Estimating the probability law of the codelength as a function of the approximation error in image compression

François Malgouyres†

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Abstract

After some recollections on compression of images using a projection onto a polyhedral set (which generalizes the compression by coordinate quantization), we express, in this framework, the probability that an image is coded with K coefficients as an explicit function of the approximation error.

1 Introduction

In the past twenty years, many image processing tasks have been approached using two distinct mathematical tools: image decomposition in a basis and optimization.

The first mathematical approach has proved very useful and is supported by solid theoretical foundations: these guarantee its efficiency, as long as the basis captures the information contained in images. Modelling the image content by appropriate function spaces (of infinite dimension), the mathematical theory tells us how the coordinates of an image, in a given basis, behave. For example, it is possible to characterize Besov spaces (see [12]) and the space of bounded variation (which is "almost characterized" in [3]) with wavelet coefficients. As a consequence of these characterizations, one can obtain performance estimates for practical algorithms (see Th 9.6, pp. 386, in [11] and [5, 4] for some analyses in more complex situations). Image compression and restoration are the typical applications where such analyses are meaningful.

The optimization methods which have been applied to solve those practical problems have also proved very efficient (see [14], for a very famous example).

†LAGA/L2TI, Université Paris 13, 99 avenue Jean-Batiste Clément, 93430 Villetaneuse, France.
malgouy@math.univ-paris13.fr
http://www.math.univ-paris13.fr/∼malgouy/

However, the theory is not able to assess how well they perform, given an image model.

Interestingly, many in the community who were primarily involved in the image decomposition approach are now focusing on optimization models (see, for instance, the work on Basis Pursuit [2] or compressed sensing [6]). The main reason for this is probably that optimization provides a more general framework ([1, 7, 8]).

The framework which seems to allow both a good flexibility for practical applications (see [2] and other papers on Basis Pursuit) and good properties for theoretical analysis is the method of projection onto polyhedra or polytopes. For theoretical studies, it shares simple geometrical properties with the usual image decomposition models (see [10]); this should allow the derivation of approximation results.

The aim of this paper is to state a theorem[1] which relates, asymptotically as the precision grows, the approximation error and the number of coefficients which are coded (which we abusively call codelength, for simplicity). More precisely, when the initial datum is assumed random in a convex set, we give the probability for the datum to be coded by K coefficients, as a function of the approximation error (see Theorem 3.1 for details).

This result is given in a framework which generalizes the usual coding of the quantized coefficients ("non-linear approximation"), as usually performed by compression standards (for instance, JPEG and JPEG2000).

2 Recollection on variational compression

Here and throughout the paper N is a positive integer, $I = \{1, \ldots, N\}$ and $\mathcal{B} = (\psi_i)_{i \in I}$ is a basis of $\mathbb{R}^N$. We will also denote, for $\tau > 0$ (throughout the paper τ stands for a positive real number) and for all $k \in \mathbb{Z}$, $\tau_k = \tau(k - \frac{1}{2})$.

For any $(k_i)_{i \in I} \in \mathbb{Z}^N$, we set

$$\mathcal{C}\left((k_i)_{i \in I}\right) = \left\{ \sum_{i \in I} u_i \psi_i, \forall i \in I, \tau_{k_i} \leq u_i \leq \tau_{k_i + 1} \right\}. \tag{1}$$

We then consider the optimization problem

$$(\tilde{P})\left((k_i)_{i \in I}\right) : \begin{cases} \text{minimize } f(v) \\ \text{under the constraint } v \in \mathcal{C}\left((k_i)_{i \in I}\right), \end{cases}$$

where f is a norm which is continuously differentiable away from 0 and has strictly convex level sets. In order to state Theorem 3.1, we also need f to be *curved*. This means that the inverse of the homeomorphism h below[2] is

[1] The theorem concerning compression in [10] is incorrect. The situation turns out to be more complex than we thought at the time that [10] was written.

[2] We prove in [10] that, under the above hypotheses, h actually is an homeomorphism.

Lipschitz.

$$\begin{array}{rcl} h:\{u\in\mathbb{R}^N,\ f(u)=1\} & \rightarrow & \{g\in\mathbb{R}^N,\ \|g\|_2=1\} \\ u & \mapsto & \frac{\nabla f(u)}{\|\nabla f(u)\|_2}. \end{array}$$

(The notation $\|.\|_2$ refers to the euclidean norm in $\mathbb{R}^N$.)

We denote, for any $(k_i)_{i\in I}\in\mathbb{Z}^N$,

$$\tilde{J}\left((k_i)_{i\in I}\right)=\{i\in I, u_i^*=\tau_{k_i} \text{ or } u_i^*=\tau_{k_i+1}\},$$

where $u^*=\sum_{i\in I}u_i^*\psi_i$ is the solution to $(\tilde{P})\left((k_i)_{i\in I}\right)$.

The interest in these optimization problems comes from the fact that, as explained in [8], we can recover $(k_i)_{i\in I}$ from the knowledge of $(\tilde{J},(u_i^*)_{j\in\tilde{J}})$ (where $\tilde{J}=\tilde{J}\left((k_i)_{i\in I}\right)$).

The problem $(\tilde{P})$ can therefore be used for compression. Given a datum $u=\sum_{i\in I}u_i\psi_i\in\mathbb{R}^N$, we consider the unique $(k_i(u))_{i\in I}\in\mathbb{Z}^N$ such that (for instance)

$$\forall i\in I, \tau_{k_i(u)}\leq u_i<\tau_{k_i(u)+1}. \tag{2}$$

The information $(\tilde{J},(u_i^*)_{j\in\tilde{J}})$, where $\tilde{J}=\tilde{J}\left((k_i(u))_{i\in I}\right)$, is then used to encode u. In the following, we denote the set of indexes that need to be coded to describe u by $\tilde{J}(u)=\tilde{J}\left((k_i(u))_{i\in I}\right)$.

Notice that we can also show (see [8]) that the coding performed by the standard image processing compression algorithms (JPEG and JPEG2000) corresponds to the above model when, for instance,

$$f(\sum_{i\in I}u_i\psi_i)=\left(\sum_{i\in I}|u_i|^2\right)^{\frac{1}{2}}.$$

Observe that the above compression scheme works for any quantization table (see [8]); we restrict to the uniform quantization because Theorem 3.1 only applies in this context. However, several levels of generalization are possible, if one wants to generalize it to more general quantization tables. Notice that, in the theorem, we assume that the data belong to a given level set, denoted $\mathcal{L}_{f_d}(\tau')$, of a norm f_d. Therefore, the code attributed to each coefficient need not to be infinite.

