A noise selection approach of image restoration

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ABSTRACT

This paper deals with a restoration (both denoising and deblurring) method. For instance in the case of denoising, this latter is only a small modification from the usual wavelet thresholding. However, it has the significant advantage to allow the use of several bases in such a way that we select what is considered as information by a basis or another basis or another basis, and so on for as many bases as we want. The computational cost of the method is mainly the computation of the coordinates of the signal (or image) in the bases.

Keywords: wavelet, thresholding, dictionary, pursuit, image, denoising, restoration, signal, deblurring, wavelet packet

1. INTRODUCTION

This paper is mainly concerned with image or signal restoration. More precisely, we are going to introduce a method which generalizes the usual wavelet thresholding procedure to a kind of thresholding of the projection of the signal (image) on elements of a dictionary. Basically, the dictionary will be a set of functions which does not necessarily have to be a basis. Our method permits to preserve information as soon as it yields a large scalar product with one of the element of the dictionary. In this sense it is heuristically close to best basis and matching pursuit algorithms. However, our method can only be used for the restoration (not compression) but it has the advantage of being fast and easy to implement.

Here, we will understand restoration as methods whose aim is to recover a signal (or image) $u \in \mathbb{R}^N$, from a data

$$v_l = (h * u)_l + b_l$$

for N an integer, $l \in \{0, ..., N-1\}$, $h \in \mathbb{R}^N$ and b a noise (we assume it Gaussian).

There is an large number of papers dealing with this problem. Among these papers two "families" are often opposed : the wavelet and variational approaches. Among variational approaches, the ones based on the minimization of the total variation[1] appears to be the most efficient. Since then it has been studied under various aspects.[2–5] Wavelet methods was introduced by Donoho and Johnstone and are studied and extended in several papers.[6–10]

Concerning the particular aspect of using several bases for image restoration, we can first mention the average over translations[11,9] and a multilevel thresholding[9] for image deblurring. There also exists some papers on methods which use several bases whose results are probably close to the method we are proposing here. For instance the best basis and matching pursuit aims at selecting the information with regards to its projection on elements of several bases.[12,13,8,14] These methods cost more computations and are more complex to implement than the one we propose here. However, our "noise selection" approach does not make sense in the framework of signal (or image) compression.

The paper is organized as follows :

- In section 2, we describe the method under consideration. This latter is similar to a thresholding but can be applied in a dictionary (instead of a basis).
- In section 3, we give an experiment in the case of image denoising which shows to evidence that the method permits a better preservation of some textures.
- In section 4, we show an experiment in the case of image deblurring which show to evidence the importance of the order of the elements of the dictionary as well as the advantage of the method compared to usual wavelet packet deblurring.

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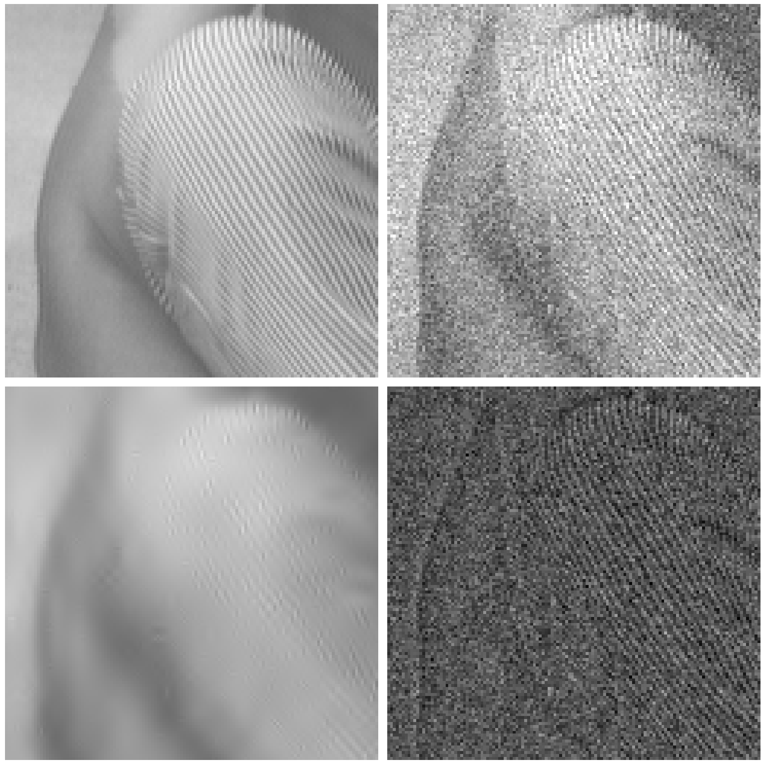

Figure 1. Up-Left : The initial image. Up-Right : The noisy image. Down-Left : translation invariant wavelet soft thresholding. Down-Right : What is consider like noise by the wavelet thresholding.

