Rank related properties for Basis Pursuit and total variation regularization

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Abstract

This paper focuses on optimization problems containing an l^1 kind of regularity criterion and a smooth data fidelity term. A general theorem is applied in this context; it gives an estimate of the distribution law of the "rank" of the solution to optimization problems, when the initial datum follows a uniform (in a convex compact set) distribution law. It says that, asymptotically, solutions with a large rank are more and more likely.

The main goal of this paper is to understand the meaning of this notion of rank for some energies which are commonly used in image processing. We study in detail the energy whose level sets are defined as the convex hull of a finite subset of $\mathbb{R}^N$ (c.f. Basis Pursuit) and the total variation. For these energies, the notion of rank relates respectively to sparse representation and staircasing.

In all cases but the 2D total variation, we are able to adapt the general theorem mentioned above to the energies under consideration.

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1 Introduction

This paper focuses on some properties of energies whose level sets are polyhedral. To be specific, we consider an energy of the form

$$E(u) = \max_{\psi \in \mathcal{D}} \langle u, \psi \rangle, \tag{1}$$

where $u \in \mathbb{R}^N$, $\mathcal{D}$ is a dictionary in $\mathbb{R}^N$ (i.e. a finite subset of $\mathbb{R}^N$) and $\langle ., . \rangle$ stands for the scalar product in $\mathbb{R}^N$.

The interest in such energies comes from their wide use in scientific computing, today. Almost all our examples come from image processing; in this field, such energies are used in: total variation regularization (for references, see the introduction of Section 4), in Basis Pursuit and likewise algorithms (see references in Section 3), wavelet soft thresholding (see [1,2], where soft-thresholding is proved to be the solution to optimization problems involving energies whose level sets are polyhedral), image compression (see [3,4], where JPEG and JPEG2000 are interpreted in terms of solutions to optimization problems involving an energy whose level sets are polyhedral),...

An analysis of such optimization problems is presented in [5]. Roughly speaking, it says that asymptotically, as $E(u)$ becomes small (u is the solution to the optimization problem), u is more and more likely to have a large rank. This latter notion is defined as follows: for all $u \in \mathbb{R}^N$, we define

$$\mathcal{F}(u) \triangleq \{\psi \in \mathcal{D}, E(u) = \langle u, \psi \rangle\},$$

and

$$\mathrm{rk}(u) \triangleq \dim\left(\mathrm{Span}((\psi)_{\psi \in \mathcal{F}(u)})\right).$$

We call $\mathrm{rk}(u)$, the rank of u. We can prove (see [5]) that $\mathrm{rk}(u)$ does not depend on the choice of $\mathcal{D}$ defining E (see (1)), it only depends on E. More precisely, if we consider two dictionaries such that (1) holds for all $u \in \mathbb{R}^N$, the notion of rank defined with respect to those dictionaries coincide.

Notice that, $\mathrm{rk}(u) \in \{1, \ldots, N\}$ and, for almost every $u \in \mathbb{R}^N$, $\mathrm{rk}(u) = 1$. Another trivial situation is $\mathrm{rk}(0) = N$.

Denoting the τ level set of the function E by

$$\mathcal{L}_E(\tau) = \{u \in \mathbb{R}^N, E(u) \le \tau\},$$

we define, for any $u \in \mathbb{R}^N$, the facet of $\mathcal{L}_E(E(u))$ at u by

$$\overline{u} = \{v \in \mathbb{R}^N, E(v) = E(u) \text{ and } \forall \psi \in \mathcal{D}, \psi \in \mathcal{F}(u) \Leftrightarrow \langle v, \psi \rangle = E(v)\}.$$

Geometrically, $\overline{u}$ is an open polyhedron of an affine manifold. The rank of u is simply the codimension of $\overline{u}$ (or equivalently of the affine manifold).

In fact, the meaning of the rank is quite explicit when the energy is expressed under the form (1). The importance of the notion of rank has been established in [3], where it is shown that the compression standards JPEG and JPEG2000 can be considered particular cases of more general compression schemes for which the number of coefficients that need to be coded is the rank of the solution to an optimization problem involving an energy whose level sets are polyhedral. The modification of the result of [5] to the context of compression is given in [4].