3 The estimate

Theorem 3.1 *Let $\tau'>0$ and U be a random variable whose low is uniform in $\mathcal{L}_{f_d}(\tau')$, for a norm f_d. Assume f satisfies the hypotheses given in Section 2. For any norm $\|.\|$ and any $K\in\{1,\dots N\}$ there exists D_K such that for all $\varepsilon>0$, there exists $T>0$ such that for all $\tau<T$*

$$\mathbb{P}\left(\#\tilde{J}(U)=K\right)\leq D_K E^{\frac{N-K}{N+1}}+\varepsilon,$$

where E is the approximation error[3]*:*

$$E = \mathbb{E}\left(\|U - \tau \sum_{i\in I} k_i(U)\psi_i\| \right).$$

Moreover, if $f(\sum_{i\in I} u_i\psi_i) = \left(\sum_{i\in I} |u_i|^2\right)^{\frac{1}{2}}$*, we also have*[4]

$$\mathbb{P}\left(\#\tilde{J}\left(U\right) = K\right) \geq D_K E^{\frac{N-K}{N+1}} - \varepsilon.$$

The proof of the above theorem is given in [9]. Its two main steps are: the characterization of all the $(k_i)_{i\in I} \in \mathcal{L}_{f_d}(\tau')$ which are coded with K coefficients, for any given $K \in \{1, \ldots N\}$; the census, for each K, of $(k_i)_{i\in I}$ obtained at the first step.

When the above theorem differs from the results evoked in Section 1 in several ways.

First, it concerns variational models which are more general than the model for which the results of Section 1 are usually stated. This is probably the main interest of the current result. For instance, by a reasoning similar to the one used in the proof of Theorem 3.1, it is probably possible to obtain approximation results for redundant transforms.

Secondly, it expresses the distribution of the number of coefficients as a function of the approximation error, whereas earlier results do the opposite. Typically, they bound the approximation error (quantified by the L^2 norm) by a function of the number of coefficients that are coded. The advantages and drawbacks of the different kinds of statements is not very clear. In the framework of Theorem 3.1, the larger D_K (for K small), the better the model compresses the data. However, it is clear that, as the approximation error goes to 0, it is more and more likely to obtain a code of size N. In this respect, the constant D_{N-1} seems to play an important role, since it dominates (asymptotically as τ goes to 0) the probability not to obtain a code of length N.

Thirdly, the theorem is stated for data leaving in a finite dimension vector space and, as a consequence, it does not impose a priori links between the data distribution (the function f_d) and the model (the function f and the basis $\mathcal{B}$). The ability of the model to represent the data is always assessed by the C_K. Of course, an analogue of Theorem 3.1 for data leaving in infinite dimension space would be interesting.

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[4]This assumption is very pessimistic. For instance, the lower bound seems to hold for almost every basis $\mathcal{B}$ of $\mathbb{R}^N$, when f is fixed. However, we have not worked out the details of the proof of such a statement.

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4 Proof of Theorem 3.1

4.1 First properties and recollection

4.1.1 Rewriting $(\tilde{P})$

For any $u \in \mathbb{R}^N$, $(P)(u)$ denotes the optimization problem

$$(P)(u) : \begin{cases} \text{minimize } f(v-u) \\ \text{under the constraint } v \in \mathcal{C}(0), \end{cases}$$

where 0 denotes the origin in $\mathbb{Z}^N$ and $\mathcal{C}(.)$ is defined by (1).

We then denote, for any $u = \sum_{i\in I} u_i \psi_i \in \mathcal{C}(0)$,

$$J(u) = \{i \in I, u_i = \frac{\tau}{2} \text{ or } u_i = -\frac{\tau}{2}\}.$$

With this notation, the set of active constraints of the solution u^* to $(P)(u)$ is simply $J(u^*)$.

Proposition 4.1 *For any $(k_i)_{i\in I} \in \mathbb{Z}^N$*

$$\tilde{J}((k_i)_{i\in I}) = J(u^*),$$

where u^* is the solution to $(P)\left(\tau \sum_{i\in I} k_i\psi_i\right)$.

Proof. Denoting $\tilde{u}^*$ the solution of $(\tilde{P})((k_i)_{i\in I})$ and u^* the solution to $(P)\left(\tau \sum_{i\in I} k_i\psi_i\right)$, we have

$$\tilde{u}^* = u^* + \tau \sum_{i\in I} k_i\psi_i. \quad (3)$$

This can be seen from the fact that $(P)\left(\sum_{i\in I} k_i\psi_i\right)$ is exactly $(\tilde{P})((k_i)_{i\in I})$, modulo a "global translation" by $\tau \sum_{i\in I} k_i\psi_i$. (The rigorous proof of (3) can easily be established using Kuhn-Tucker conditions, see [13], Th 28.3, pp. 281.)

The proposition is then obtained by identifying the coordinates of $\tilde{u}^*$ and u^* in the basis $\mathcal{B}$. □

4.1.2 On projection onto polytopes

We can now adapt the definitions and notations of [10] to the problems $(P)(.)$. Beside Proposition 4.6, all the results stated in this section are proved in [10].

We consider a norm f_d (which will be used latter on to define the data distribution law) and define for any $C \subset \mathbb{R}^N$ and any $A \subset \mathbb{R}$

$$\mathcal{S}_C^A = \left\{u \in \mathbb{R}^N, \exists u^* \in C, u^* \text{ is solution to } (P)(u) \text{ and } f_d(u-u^*) \in A\right\}.$$

This corresponds to all the optimization problems whose solution is in C (we also control the distance between u and the result of $(P)(u)$). Notice that $\mathcal{S}_C^A$

depends on τ. We do not make this dependence explicit since it does not create any confusion, in practice.

We also define the equivalence relationship over $\mathcal{C}(0)$

$$u \sim v \Longleftrightarrow J(u) = J(v).$$

For any $u \in \mathcal{C}(0)$, we denote $\overline{u}$ the equivalence class of u.