2. NOISE SELECTION

Let us now introduce the new method under consideration in the case of denoising. It is based on the remark that in order to separate the noise and the information it is simpler to characterize the noise than the information.

Indeed, usual similar methods (for instance wavelet ones) generally use the fact that a given basis gives a sparse representation of an image. Knowing that and the fact that a Gaussian noise has, in practice, a bounded l^∞ norm in the considered basis. These methods basically only preserve the large coefficients in this appropriate basis. They are generally supported by arguments claiming that a given basis yields a sparse representation of a given class of functions.

The method we are proposing here is in fact only a small modification of these latest. It consists simply in taking a "noise point of view". More precisely, another interpretation of the preceding methods consists in saying that we want to remove from the image all that is not considered as information by a given basis. Having said that, it is clear that we do not need to restrict ourselves to a basis and that we can simply remove all what is not considered as information by a given dictionary. As a consequence, our algorithm does not try to recover the information present in an image, it tries to determine the noise (and then we subtract it to the initial data).

This is illustrated on Figure 1. Indeed, it is clear that what is considered like noise by the wavelet thresholding (image Down-Right on Figure 1) has some large coefficients when it is expressed in an adapted basis (for instance a dictionary of texture, a Gabor dictionary or a very decomposed wavelet packet basis). Therefore, if we only consider like noise the small coefficients of this noise in this adapted basis we can preserve the texture. As a consequence, our algorithm does not try to recover the information present in an image, it tries to determine the noise. Let us generalize this to dictionaries and state this mathematically.

Let $\mathcal{D} = \{\psi_l\}_{0 \leq l \leq P}$ be a dictionary of functions $\psi_l \in \mathbb{R}^N$ which are not identically zero (typically, $\mathcal{D}$ can be a union of bases). For simplicity, we assume that $\|\psi_l\|_2 = 1$, for all $l = 0, ..., P$. For any $\sigma > 0$, we define, for $v \in \mathbb{R}^N$, the operator B_σ, which goes from $\mathbb{R}^N$ into itself, by the result of the following iterative process:

$$v_0 = v,$$

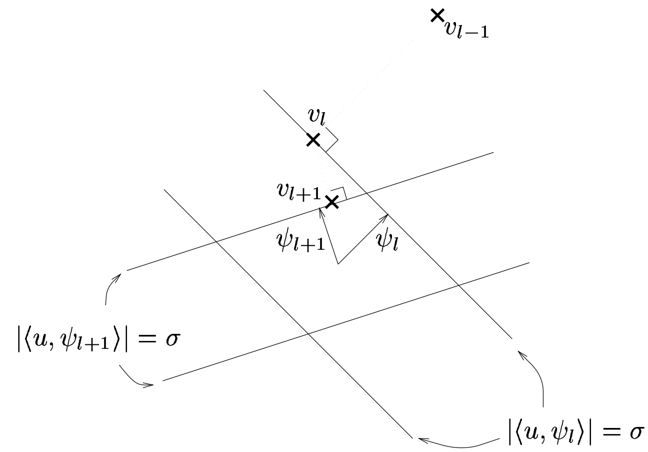

Figure 2. Three iterations of the process which defines $B_\sigma(v)$.

and for $l \in \{0, \cdots, P-1\}$, we let $v_{l+1} = v_l - \delta_{l+1}\psi_{l+1}$ with

$$\delta_{l+1} = \begin{cases} 0 & , \text{if } |\langle v_l, \psi_{l+1}\rangle| \leq \sigma \\ \langle v_l, \psi_{l+1}\rangle - \sigma & , \text{if } \sigma \leq \langle v_l, \psi_{l+1}\rangle \\ \langle v_l, \psi_{l+1}\rangle + \sigma & , \text{if } \langle v_l, \psi_{l+1}\rangle \leq -\sigma\,. \end{cases} \tag{1}$$

Then, we let $B_\sigma(v) = v_P$. We display on Figure 2 three iterations of the above process.

