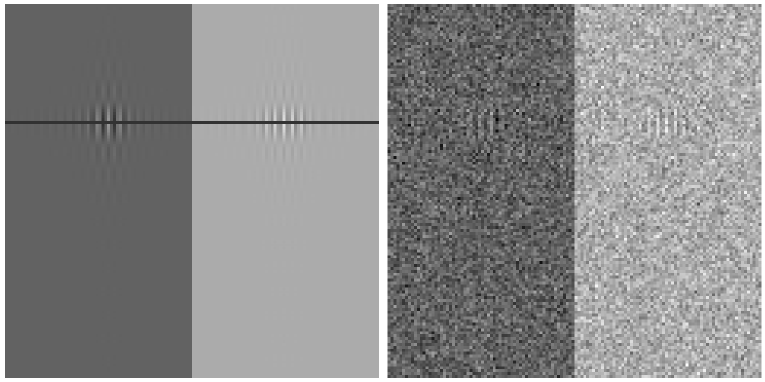

Figure 1: Left: the initial image (the black line is the one extracted for Figure 2 and 3). Right: The noisy image.

So, w_0 is a solution to (1). $\square$

3 Experiments

In this section we will describe a denoising experiment with a synthetic image. Readers interested in deblurring or in seeing experiments on natural images are referred to [16]. The algorithm used to solve (1) is also described in [16].

In the experiments, we start from an initial image which is made of a Heavyside function (of amplitude 130). To this Heavyside function, we add two wavelet packets of approximative amplitude 160 (both wavelet packets are located in high frequency). This is our initial image which is displayed on the left side of Figure 1. The noisy image is obtained by adding to the initial image a Gaussian noise of standard deviation $\sigma = 40$. It is displayed on the right side of Figure 1.

Then, we restore this image. For the display, we only represent an extracted line of reconstructed images. The location of the extracted line is represented by the black line on Figure 1. The extracted lines are then displayed on Figure 2 and 3. Note that we used the lower part of the images to tune the methods parameters in such a way that homogeneous zones look about the same (when possible) or have about the same mean squared error with the initial image.

On Figure 2, we would like to illustrate how badly wavelet basis are adapted to the reconstruction of such an image. This is basically due to the fact that the two wavelet packets we added to the Heavyside function are badly represented in a wavelet basis. Typically, they are represented by a lot of small correlated coefficients (versus one large coefficient in an appropriate wavelet packet basis). Therefore, the comparison between Figure 2 and Figure 3 highlights the advantage of considering a dictionary. Indeed, this latter enables us to preserve several kinds of structure. Note that we do not claim here that wavelet packet bases are better than wavelet bases. We only claim that for images with different kinds of structure constraints using dictionaries are better than constraints using one basis.

Here is the description of Figure 2. Once again, the represented signals are extracted lines from images. The images are (from up to down):

- The initial image.
- The noisy image.
- The image obtained when doing a cycle-spinning wavelet soft-thresholding of wavelet coefficients in a basis constructed with a cubic spline wavelet. The basis is of depth 3 and the threshold values 140.
- A solution of (1) with a dictionary containing only the wavelet basis described for the preceding image. The parameter τ also values 140.