In the context of this paper, we obviously have for all $u = \sum_{i\in I} u_i\psi_i \in \mathcal{C}(0)$

$$\overline{u} = \left\{ u^c + \tau \sum_{j\notin J(u)} \beta_j\psi_j, \forall j \notin J(u), -\frac{1}{2} < \beta_j < \frac{1}{2} \right\}, \tag{4}$$

where

$$u^c = \sum_{j\in J(u)} u_j\psi_j.$$

(Here and all along the paper the notation $j \notin J$ stands for $j \in I \setminus J$.)

Let us give some descriptions of $\mathcal{S}_{\cdot}^{\cdot}$.

Proposition 4.2 *For any $u^* \in \partial\mathcal{C}(0)$ and any $v \in \overline{u^*}$,*

$$\mathcal{S}_v^1 = (v - u^*) + \mathcal{S}_{u^*}^1.$$

In words, $\mathcal{S}_v^1$ is a translation of $\mathcal{S}_{u^*}^1$.

Proposition 4.3 *For any $u^* \in \partial\mathcal{C}(0)$, any $v \in \mathcal{S}_{u^*}^{]0,+\infty[}$ and any $\lambda > 0$*

$$u^* + \lambda(v - u^*) \in \mathcal{S}_{u^*}^{]0,+\infty[}.$$

Theorem 4.4 *For any $u^* \in \partial\mathcal{C}(0)$ and any $\tau' > 0$,*

$$\mathcal{S}_{\overline{u^*}}^{]0,\tau']} = \left\{ v + \lambda(u - u^*), \textit{for } v \in \overline{u^*}, \lambda \in]0,\tau'] \textit{ and } u \in \mathcal{S}_{u^*}^1 \right\}$$

We also have (see [10])

Proposition 4.5 *If f satisfies the hypotheses given in Section 2, for any $u^* \in \partial\mathcal{C}(0)$, $\mathcal{S}_{u^*}^1$ is a non-empty, compact Lipschitz manifold of dimension $\#J(u^*)-1$.*

Another useful result for the purpose of this paper is the following.

Proposition 4.6 *If f satisfies the hypotheses given in Section 2, for any $u^* \in \partial\mathcal{C}(0)$ and any $\tau' > 0$, $\mathcal{S}_{u^*}^{]0,\tau']}$ is a non-empty, bounded Lipschitz manifold of dimension $\#J(u^*)$.*

Proof. In order to prove the proposition, we consider $u^* = \sum_{i\in I} u_i^*\psi_i \in \partial\mathcal{C}(0)$ and $u^c = \sum_{i\in J(u^*)} u_i^*\psi_i$. We are going to prove the proposition in the particular case where $u^c = u^*$. Proposition 4.2 and 4.3 permit indeed to generalize the latter result obtained to any $\mathcal{S}_{u^*}^{]0,\tau']}$, for $u^* \in \overline{u^c}$. (They indeed guarantee that $\mathcal{S}_{u^*}^{]0,\tau']}$ is obtained by translating $\mathcal{S}_{u^c}^{]0,\tau']}$.)

In order to prove that $\mathcal{S}_{u^c}^{]0,\tau']}$ is a bounded Lipschitz manifold of dimension $\#J(u^*)$, we prove that the mapping h' defined below is a Lipschitz homeomorphism.

$$\begin{array}{rrcl} h': & \mathcal{S}_{u^c}^1 \times]0,\tau'] & \longrightarrow & \mathcal{S}_{u^c}^{]0,\tau']} \\ & (u,\lambda) & \longmapsto & u^c + \lambda(u-u^c). \end{array} \tag{5}$$

The conclusion then directly follows from Proposition 4.5.

Notice first that we can deduce from Proposition 4.3, that h' is properly defined.

Let us prove that h' is invertible. For this purpose, we consider λ_1 and λ_2 in $]0,\tau']$ and u_1 and u_2 in $\mathcal{S}_{u^c}^1$ such that

$$u^c + \lambda_1(u_1 - u^c) = u^c + \lambda_2(u_2 - u^c). \tag{6}$$

We have

$$\begin{array}{rcl} \lambda_1 & = & f_d(\lambda_1(u_1-u^c)) \\ & = & f_d(\lambda_2(u_2-u^c)) \\ & = & \lambda_2. \end{array}$$

Using (6), we also obtain $u_1 = u_2$ and h' is invertible.

Finally, h' is Lipschitz since, for any λ_1 and λ_2 in $]0,\tau']$ and any u_1 and u_2 in $\mathcal{S}_{u^c}^1$,

$$\begin{array}{rcl} \|\lambda_1(u_1-u^c) - \lambda_2(u_2-u^c)\|_2 & = & \|\lambda_1(u_1-u_2) + (\lambda_1-\lambda_2)(u_2-u^c)\|_2, \\ & \leq & \tau'\|u_1-u_2\|_2 + C|\lambda_1-\lambda_2|, \end{array}$$

where C is such that for all $u \in \mathcal{S}_{u^c}^1$,

$$\|u - u^c\|_2 \leq C.$$

(Remember $\mathcal{S}_{u^c}^1$ is compact, see Proposition 4.5.) $\square$

4.2 The estimate

We denote the discrete grid by

$$\mathcal{D} = \{\tau \sum_{i\in I} k_i\psi_i, (k_i)_{i\in I} \in \mathbb{Z}^N\},$$

and, for $u^* \in \partial\mathcal{C}(0)$ and $(k_j)_{j\in J(u^*)} \in \mathbb{Z}^{J(u^*)}$,

$$\mathcal{D}\left((k_j)_{j\in J(u^*)}\right) = \{\tau \sum_{j\in J(u^*)} k_j\psi_j + \tau \sum_{i\notin J(u^*)} k_i\psi_i, \text{ where } (k_i)_{i\notin J(u^*)} \in \mathbb{Z}^{I\setminus J(u^*)}\}.$$

The set $\mathcal{D}\left((k_j)_{j\in J(u^*)}\right)$ is a slice in $\mathcal{D}$.