Remark that, when $\mathcal{D}$ contains only one wavelet basis, the function $v - B_\sigma(v)$ is exactly the usual wavelet soft thresholding. Moreover, we could modify (1) in such a way that $v - B_\sigma(v)$ is the hard thresholding (we simply have to set $\delta_{l+1} = \langle v_l, \psi_{l+1}\rangle$ when $|\langle v_l, \psi_{l+1}\rangle| > \sigma$).

Note also that we have

$$|\langle B_\sigma(v), \psi_l\rangle| \leq \sigma\,, \tag{2}$$

for all $l \in \{0, ..., P\}$. This is highlighted by Figure 2 and can be rigorously proved.

Moreover, it is clear that the result of B_σ strongly depends on the order in which the functions are stored in the dictionary $\mathcal{D}$. For instance, if we take for the first elements of $\mathcal{D}$ the basis made of Dirac delta functions on sampling points, for any bright image, after the noise selection, we get a noise which is uniformly equal to σ. It seems therefore advisable to first use functions which highly decorrelate information and noise (such as the elements of a usual wavelet basis). However, it is clear that we can switch any consecutive elements of the dictionary such that

$$\langle \psi_l, \psi_{l+1}\rangle = 0\,.$$

First, this means that, if $\mathcal{D}$ is a union of orthonormal bases, we can compute the coefficient of our image in the first basis, select the noise on all coefficients, without taking care of their order, and then redo this for the other bases. (Instead of computing one coefficient, modify it, compute the second coefficient, ... as described in (1).) Second, this can be used to modify the order of the functions of $\mathcal{D}$ in order to accelerate the process yielding the result. For instance, if $\mathcal{D}$ contains a wavelet basis and a wavelet packet basis, we can use the wavelet decomposition of the image in order to compute its wavelet packet decomposition without changing the result. This yields a “multilevel” kind of wavelet packet noise selection which is equivalent to the composition of a wavelet noise selection and several wavelet packet noise selection.

Remark that, when adding a new function to our dictionary, the computational cost is basically increased by the computation of one scalar product with this function. It even happens that the computation of this scalar product is an intermediate step for the computation of another scalar product; in which case it does almost cost nothing. For instance, if we take the example of the wavelet packet basis based on an admissible binary tree of full depth J, it almost costs nothing to make the noise selection in all the intermediate bases of levels lower than J (of course, here, we do not make the noise selection on the elements of the bases which correspond to low frequencies).

The simplicity of this method permits to envisage the use of very large dictionaries. We can for instance quote the dictionaries generally used in matching pursuit and best basis pursuit[8] : wavelet packets, Gabor dictionary, translations of wavelet and wavelet packet bases. However, it is clear that this method yields better results when used with a dictionary made of different kind of bases. For instance, it could be interesting to add a wavelet packet dictionary with other bases such as the ridgelets,[10] or even the Fourier basis. It could also be interesting to define some criterion (probably derived from the usual criterion saying that a wavelet basis yields a sparse representation of the information) on dictionaries to characterize whether they sufficiently select a decorrelated noise or not. Moreover, we can also use texture dictionaries, dictionaries made of characteristic functions (maybe modified to have a mean equal to 0), dictionaries adapted to particular type of images (for instance if we are only interested in denoising images of faces, it seems accurate to use a dictionary made of eyes, noses, mouths,...), we are mainly restricted by our imagination, computation issues and the fact that the elements of our dictionary must be "autocorrelated".

Note also that it is possible to compose this noise selection with some algorithms of other types. Indeed, (for simplicity we only state this in the case of denoising even thought it is also true for deblurring) given a denoising algorithm yielding a result $Algo(v)$, we can apply the noise selection described above to the function $v - Algo(v)$ and then remove $B_\sigma(v - algo(v))$ to v as a final result. We present in the next section some results when combining a noise selection with the Rudin-Osher-Fatemi* denoising. Note that doing so, we simply denoise the main structure with the Rudin-Osher-Fatemi method and then denoise the texture with our noise selection. This is quit similar to the idea of representing an image as a sketch plus some texture.[15]

3. NUMERICAL RESULT FOR IMAGE DENOISING

We present here some experiments for the noise selection method. All the parameters (the thresholds and the parameter of the Rudin-Osher-Fatemi method) have been estimated empirically. However, when using a larger dictionary we have observed that the method is much more robust to the increase of the threshold than wavelet thresholding. The considered wavelet basis is obtained with a cubic spline wavelet and is of depth 4.