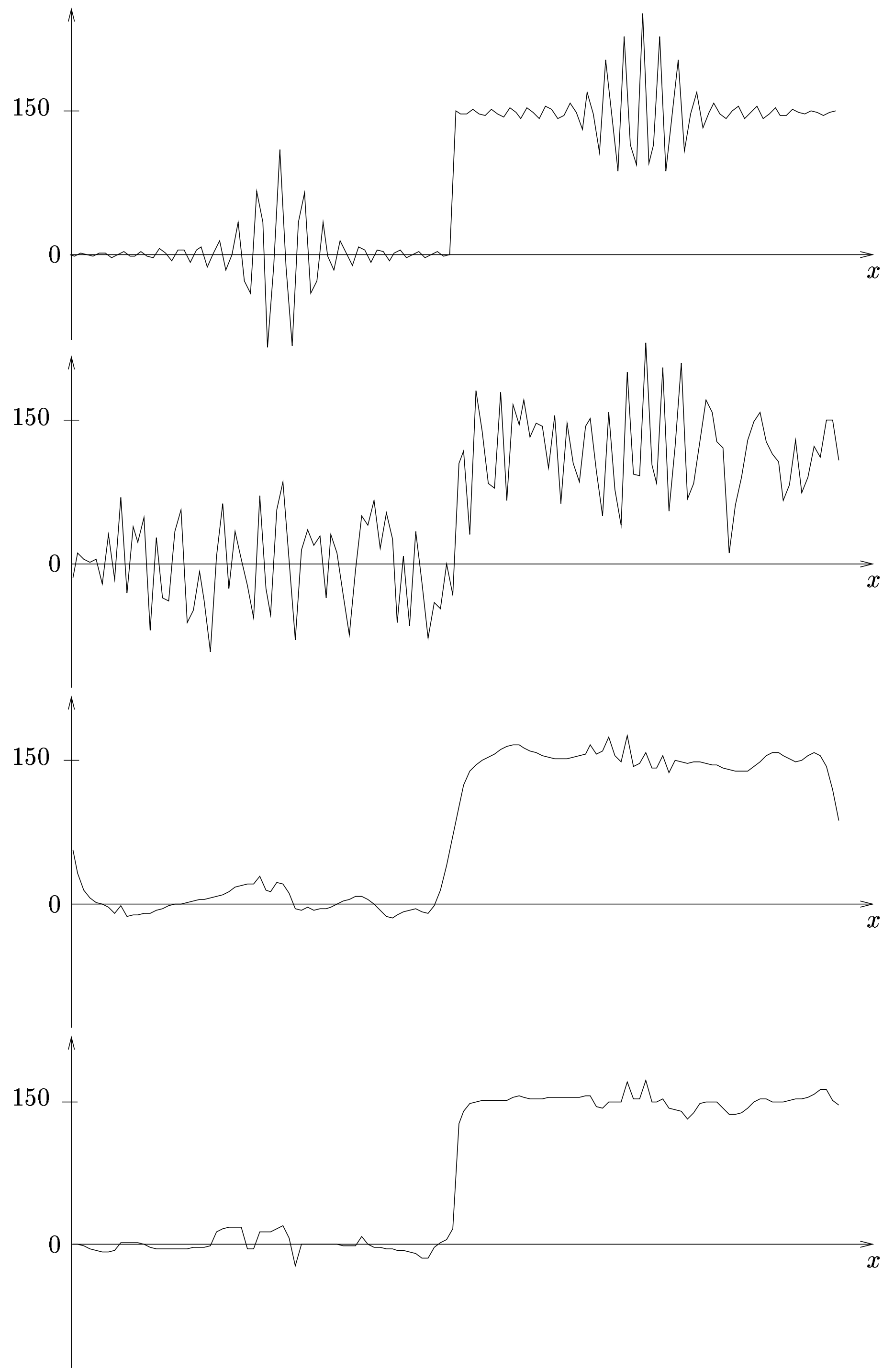

Figure 2: Extracted lines from images. From up to down: the initial image; the noisy image; the denoised image with a wavelet thresholding; a solution of (1) when the dictionary is made of one wavelet basis.