Proposition 4.7 *Let $\tau' > 0$, $u^* \in \partial\mathcal{C}(0)$ and $(k_j)_{j\in J(u^*)} \in \mathbb{Z}^{J(u^*)}$,*

$$\#\left(\mathcal{S}_{\overline{u^*}}^{]0,\tau']} \cap \mathcal{D}\left((k_j)_{j\in J(u^*)}\right)\right) \leq 1.$$

Proof. Taking the notations of the proposition and assuming $\mathcal{S}_{\overline{u^*}}^{]0,\tau']} \cap \mathcal{D}\left((k_j)_{j\in J(u^*)}\right) \neq \emptyset$, we consider $(k_i^1)_{i\in I}$ and $(k_i^2)_{i\in I}$ such that

$$\tau \sum_{i\in I} k_i^1 \psi_i \in \mathcal{S}_{\overline{u^*}}^{]0,\tau']} \cap \mathcal{D}\left((k_j)_{j\in J(u^*)}\right)$$

and

$$\tau \sum_{i\in I} k_i^2 \psi_i \in \mathcal{S}_{\overline{u^*}}^{]0,\tau']} \cap \mathcal{D}\left((k_j)_{j\in J(u^*)}\right).$$

Theorem 4.4 guarantees there exist v_1 and v_2 in $\overline{u^*}$, λ_1 and λ_2 in $]0,\tau']$ and u_1 and u_2 in $\mathcal{S}_{u^*}^1$ such that

$$\tau \sum_{i\in I} k_i^1 \psi_i = v_1 + \lambda_1(u_1 - u^*)$$

and

$$\tau \sum_{i\in I} k_i^2 \psi_i = v_2 + \lambda_2(u_2 - u^*).$$

So

$$v_1 + \lambda_1(u_1 - u^*) = v_2 + \lambda_2(u_2 - u^*) + \tau \sum_{i\notin J(u^*)} (k_i^1 - k_i^2)\psi_i.$$

Using (4), we know there exists $(\beta_i^1)_{i\notin J(u^*)}$ and $(\beta_i^2)_{i\notin J(u^*)}$ such that

$$\forall i \notin J(u^*), -\frac{1}{2} < \beta_i^1 < \frac{1}{2} \text{ and } -\frac{1}{2} < \beta_i^2 < \frac{1}{2},$$

$$v_1 = u^c + \tau \sum_{i\notin J(u^*)} \beta_i^1 \psi_i$$

and

$$v_2 = u^c + \tau \sum_{i\notin J(u^*)} \beta_i^2 \psi_i,$$

with $u^c = \sum_{j\in J(u^*)} u_j^* \psi_j$, where $u^* = \sum_{i\in I} u_i^* \psi_i$.

So, letting for all $i \notin J(u^*)$, $\alpha_i = k_i^1 - k_i^2 + \beta_i^2 - \beta_i^1$, we finally have

$$\lambda_1(u_1 - u^*) = \lambda_2(u_2 - u^*) + \tau \sum_{i\notin J(u^*)} \alpha_i \psi_i. \tag{7}$$

Let us assume

$$\max_{i\notin J(u^*)} |\alpha_i| > 0, \tag{8}$$

and consider $0 < \lambda \leq 1$ such that

$$\lambda < \frac{1}{2\max_{i \notin J(u^*)} |\alpha_i|}. \tag{9}$$

We have, using (7),

$$\begin{aligned} u^c + \lambda\lambda_1[(u_1 - u^* + u^c) - u^c] &= u^c + \lambda\lambda_1(u_1 - u^*) \\ &= u^c + \lambda\tau \sum_{i \notin J(u^*)} \alpha_i\psi_i + \lambda\lambda_2(u_2 - u^*) \\ &= v + \lambda\lambda_2[(u_2 - u^* + v) - v], \end{aligned}$$

where $v = u^c + \lambda\tau \sum_{i \notin J(u^*)} \alpha_i\psi_i$. Moreover, using (4) and (9), we know that $v \in \overline{u^c}$. Using Proposition 4.2, we know that

$$u_1 - u^* + u^c \in \mathcal{S}^1_{u^c} \text{ and } u_2 - u^* + v \in \mathcal{S}^1_v.$$

Finally, applying Theorem 4.4, we obtain

$$u^c + \lambda\lambda_1(u_1 - u^*) \in \mathcal{S}^{]0,\tau']}_{u^c} \cap \mathcal{S}^{]0,\tau']}_v.$$

Since the solution to $(P)(u^c + \lambda\lambda_1(u_1 - u^*))$ is unique, we necessarily have $u^c = v$ and therefore $\max_{i \notin J(u^*)} |\alpha_i| = 0$. This contradicts (8) and guarantees that

$$\max_{i \notin J(u^*)} |\alpha_i| = 0.$$

Using the definition of α_i, we obtain, for all $i \notin J(u^*)$,

$$|k_i^1 - k_i^2| = |\beta_i^1 - \beta_i^2| < 1.$$

This implies $k_i^1 = k_i^2$, for all $i \in I$. $\square$

Let us denote, for $u^* \in \partial\mathcal{C}(0)$, the projection onto $Span\,(\psi_j, j \in J(u^*))$ by

$$\begin{array}{rccc} p: & \mathbb{R}^N & \longrightarrow & Span\,(\psi_j, j \in J(u^*)) \\ & \sum_{i \in I} \alpha_i\psi_i & \longmapsto & \sum_{j \in J(u^*)} \alpha_j\psi_j. \end{array}$$

It is not difficult to see that, for any $\tau' > 0$, $u^* \in \partial\mathcal{C}(0)$ and $(k_j)_{j \in J(u^*)} \in \mathbb{Z}^{J(u^*)}$,

$$\#\left(\mathcal{S}^{]0,\tau']}_{u^*} \cap \mathcal{D}\left((k_j)_{j \in J(u^*)}\right)\right) = 1 \Longrightarrow \tau \sum_{j \in J(u^*)} k_j\psi_j \in p\left(\mathcal{S}^{]0,\tau']}_{u^*}\right). \tag{10}$$