We display on Figure 3 an extracted part of some experiments made on the image "Barbara". They correspond to:

- Up-Left : The initial image.
- Up-Right : The preceding image plus a Gaussian noise of standard variation 30.
- Middle-Left : An image denoised with a wavelet soft thresholding for a threshold $\sigma = 75$.
- Middle-Right : An image denoised with the Rudin-Osher-Fatemi method with a parameter $\lambda = 0.0005$.
- Down-Left : The noise selection with a dictionary† containing a wavelet basis, all the wavelet packet bases of full depth 2, 3 and 4 (again with a cubic spline wavelet) and a Fourier basis. We choose a parameter $\sigma = 95$.
- Down-Right : The noise selection similar to the preceding one computed on what has been considered like noise by Rudin-Osher-Fatemi method (with $\lambda = 0.0005$).

It is clear that the use of several bases permits a better preservation of the textures. Indeed, the texture on the pants yields small wavelet coefficients and is considered like noise by such a basis. However, when we test this noise in a wavelet packet basis with a better frequencial localization, this texture is this time considered as information and is therefore preserved. Moreover, when preliminary using the Rudin-Osher-Fatemi method, the noise selection approach permits to properly restore edges. Furthermore, the difference between this result (Down-Right) and the preceding one (Down-Left) is mostly concentrated on large wavelet coefficients. It can be understood as an alternative to soft and hard thresholding.

Note that all the images are available at http://www.math.ucla.edu/~malgouy

*This method is based on the minimization of

$$\|\nabla u\|_1 + \lambda \|u - v\|_2^2 \,, \tag{3}$$

among $u \in \mathbb{R}^N$, and was introduced in Ref. 1.

†Remark that despite the size of the dictionary the computational cost is almost the one of the calculation of the wavelet packet coefficients in the basis of full depth equal to 4 and the Fourier basis.