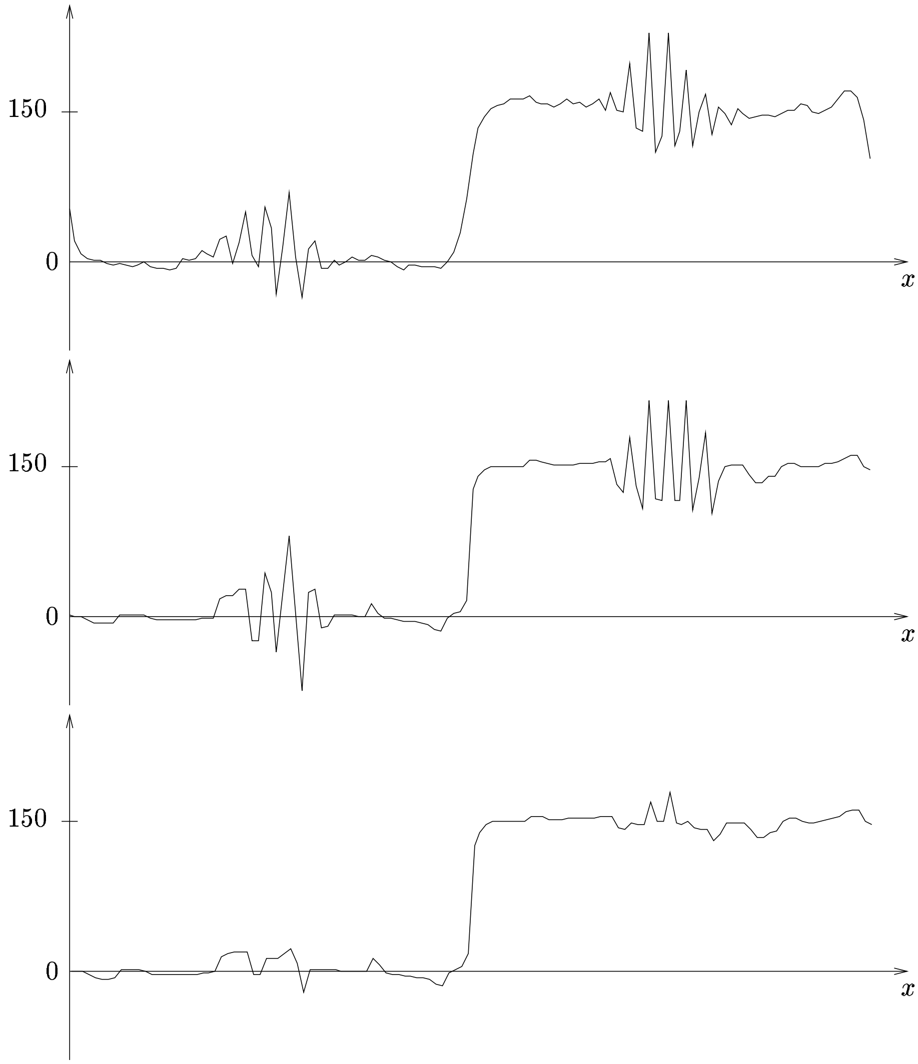

Figure 3: Extracted lines from images. From up to down: the noise selection approach in a wavelet packet dictionary; a solution of (1) with a wavelet packet dictionary; Rudin-Osher-Fatemi method.

We can see on this figure that none of the restoration retrieves the wavelet packets. However, the restoration with (1) permits to obtain a sharper edge. This is basically due to the fact that it does not fill with 0 all the small wavelet coefficients and that some of them contributes to both the sharpness of the edge and the erasure of the ringing artifact.

On Figure 3, we display extracted lines of the following images (from up to down):

- The restoration using the "noise selection" approach which is described in [15]. We would like to mention that since we wrote this paper, we discovered that identical ideas had been published in [11] and [22]. This method is basically an adaptation of wavelet soft-thresholding to dictionaries. We apply it with a wavelet packet dictionary made of all the wavelet packet bases of full depth 1, 2 and 3. Once again, we use a cubic spline. The parameter τ values 160.
- The restoration according to (1) with the dictionary described for the preceding method. τ values 160.

- The Rudin-Osher-Fatemi (see [20])method [1] with a parameter $\lambda = 0.05$. Note that we tuned this parameter in order to have about the same result as in the preceding image far from the wavelet packets. We recall that Rudin-Osher-Fatemi method corresponds to a solution of (1) when taking $\mathcal{D} = \{w \in L^2(\mathbb{T}), \|w\|_2 = 1\}$.

We can see on this results that both methods which use a wavelet packet dictionary somewhat retrieves the wavelet packets. However, the one using the total variation once again has a sharper edge. On the restoration with $\mathcal{D} = \{w \in L^2(\mathbb{T}), \|w\|_2 = 1\}$ (the Rudin-Osher-Fatemi method), we see that since we restrict the move identically in interesting direction (here the two wavelet packets) and in uninteresting directions (for instance $\frac{b}{\|b\|_2}$), we need to erase the wavelet packets to erase the noise. However, it is sharper than the image of the top. This highlights that taking a too large dictionary (especially when it contains uninteresting directions) yields to a large parameter τ. (1) becomes then less efficient.

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