Remark 1 Notice that the converse implication does not hold in general. It is indeed possible to build counter examples where $\mathcal{S}^{]0,\tau']}_{u^*}$ *passes between the points of the discrete grid* $\mathcal{D}$*. However, it is not difficult to see that, if* $\tau \sum_{j \in J(u^*)} k_j\psi_j \in p\left(\mathcal{S}^{]0,\tau']}_{u^*}\right)$ *and* $\mathcal{S}^{]0,\tau']}_{u^*} \cap \mathcal{D}\left((k_j)_{j \in J(u^*)}\right) = \emptyset$*, we can build* $(k_i)_{i \notin J(u^*)} \in \mathbb{Z}^{J \setminus J(u^*)}$ *such that*

$$\tau \sum_{j \in J(u^*)} k_j\psi_j + \tau \sum_{i \notin J(u^*)} (k_i + \frac{1}{2})\psi_i \in \mathcal{S}^{]0,\tau']}_{u^c},$$

where

$$u^c = \sum_{j\in J(u^*)} u_j^*\psi_j.(u_j^* \text{ are the coordinates of } u^*)$$

This means that the set $\mathcal{S}_{u^c}^{]0,\tau']}$, which is a manifold of dimension $\#J(u^c)$ living in $\mathbb{R}^N$, intersects a discrete grid. This is obviously a very rare event. Typically, adding to the basis $\mathcal{B}$ some kind of randomness (for instance adding a very small Gaussian noise to every ψ_i) would make it an event of probability 0.

Notice, with this regard, that when $f(\sum_{i\in I} u_i\psi_i) = \sum_{i\in I} |u_i|^2$, we trivially have the equivalence in (10).

A simple consequence of (10) is that

$$\#\left(\mathcal{S}_{\overline{u^*}}^{]0,\tau']}\cap\mathcal{D}\right) \leq \#\left(p\left(\mathcal{S}_{\overline{u^*}}^{]0,\tau']}\right)\cap\left\{\tau\sum_{j\in J(u^*)} k_j\psi_j, (k_j)_{j\in J(u^*)}\in\mathbb{Z}^{J(u^*)}\right\}\right). \tag{11}$$

Notice finally that, for $u^* = \sum_{i\in I} u_i^*\psi_i \in \partial\mathcal{C}(0)$, Proposition 4.2 and Equation (4) guarantees that

$$p\left(\mathcal{S}_{u^c}^1\right) = p\left(\mathcal{S}_{u^*}^1\right),$$

for $u^c = \sum_{j\in J(u^*)} u_j^*\psi_j$.

We therefore have, using also Theorem 4.4, Proposition 4.3 and Equation (4),

$$\begin{aligned} p\left(\mathcal{S}_{\overline{u^*}}^{]0,\tau']}\right) &= \{p(v)+\lambda(p(u)-p(u^*)),\text{ for } v\in\overline{u^*},\lambda\in]0,\tau'] \text{ and } u\in\mathcal{S}_{u^*}^1\},\\ &= \{u^c+\lambda(p(u)-u^c),\text{ for } \lambda\in]0,\tau'] \text{ and } u\in\mathcal{S}_{u^c}^1\},\\ &= p\left(\mathcal{S}_{u^c}^{]0,\tau']}\right). \end{aligned}$$

Finally,

$$\#\left(\mathcal{S}_{\overline{u^*}}^{]0,\tau']}\cap\mathcal{D}\right) \leq \#\left(p\left(\mathcal{S}_{u^c}^{]0,\tau']}\right)\cap\left\{\tau\sum_{j\in J(u^c)} k_j\psi_j, (k_j)_{j\in J(u^c)}\in\mathbb{Z}^{J(u^c)}\right\}\right). \tag{12}$$

Proposition 4.8 *If f satisfies the hypotheses given in Section 2 then, for any $u^* = \sum_{i\in I} u_i^*\psi_i \in \partial\mathcal{C}(0)$, $p\left(\mathcal{S}_{u^c}^{]0,\tau']}\right)$ (where $u^c = \sum_{j\in J(u^*)} u_j^*\psi_j$) is a non-empty, bounded Lipschitz manifold of dimension $\#J(u^*)$.*

Proof. Thanks to Proposition 4.6, it suffices to establish that the restriction of p :

$$\begin{array}{rccc} p' : & \mathcal{S}_{u^c}^{]0,\tau']} & \longrightarrow & p\left(\mathcal{S}_{u^c}^{]0,\tau']}\right)\\ & u & \longmapsto & p(u). \end{array}$$

is a Lipschitz homeomorphism. This latter result is immediate once we have established that p' is invertible.

This proof is similar to the one of Proposition 4.7. Taking the notations of the proposition, we assume that there exist u_1 and u_2 in $\mathcal{S}_{u^c}^{]0,\tau']}$ and $(\alpha_i)_{i\notin J(u^*)} \in \mathbb{R}^{J(u^*)}$ satisfying

$$u_1 = u_2 + \tau \sum_{i\notin J(u^*)} \alpha_i \psi_i.$$

If we assume $\max_{i\notin J(u^*)} |\alpha_i| \neq 0$, we have for $0 < \lambda < \min(1, \frac{1}{2\max_{i\notin J(u^*)} |\alpha_i|})$,

$$\begin{aligned} u^c + \lambda(u_1 - u^c) &= u^c + \tau \sum_{i\notin J(u^*)} \lambda\alpha_i\psi_i + \lambda(u_2 - u^c) \\ &= v + \lambda \left(u_2 + \tau \sum_{i\notin J(u^*)} \lambda\alpha_i\psi_i - v \right) \end{aligned}$$

for $v = u^c + \tau \sum_{i\notin J(u^*)} \lambda\alpha_i\psi_i$. Since $v \in \overline{u^c}$ (see (4)), Proposition 4.2 guarantees that $u_2 + \tau \sum_{i\notin J(u^*)} \lambda\alpha_i\psi_i = u_2 + v - u^c \in \mathcal{S}_v^{]0,\tau]}$. As a consequence, applying Proposition 4.3, we know that

$$u^c + \lambda(u_1 - u^c) \in \mathcal{S}_{u^c}^{\lambda} \cap \mathcal{S}_v^{]0,+\infty[}.$$

Since $(P)(u^c + \lambda(u_1 - u^c))$ has a unique solution, we obtain a contradiction and can conclude that for all $i \notin J(u^*)$, $\max_{i\notin J(u^*)} |\alpha_i| = 0$.