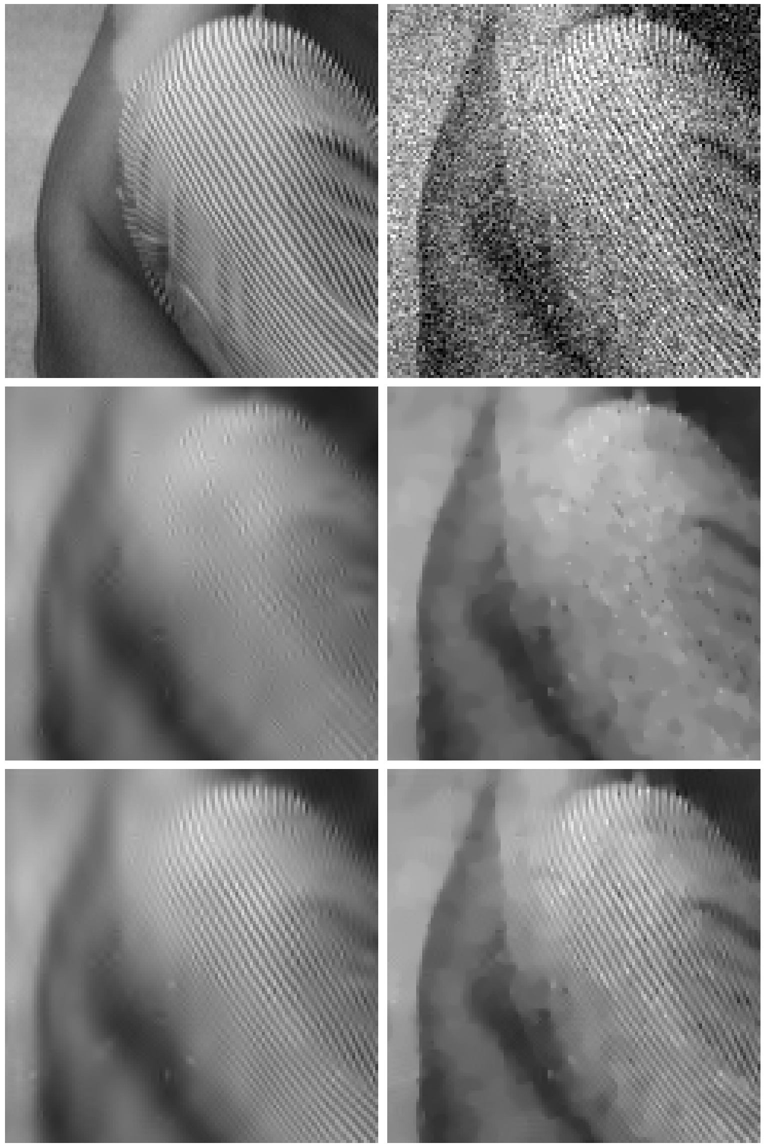

Figure 3. Up-Left : The initial image. Up-Right : The noisy image. Middle-Left : wavelet soft thresholding. Middle-Right : Rudin-Osher-Fatemi method. Down-Left : The noise selection in a wavelet basis, all the wavelet packet bases of full depth2, 3 and 4 and a Fourier basis. Down-Right : Rudin-Osher-Fatemi method "composed" with the preceding noise selection.

image, method or dictionary	noisy image	wavelet thresholding	wavelet + wavelet packets noise selection	W + WP + Fourier noise selection	Rudin-Osher-Fatemi method	ROF W+WP+F noise selection
SNR	6.36	9.03	10.85	11.47	10.43	13.01
MSE	893	315	214	186	239	135

Table 1. Signal to noise ratio and mean squared errors between the image and the initial image (ROF, W, WP, F refer respectively to Rudin-Osher-Fatemi, wavelet, wavelet packet and Fourier.

We also summarize some statistics between the different denoised images and the initial image in the Table 1. It is clear here that when more bases are considered we get better statistics. The best result being the one for the combination of Rudin-Osher-Fatemi method and the noise selection in several bases. Note that we have not displayed the image obtained by a noise selection in wavelet and wavelet packets bases since it is visually similar to the image displayed on Figure 3 Down-Left.

4. APPLICATION TO IMAGE DEBLURRING

We proved in Ref. 9 that it is possible to approximate a deconvolution operator by averaging over translations an operator diagonal in an appropriate wavelet packet basis. Therefore, in order to both denoise and deconvolve an image, it is reasonable to apply a thresholding in this appropriate basis with a fixed threshold and then deconvolve the image‡. So we can expect to obtain good results when deconvolving the result of a noise selection approach with regard to a dictionary which contains this appropriate basis. The following experiment deals with this possibility.

We want to recover an image $u \in \mathbb{R}^{N^2}$ from a data

$$v_{l,m} = (h * u)_{l,m} + b_{l,m},$$

where $(l, m) \in \{0, \cdots, N-1\}^2$, $h = \frac{1}{4} 1_{|\{0,1\}^2}$ and b is a white Gaussian noise of standard deviation 10. The original image u and the blurred image v are displayed on Figure 4 (up and middle-left).

We also display on Figure 4 some sharpened versions (the sharpening of "xv") of the cycle-spinning over translations of :

- Middle-Right : The soft thresholding in the mirror basis[7] (the threshold is 40) followed by a deconvolution.
- Down-Left : The noise selection (the threshold is 40) in a dictionary made of a wavelet basis of depth 4 and all the fully decomposed wavelet packet bases of depth 2, 3, 4.
- Down-Right : The noise selection (the threshold is 40) in a dictionary made of a the mirror basis and all the wavelet packet bases which are "more decomposed" than the mirror basis.

All the wavelet and wavelet packet bases are made with a cubic spline wavelet (see Ref. 8, pp 236).

We can see on Figure 4 that the noise selection improves this usual wavelet packet approach when used correctly (in the Down-Right image the texture is better preserved). Moreover, the two images on the lower row are issued from a noise selection in very similar dictionaries. The difference is mainly in the order of the elements in each dictionary. This illustrates that the first elements of the dictionary play a much more important role than the last ones and shows the importance of the choice of the order.

Note that once again all the images are available at http://www.math.ucla.edu/~malgouy

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‡Note that in general,[7,9] this is presented like a thresholding with adapted thresholds of the deconvolved image. For further detail, the reader is referred to Ref. 9 and 7.