However, the meaning of the rank might not be obvious for a functional of l^1 kind. The first purpose of this paper is to establish this meaning for two kinds of energies: those whose level sets are expressed as the convex hull of a finite subset of $\mathbb{R}^N$ and the total variation. Once this meaning is established, an adaptation of the results in [5] is sometimes possible (see below).

The paper is organized as follows:

- In Section 2, we summarize the result established in [5].
- In Section 3, we study energies whose level sets are defined as scaled versions of the convex hull of a finite subset $\mathcal{B} \subset \mathbb{R}^N$. Examples of use of such energies are found in Basis Pursuit algorithms. We show that, for any $u \in \mathbb{R}^N$, u can be expressed with $N - \mathrm{rk}(u) + 1$ non-zero coordinates, where $\mathrm{rk}(u)$ denotes the rank of u. In this context, we are able to adapt the results established in [5]. The resulting theorem says that algorithms involving such a regularity criterion are very likely to provide a solution which is represented with only few coordinates in $\mathcal{B}$. Since it is easy to obtain, we also state a result saying that, with probability 1, the decomposition of the result of a Basis Pursuit model is the sparsest decomposition of this result.
- In Section 4, we study an approximation of the total variation (with full details in dimension 2 and indications in dimension 1) whose level sets are polyhedral. We establish some links between the rank of an element $u \in \mathbb{R}^{N^2}$ and the size of the set

$$I_2 = \{(i,j) \in \mathbb{R}^{N^2}, \nabla u_{i,j} = (0,0)\}.$$

  These results suggest that staircasing (which is usually quantified by the size of I_2) relates to the notion of rank. However, the notion of rank, by itself, does not permit to quantify the staircasing artifact in dimension 2. A deeper analysis is needed to state a formal theorem stating that, in 2D, the solutions containing staircasing are very likely to appear. This shows the limitations of the notion of rank for energies like the 2D total variation. However, we are able to state such a theorem for 1D signals.

The paper is organized so that a reader can skip Section 3 or 4. They are completely independent and both contain a short introduction to their specific research area.

2 A rank likelihood estimate

As a particular case, [5] studies an optimization problem of the form

$$(P) : \begin{cases} \text{Minimize } D(w - u_0) \\ \text{under the constraint } E(w) \leq \tau \end{cases}$$

for a datum $u_0 \in \mathbb{R}^N$, such that $E(u_0) > \tau > 0$, a norm E defined by (1) and a functional D, defined over $\mathbb{R}^N$. In the text, we will refer to D as the "data fidelity term" and to E as the "regularization term".

In [5], it is assumed that D is a regular (i.e. continuously differentiable away from 0) norm. (The boundary of the level sets of D also needs to be slightly "curved", we refer the reader to [5] for details.)

Notice that, such optimisation problems are explicitly expressed under the form (P) (see [6], for instance). However, most of the time, they are written in the form

$$\begin{cases} \text{Minimize } E(w) \\ \text{under the constraint } D(w - u_0) \leq \tau' \end{cases}$$

for $D(-u_0) > \tau' > 0$, or

$$\text{Minimize } E(w) + \lambda D(w - u_0),$$

for $\lambda > 0$.

However, under the hypotheses given above, these optimization problems are equivalent to (P). Indeed, their solutions are always characterized by

$$-\lambda' D(w - u_0) \in \partial E(w),$$

for an adequate $\lambda' > 0$, where ∂E denotes the subgradient of E.

Returning to (P), the following theorem is proved in [5].

Theorem 1 *Let f be a norm over $\mathbb{R}^N$ and U be a random variable whose distribution law is uniform in a set $\mathcal{A}$ satisfying*

$$\mathcal{L}_f(\tau_1) \subset \mathcal{A} \subset \mathcal{L}_f(\tau_2),$$

for $\tau_2 \geq \tau_1 > 0$.