As a consequence, p' is invertible. It is then obviously a Lipschitz homeomorphism. $\square$

Proposition 4.8 guarantees that $p\left(\mathcal{S}_{u^c}^{]0,\tau']}\right)$ is Lebesgue measurable in $\mathbb{R}^{\#J(u^*)}$. Moreover, its Lebesgue measure in $\mathbb{R}^{\#J(u^*)}$ (denoted $\mathbb{L}_{\#J(u^*)}\left(p\left(\mathcal{S}_{u^c}^{]0,\tau']}\right)\right)$) is finite and strictly positive :

$$0 < \mathbb{L}_{\#J(u^*)}\left(p\left(\mathcal{S}_{u^c}^{]0,\tau']}\right)\right) < \infty.$$

Another consequence takes the form of the following proposition.

Proposition 4.9 *Let $\tau' > 0$ and $u^* \in \partial\mathcal{C}(0)$*

$$\lim_{\tau\to 0} \tau^K \# \left(\mathcal{S}_{\frac{u^*}{\tau}}^{]0,\tau']} \cap \mathcal{D}\right) \leq \mathbb{L}_K\left(p\left(\mathcal{S}_{u^c}^{]0,\tau']}\right)\right)$$

where $K = \#J(u^*)$.

Moreover, if the equality holds in (11) *(or equivalently : the equality holds in* (12)*)*

$$\lim_{\tau\to 0} \tau^K \# \left(\mathcal{S}_{\frac{u^*}{\tau}}^{]0,\tau']} \cap \mathcal{D}\right) = \mathbb{L}_K\left(p\left(\mathcal{S}_{u^c}^{]0,\tau']}\right)\right).$$

Proof. In order to prove the proposition, we are going to prove that, denoting $K = \#J(u^c)$,

$$\lim_{\tau\to 0} \tau^K \# \left(p\left(\mathcal{S}_{u^c}^{]0,\tau']}\right) \cap \left\{ \tau \sum_{j\in J(u^c)} k_j\psi_j, (k_j)_{j\in J(u^c)} \in \mathbb{Z}^{J(u^c)} \right\} \right) = \mathbb{L}_K\left(p\left(\mathcal{S}_{u^c}^{]0,\tau']}\right)\right) \tag{13}$$

The conclusion follows from (12).

Let us first remark that, unlike $\mathcal{S}_{u^c}^{]0,\tau']}$, the set

$$A = p\left(\mathcal{S}_{u^c}^{]0,\tau']}\right) - u^c$$

does not depend on τ. This is due to Proposition 9[5], in [10]. Notice also that, because of Proposition 4.8, both A and $p\left(\mathcal{S}_{u^c}^{]0,\tau']}\right)$ are Lebesgue measurable (in $\mathbb{R}^K$) and that

$$\mathbb{L}_K(A) = \mathbb{L}_K\left(p\left(\mathcal{S}_{u^c}^{]0,\tau']}\right)\right).$$

In order to prove the upper bound in (13), we consider the sequence of functions, defined over $\mathbb{R}^K$

$$f_n(u) = \max\left(0, 1 - n \inf_{v\in A} \|u - v\|_2\right).$$

This is a sequence of functions which are both Lebesgue and Riemann integrable and the sequence converges in $L^1(\mathbb{R}^K)$ to $\mathbb{1}_A$ (the indicator function of the set A). So, for any $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that

$$\int f_n \le \int \mathbb{1}_A + \varepsilon.$$

Moreover, we have, for all $u \in \mathbb{R}^K$ and all $n \in \mathbb{N}$,

$$\mathbb{1}_A(u) \le f_n(u).$$

So, denoting $V_\tau = \left\{\tau \sum_{j\in J(u^c)} k_j\psi_j - u^c, (k_j)_{j\in J(u^c)} \in \mathbb{Z}^{J(u^c)}\right\}$,

$$\begin{aligned}
\lim_{\tau\to 0} \tau^K \# \left(p\left(\mathcal{S}_{u^c}^{]0,\tau']}\right) \cap \left\{ \tau \sum_{j\in J(u^c)} k_j\psi_j, (k_j)_{j\in J(u^c)} \in \mathbb{Z}^{J(u^c)} \right\} \right) &= \lim_{\tau\to 0} \tau^K \sum_{v\in V_\tau} \mathbb{1}_A(v) \\
&\le \lim_{\tau\to 0} \tau^K \sum_{v\in V_\tau} f_n(v) \\
&\le \int f_n \\
&\le \int \mathbb{1}_A + \varepsilon \\
&\le \mathbb{L}_K\left(p\left(\mathcal{S}_{u^c}^{]0,\tau']}\right)\right) + \varepsilon.
\end{aligned}$$

[5] The definition of $\mathcal{S}_C^A$ given in the current paper does not allow the rewriting of the proposition 9 of [10]. This is why we have not adapted it in Section 4.1.2.

So,

$$\lim_{\tau\to 0} \tau^K \# \left(p\left(\mathcal{S}_{u^c}^{]0,\tau']}\right) \cap \left\{ \tau \sum_{j\in J(u^c)} k_j \psi_j, (k_j)_{j\in J(u^c)} \in \mathbb{Z}^{J(u^c)} \right\} \right) \leq \mathbb{L}_K \left(p\left(\mathcal{S}_{u^c}^{]0,\tau']}\right)\right)$$

The lower bound in (13) is obtained in a similar way, by considering an approximation of $\mathbb{1}_A$ by a function smaller than $\mathbb{1}_A$ which is Riemann integrable. (For instance : $f_n(u) = 1 - \max(0, 1 - n \ \inf_{v\notin A} \|u - v\|_2))$.) $\square$

From now on , we will denote for all $K \in \{1, \ldots, N\}$

$$C_K = \left\{ \tau \sum_{j\in J} u_j \psi_j, \text{ where } J \subset I, \#J = K \text{ and } \forall j \in J, u_j = -\frac{1}{2} \text{ or } u_j = \frac{1}{2} \right\}$$

The set C_K contains all the "centers" of the equivalence classes of codimension K.