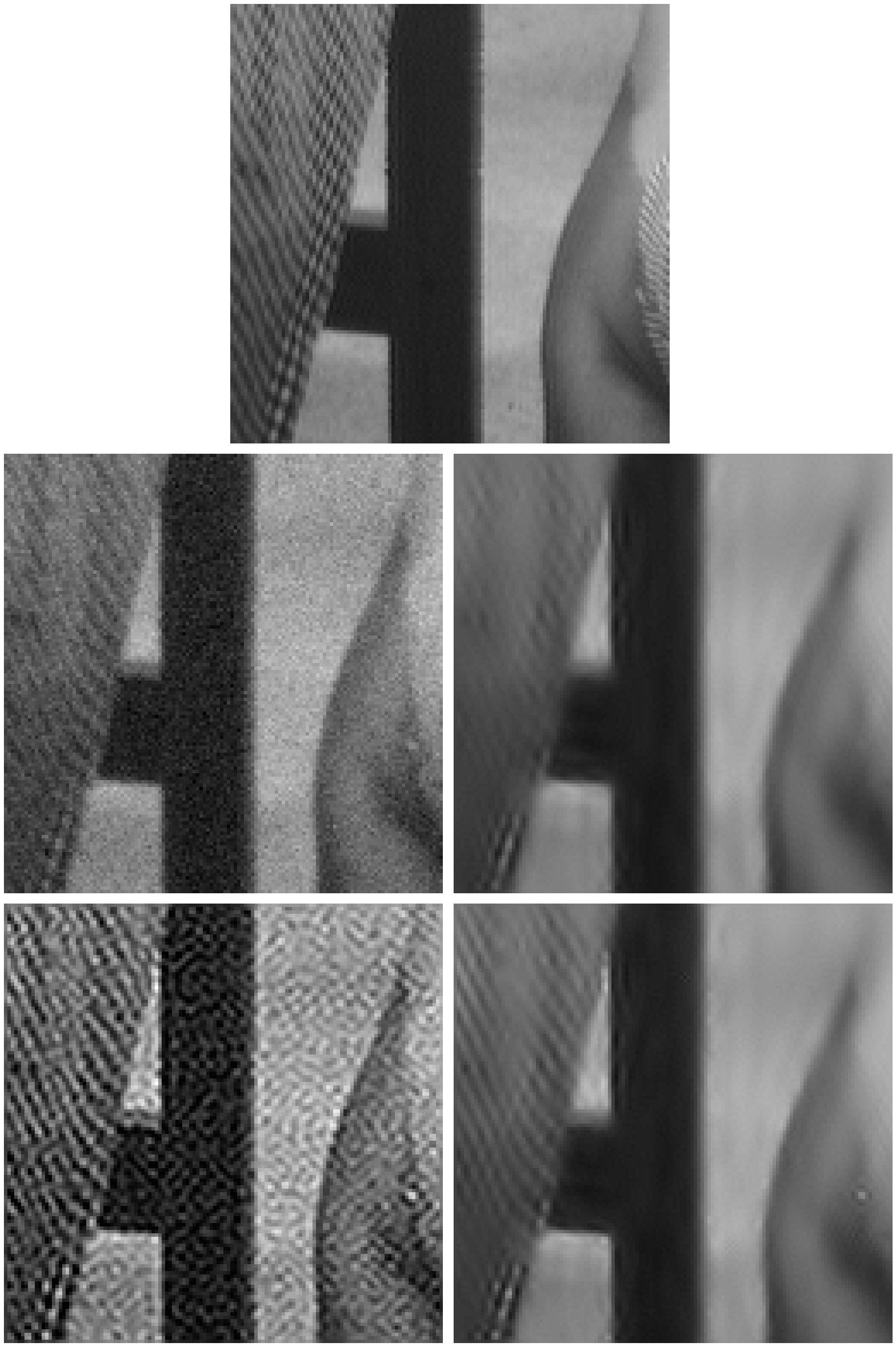

Figure 4. The three restored images have been sharpened. Up: The initial image. Middle-Left : The blurred and noisy image. Middle-Right : Thresholding in the mirror basis. Down-Left : The noise selection in a wavelet basis and then all the wavelet packet bases of full depth 2, 3 and 4. Down-Right : The noise selection in the mirror bases and all the more decomposed wavelet packet bases (with a depth smaller than 4).

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