There exist non-negative real numbers $(C_K)_{1\leq K\leq N}$ *such that, for all* $K \in \{1,\dots,N\}$,

$$C_K \tau^{N-K} \frac{\tau_1^K}{\mathbb{L}(\mathcal{A})} + \circ(\tau^{N-K}) \leq \mathbb{P}(\mathrm{rk}(U_s) = K) \leq C_K \tau^{N-K} \frac{\tau_2^K}{\mathbb{L}(\mathcal{A})} + \circ(\tau^{N-K})$$

as τ *goes to* 0*, where* $\mathbb{L}(\mathcal{A})$ *is the Lebesgue measure of* $\mathcal{A}$ *and* U_s *denotes the solution to* (P) *when* u_0 *is a realization of* U*. (Where* E *and* D *satisfy the hypotheses given after the definition of* (P)*.)*

This theorem estimates the probability law of $\mathrm{rk}(u_s)$, when u_s is a solution to (P) and τ is small. It tells us that, in this case, $\mathrm{rk}(u_s)$ is very likely to be large. This holds under quite general assumptions on the data distribution law and for a large class of data fidelity terms. Notice that, depending on the function E (see the next two sections), the property of having a large rank might be an advantage or a drawback of the method.

Of course, it would be absurd to claim that sampled images are uniformly distributed in a convex set. This is a limitation of the theorem as stated above. However, as suggested by the issues encountered in the applications (compression, Basis Pursuit and total variation regularization), the general philosophy of the theorem agrees with the common observations[1] made on these algorithms. Also, as is explained in [5], one could in principle integrate other kinds of distribution laws. The calculation might be difficult to set up though.

Finally, for a fixed regularization term, we can increase the likelihood of getting a small or large rank by designing E so that the constants C_K are improved. When expressing C_K (see [5]), we see that they depend strongly on D. Modifying D could therefore also lead to improvements of the model. However, we can expect improvements but cannot expect to go beyond the estimation given in Theorem 1. The only way to go beyond those estimates would be to chose D failing to comply with the hypotheses.

[1] Image compression standards (JPEG and JPEG 2000) use a projection onto a polytope. The Basis Pursuit algorithm is used to obtain sparse decomposition of images. When applied to images, the total variation regularization is know to provide solutions which contain staircasing.

3 The Basis Pursuit regularization term

In this section, we consider the energy defined for $u \in \mathbb{R}^N$ by

$$E(u) = \min_{(\alpha_j)_{j\in J} \in \mathcal{C}(u)} \sum_{j\in J} \alpha_j \tag{2}$$

where

$$\mathcal{C}(u) = \{(\alpha_j)_{j\in J}, \text{ such that } \forall j \in J, \alpha_j \geq 0 \text{ and } u = \sum_{j\in J} \alpha_j \varphi_j\}$$

where $\mathcal{B} = (\varphi_j)_{j\in J}$ is a finite subset of $\mathbb{R}^N$ such that,

$$\text{for all } u \in \mathbb{R}^N, \mathcal{C}(u) \neq \emptyset. \tag{3}$$

Note that, throughout the section, we will use the φ letter for the elements of $\mathcal{B}$. Our purpose is to avoid possible confusion with the elements of a dictionary $\mathcal{D}$ which will soon be used to characterize E by a formula of the type

$$E(u) = \max_{\psi\in\mathcal{D}} \langle u, \psi\rangle, \forall u \in \mathbb{R}^N.$$

The letter ψ is reserved for the elements of $\mathcal{D}$.

Also, we will often abuse the notation, by writing $(\alpha_i)_{i\in I} \in \mathcal{C}(u)$, for $I \subset J$. In such a case, one should understand that the numbers $(\alpha_j)_{j\in J\setminus I}$ are implicitly set to 0.

Notice that we could easily avoid the condition (3). In this case E is only defined over

$$\left\{ u \in \mathbb{R}^N, \exists (\alpha_j)_{j\in J} \in \mathbb{R}^J \text{ such that } u = \sum_{j\in J} \alpha_j \varphi_j \text{ and } \forall j \in J, \alpha_j \geq 0 \right\}.$$

For simplicity, we will always suppose that the condition (3) holds.

One motivation for considering such an energy is that it is used in Basis Pursuit [7], for source separation [8], and for feature selection in classification [9]. Let us also mention the theoretical work in [10,11] which specifies the geometry of $\mathcal{B}$ so that the $(\alpha_j)_{j\in J}$ solving $E(u)$ is the sparsest in $\mathcal{C}(u)$. The Basis Pursuit norm considered in [7,8] is of the form E above, for a dictionary $\mathcal{B} = \widetilde{\mathcal{B}} \cup \{-\varphi, \varphi \in \widetilde{\mathcal{B}}\}$, where $\widetilde{\mathcal{B}}$ is a given dictionary.