Similarly, we denote

$$\mathcal{C}l_K = \{u^* \in \partial\mathcal{C}(0), \#J(u^*) = K\}.$$

We obviously have, for all $K \in \{1, \ldots, N\}$,

$$\mathcal{C}l_K = \cup_{u^c\in C_K} \overline{u^c}.$$

Since, for all $K \in \{1, \ldots, N\}$, C_K is finite, it is clear from Proposition 4.9 that, for any $\tau' > 0$,

$$\lim_{\tau\to 0} \tau^K \# \left(\mathcal{S}_{\mathcal{C}l_K}^{]0,\tau']} \cap \mathcal{D}\right) \leq \sum_{u^c\in C_K} \mathbb{L}_K \left(p\left(\mathcal{S}_{u^c}^{]0,\tau']}\right)\right) < +\infty$$

Moreover, we have an equality between the above two terms, as soon as the equality holds in (11).

We can finally express the following estimate.

Proposition 4.10 *Let $\tau' > 0$*

$$\lim_{\tau\to 0} \tau^K \# \left(\mathcal{S}_{\mathcal{C}l_K}^{]0,\infty[} \cap \mathcal{L}_{f_d}(\tau') \cap \mathcal{D}\right) \leq \sum_{u^c\in C_K} \mathbb{L}_K \left(p\left(\mathcal{S}_{u^c}^{]0,\tau']}\right)\right)$$

where $K = \#J(u^*)$.

Moreover, if the equality holds in (11) for all $u^c \in C_K$ (or equivalently : the equality holds in (12))

$$\lim_{\tau\to 0} \tau^K \# \left(\mathcal{S}_{\mathcal{C}l_K}^{]0,\infty[} \cap \mathcal{L}_{f_d}(\tau') \cap \mathcal{D}\right) = \sum_{u^c\in C_K} \mathbb{L}_K \left(p\left(\mathcal{S}_{u^c}^{]0,\tau']}\right)\right).$$

Proof. We consider

$$M = \sup_{\{u=\sum_{i\in I} u_i\psi_i, \forall i\in I, |u_i|\leq\frac{1}{2}\}} f_d(u)$$

We have, for all $u^* \in \partial\mathcal{C}(0)$,

$$f_d(u^*) \leq M\tau. \tag{14}$$

We therefore have for all $u \in \mathcal{L}_{f_d}(\tau')$ and for u^* the solution to $(P)(u)$,

$$\begin{array}{rcl} f_d(u-u^*) & \leq & f_d(u) + f_d(u^*) \\ & \leq & \tau' + M\tau. \end{array}$$

So

$$\mathcal{S}_{\mathcal{C}l_K}^{]0,\infty[} \cap \mathcal{L}_{f_d}(\tau') \subset \mathcal{S}_{\mathcal{C}l_K}^{]0,\tau'+M\tau]}.$$

Moreover, it is not difficult to see that (remember h' defined by (5) is an homeomorphism)

$$\lim_{\tau\to 0} \sum_{u^c\in C_K} \mathbb{L}_K\left(p\left(\mathcal{S}_{u^c}^{]0,\tau'+M\tau]}\right)\right) = \sum_{u^c\in C_K} \mathbb{L}_K\left(p\left(\mathcal{S}_{u^c}^{]0,\tau']}\right)\right).$$

We can therefore deduce (from Proposition 4.9) that

$$\lim_{\tau\to 0} \tau^K \#\left(\mathcal{S}_{\mathcal{C}l_K}^{]0,\infty[} \cup \mathcal{L}_{f_d}(\tau') \cap \mathcal{D}\right) \leq \sum_{u^c\in C_K} \mathbb{L}_K\left(p\left(\mathcal{S}_{u^c}^{]0,\tau']}\right)\right)$$

In order to prove the last statement of the proposition, we consider $u^* \in \partial\mathcal{C}(0)$ and $u \in \mathcal{S}_{u^*}^{]0,\tau']}$, we know that

$$\begin{array}{rcl} f_d(u) & \leq & f_d(u-u^*) + f_d(u^*) \\ & \leq & \tau' + M\tau \end{array}$$

So

$$\mathcal{S}_{\mathcal{C}l_K}^{]0,\tau'-M\tau]} \subset \mathcal{S}_{\mathcal{C}l_K}^{]0,\infty[} \cap \mathcal{L}_{f_d}(\tau').$$

Since (again)

$$\lim_{\tau\to 0} \sum_{u^c\in C_K} \mathbb{L}_K\left(p\left(\mathcal{S}_{u^c}^{]0,\tau'-M\tau]}\right)\right) = \sum_{u^c\in C_K} \mathbb{L}_K\left(p\left(\mathcal{S}_{u^c}^{]0,\tau']}\right)\right),$$

we know that the second statement of the proposition holds. $\square$

Another immediate result is useful to state the final theorem. Notice first that we have, for any $(k_i)_{i\in I} \in \mathbb{Z}^N$ and any norm $\|.\|$,

$$\int_{v\in\mathcal{C}((k_i)_{i\in I})} \|v - \tau\sum_{i\in I} k_i\psi_i\| dv = C\tau^{N+1},$$

where

$$C=\int_{\{v=\sum_{i\in I}v_i\psi_i,\forall i\in I,|v_i|\leq\frac{1}{2}\}}\|v\|dv$$

only depends on the particular norm $\|.\|$ and the basis $(\psi_i)_{i\in I}$.

So, denoting U a random variable whose law is uniform in $\mathcal{L}_{f_d}(\tau')$ and $(k_i(U))_{i\in I}$ the discrete point defined by (2), we have

$$\lim_{\tau\to 0}\frac{\mathbb{E}\left(\|U-\tau\sum_{i\in I}k_i(U)\psi_i\|\right)}{\tau^{N+1}}=C. \tag{15}$$

This follows from the fact that the number of points $(k_i)_{i\in I}$ such that $\mathcal{C}((k_i)_{i\in I})$ intersects both $\mathcal{L}_{f_d}(\tau')$ and its complement in $\mathbb{R}^N$ becomes negligible with regard to the number of points $(k_i)_{i\in I}$ such that $\mathcal{C}((k_i)_{i\in I})$ is included in $\mathcal{L}_{f_d}(\tau')$, when τ goes to 0.

We can now state the final result.

Theorem 4.11 *Let $\tau'>0$ and U be a random variable whose low is uniform in $\mathcal{L}_{f_d}(\tau')$, for a norm f_d. For any norm $\|.\|$, any $K\in\{1,\dots N\}$ and any $\varepsilon>0$, there exists $T>0$ such that for all $\tau<T$*

$$\mathbb{P}\left(\#\tilde{J}(U)=K\right)\leq D_K E^{\frac{N-K}{N+1}}+\varepsilon,$$

where E is the approximation error[6] *:*

$$E=\mathbb{E}\left(\|U-\tau\sum_{i\in I}k_i(U)\psi_i\|\right),$$

Moreover, if the equality holds in (11) *(or equivalently : the equality holds in* (12)*) for all $u^c\in C_K$, then we also have*

$$\mathbb{P}\left(\#\tilde{J}(U)=K\right)\geq D_K E^{\frac{N-K}{N+1}}-\varepsilon.$$

The constant D_K is given by

$$D_K=\frac{A_K}{BC^{\frac{N-K}{N+1}}},$$

with

$$A_K=\sum_{u^c\in C_K}\mathbb{L}_K\left(p\left(\mathcal{S}_{u^c}^{]0,\tau']}\right)\right),$$

$$B=\frac{\mathbb{L}_N\left(\mathcal{L}_{f_d}(\tau')\right)}{\mathbb{L}_N\left(\{v=\sum_{i\in I}v_i\psi_i,\forall i\in I,|v_i|\leq\frac{1}{2}\}\right)}$$

and

$$C=\int_{\{v=\sum_{i\in I}v_i\psi_i,\forall i\in I,|v_i|\leq\frac{1}{2}\}}\|v\|dv.$$

[6]When computing the approximation error, we consider the center of $\mathcal{C}\left((k_i)_{i\in I}\right)$ has been chosen to represent all the elements coded by $(\tilde{P})\left((k_i)_{i\in I}\right)$.

Proof. Remark first that, for any $(k_i)_{i\in I} \in \mathbb{Z}^N$, the probability that

$$\tau_{k_i} \leq U_i \leq \tau_{k_i+1},$$

when $U = \sum_{i\in I} U_i \psi_i$ follows a uniform law in $\mathcal{L}_{f_d}(\tau')$, is

$$\frac{\mathbb{L}_N\left(\mathcal{C}((k_i)_{i\in I}) \cap \mathcal{L}_{f_d}(\tau')\right)}{\mathbb{L}_N\left(\mathcal{L}_{f_d}(\tau')\right)}.$$

Therefore, taking the notation of the theorem

$$\mathbb{P}\left(\#\tilde{J}(U) = K\right) = \sum_{(k_i)_{i\in I}\in\mathbb{Z}^N} \mathbb{1}_{\tau\sum_{i\in I} k_i\psi_i \in \mathcal{S}_{\mathcal{C}l_K}^{[0,+\infty[}} \frac{\mathbb{L}_N\left(\mathcal{C}((k_i)_{i\in I}) \cap \mathcal{L}_{f_d}(\tau')\right)}{\mathbb{L}_N\left(\mathcal{L}_{f_d}(\tau')\right)}.$$

If $(k_i)_{i\in I}$ is such that $\mathbb{L}_N\left(\mathcal{C}((k_i)_{i\in I}) \cap \mathcal{L}_{f_d}(\tau')\right) \neq 0$, there exists $v \in \mathcal{C}(0)$ such that $v + \tau\sum_{i\in I} k_i\psi_i \in \mathcal{L}_{f_d}(\tau')$. So, we have

$$\begin{aligned} f_d(\tau\sum_{i\in I} k_i\psi_i) &\leq \tau' + f_d(v) \\ &\leq \tau' + M\tau, \end{aligned}$$

where M is given by (14).

We therefore have

$$\mathbb{P}\left(\#\tilde{J}(U) = K\right) \leq \frac{\mathbb{L}_N\left(\mathcal{C}(0)\right)}{\mathbb{L}_N\left(\mathcal{L}_{f_d}(\tau')\right)} \# \left(\mathcal{S}_{\mathcal{C}l_K}^{]0,+\infty[} \cap \mathcal{L}_{f_d}(\tau' + M\tau) \cap \mathcal{D}\right).$$

The lower bound is obtained with a similar estimation and we obtain

$$\mathbb{P}\left(\#\tilde{J}(U) = K\right) \geq \frac{\mathbb{L}_N\left(\mathcal{C}(0)\right)}{\mathbb{L}_N\left(\mathcal{L}_{f_d}(\tau')\right)} \# \left(\mathcal{S}_{\mathcal{C}l_K}^{]0,+\infty[} \cap \mathcal{L}_{f_d}(\tau' - M\tau) \cap \mathcal{D}\right).$$

Notice finally that

$$\lim_{\tau\to 0} \frac{\#\left(\mathcal{S}_{\mathcal{C}l_K}^{]0,+\infty[} \cap \mathcal{L}_{f_d}(\tau') \cap \mathcal{D}\right)}{\#\left(\mathcal{S}_{\mathcal{C}l_K}^{]0,+\infty[} \cap \mathcal{L}_{f_d}(\tau' \pm M\tau) \cap \mathcal{D}\right)} = 1.$$

The proof is now a straightforward consequence of Proposition 4.10 and (15). More precisely, taking the notations of the theorem and $\varepsilon > 0$, we know that there exists $T > 0$ such that, for all $\tau < T$,

$$\tau^K \# \left(\mathcal{S}_{\mathcal{C}l_K}^{]0,\infty[} \cup \mathcal{L}_{f_d}(\tau' + M\tau) \cap \mathcal{D}\right) \leq A_K + \varepsilon,$$

and

$$\frac{E^{\frac{1}{N+1}}}{C^{\frac{1}{N+1}}} \geq \tau - \varepsilon.$$

So

$$\begin{aligned}
\mathbb{P}\left(\#\tilde{J}\left((K_i)_{i\in I}\right) = K\right) &\leq \frac{\tau^N}{B}\frac{A_K+\varepsilon}{\tau^K} \\
&\leq \frac{A_K+\varepsilon}{B}\left(\left(\frac{E}{C}\right)^{\frac{1}{N+1}}+\varepsilon\right)^{N-K} \\
&\leq \frac{A_K}{BC^{\frac{N-K}{N+1}}}E^{\frac{N-K}{N+1}}+o(1),
\end{aligned}$$

where $o(1)$ is a function of ε which goes to 0, when ε goes to 0. The first inequality of the theorem follows.

The proof of the second inequality of the theorem is similar to one above. $\square$