Another important motivation for studying energies of the form E is that its level sets are scaled version of the convex hull of $\mathcal{B}$. Indeed, under the hypotheses above, E is clearly a norm and we have

$$\{u, E(u) \leq 1\} = \left\{ u = \sum_{j\in J} \alpha_j \varphi_j, \text{ where } \forall j \in J, \alpha_j \geq 0 \text{ and } \sum_{j\in J} \alpha_j \leq 1 \right\},$$

so

$$\mathcal{L}_E(1) = \left\{ u = \sum_{j\in J} \alpha_j \varphi_j + \alpha' 0, \text{ where } \alpha' \geq 0, \forall j \in J, \alpha_j \geq 0 \right.$$
$$\left. \text{and } \alpha' + \sum_{j\in J} \alpha_j = 1 \right\}$$

As a consequence, $\mathcal{L}_E(1)$ is the convex hull of $\mathcal{B} \cup \{0\}$ (see [12], Cor. 2.3.1, p. 12). But, because of the condition (3), 0 belongs to the interior of $\mathcal{L}_E(1)$ and therefore the convex hull of $\mathcal{B} \cup \{0\}$ equals the convex hull of $\mathcal{B}$. Of course, since E is a norm, for any $\tau > 0$, $\mathcal{L}_E(\tau)$ is a scaled version of the convex hull of $\mathcal{B}$.

As a consequence, the level sets of E are polyhedral (see [12], Th. 19.1, p. 171) and the results described in Section 2 can be applied to such an energy.

The representation of a convex polyhedron as the convex hull of a finite set of elements is classical (see [12], Section 18, p. 162). This, alone, is an important motivation for studying energies whose level sets are polyhedral in our current framework.

The question we would like to address here is the relation between the rank of an element $u \in \mathbb{R}^N$ and certain properties of u. Heuristically, it seems clear that as the rank of u grows, we need less and less φ_j to represent u.

In order to establish this result, we first construct the dictionary $\mathcal{D}$, such that

$$E(u) = max_{\psi\in\mathcal{D}} \langle u, \psi \rangle.$$

Proposition 1 *Let $u \in \mathbb{R}^N$ and let $(\alpha'_j)_{j\in J} \in \mathcal{C}(u)$ be such that*

$$E(u) = \sum_{j\in J} \alpha'_j.$$

We write

$$I' = \{j \in J, \alpha'_j \neq 0\}.$$

There exists a subset $I \subset I'$ such that

(1) $(\varphi_i)_{i\in I}$ is an independent system of vectors,

(2) there exists $(\alpha_i)_{i\in I} \in \mathcal{C}(u)$ such that, for all $i \in I$, $\alpha_i > 0$, and

$$E(u) = \sum_{i\in I} \alpha_i.$$

Proof. First, if I' is such that $(\varphi_i)_{i\in I'}$ is not an independent system of vectors, there exists $i_0 \in I'$ such that

$$\varphi_{i_0} = \sum_{i\in I'\setminus\{i_0\}} \alpha_i^{i_0} \varphi_i$$

So, we can deduce from the hypotheses of the proposition that, for all $\lambda \in \mathbb{R}$,

$$\begin{aligned} u &= \sum_{i\in I'} \alpha'_i \varphi_i \\ &= \sum_{i\in I'} \alpha'_i \varphi_i + \lambda \left(\varphi_{i_0} - \sum_{i\in I'\setminus\{i_0\}} \alpha_i^{i_0} \varphi_i \right) \\ &= \sum_{i\in I'\setminus\{i_0\}} (\alpha'_i - \lambda \alpha_i^{i_0}) \varphi_i + (\alpha'_{i_0} + \lambda) \varphi_{i_0}. \end{aligned}$$

We write, for $j \in J$

$$\alpha_j^\lambda = \begin{cases} \alpha'_j - \lambda \alpha_j^{i_0} & \text{, if } j \in I' \setminus \{i_0\} \\ \alpha'_{i_0} + \lambda & \text{, if } j = i_0 \\ 0 & \text{, otherwise.} \end{cases}$$

Notice that, for λ sufficiently small (λ can be positive or negative), $(\alpha_j^\lambda)_{j\in J} \in \mathcal{C}(u)$, and

$$\sum_{j\in J} \alpha_j^\lambda = \sum_{i\in I'} \alpha'_i + \lambda \left(1 - \sum_{i\in I'\setminus\{i_0\}} \alpha_i^{i_0} \right).$$

So, since $(\alpha'_j)_{j\in J}$ is a solution to (2), we necessarily have

$$\sum_{i\in I'\setminus\{i_0\}} \alpha_i^{i_0} = 1.$$

As a consequence, as long as $\alpha_j^\lambda \geq 0$, for all $j \in J$, $(\alpha_j^\lambda)_{j\in J}$ is also a solution to (2). We take λ_0 to be equal to the largest negative value such that, for all $j \in J$, $\alpha_j^\lambda \geq 0$ (it is not difficult to see λ_0 exists and $\lambda_0 < 0$) and we obtain $(\alpha_j^{\lambda_0})_{j\in J} \in \mathcal{C}(u)$ such that

$$E(u) = \sum_{i\in I'} \alpha_i^{\lambda_0}$$

and

$$\#\{j \in J, \alpha_j^{\lambda_0} \neq 0\} < \#I',$$

where # denotes the cardinal of a set.

We can repeat the same procedure as long as the set of indexes thus obtained does not define an independent system of vectors. We write the result I. Obviously, I satisfies the conditions of Proposition 1. □

Definition 1 *We call* face *any subset $F \subset J$ such that $(\varphi_f)_{f\in F}$ is a basis and there exists $u \in \mathbb{R}^N$, which cannot be expressed as a linear combination of strictly fewer than N elements of $\mathcal{B}$, and there exists $(\alpha_f)_{f\in F} \in \mathcal{C}(u)$ satisfying*

$$E(u) = \sum_{f\in F} \alpha_f.$$

We write $\mathcal{F}$ for the set of all the faces.

For any $F \in \mathcal{F}$, we write ψ_F for the unique solution to

$$\forall f \in F, \langle \psi_F, \varphi_f \rangle = 1.$$

Throughout this section, the notation F will be used for faces. The indexes of elements of $\mathcal{B}$ belonging to a given face F will generally be denoted by f.

Notice that, for any $u \in \mathbb{R}^N$ which cannot be expressed as a linear combination of strictly fewer than N elements of $\mathcal{B}$, Proposition 1 guarantees that there exists a face $F \in \mathcal{F}$ such that

$$E(u) = \sum_{f\in F} \alpha_f,$$

for a given $(\alpha_f)_{f\in F} \in \mathcal{C}(u)$. (It is also clear that, for all $f \in F$, $\alpha_f > 0$.)

Observe also that

$$E(u) = \sum_{f\in F} \alpha_f = \sum_{f\in F} \alpha_f \langle \varphi_f, \psi_F \rangle = \langle u, \psi_F \rangle.$$

In fact, we can prove a little more.

Proposition 2 *Let F be a face. For any*

$$v \in \left\{ \sum_{f\in F} \alpha_f \varphi_f, \text{ with } \alpha_f > 0, \forall f \in F \right\},$$

there exists $(\alpha_f)_{f\in F} \in \mathcal{C}(v)$ and

$$E(v) = \sum_{f\in F} \alpha_f = \langle v, \psi_F \rangle.$$

Proof. First, since F is a face, there exists $u \in \mathbb{R}^N$ which cannot be expressed as a linear combination of strictly fewer than N elements of $\mathcal{B}$ and there exists $(\alpha_f)_{f\in F} \in \mathcal{C}(u)$ and

$$E(u) = \sum_{f\in F} \alpha_f = \langle u, \psi_F \rangle.$$

Here, for each $f \in F$, $\alpha_f > 0$.

Let us observe that, since F is a face, for any $j \in J \setminus F$, there exists $(\alpha_f^j)_{f\in F} \in \mathbb{R}^F$ such that

$$\varphi_j = \sum_{f\in F} \alpha_f^j \varphi_f.$$

So, for any $\lambda \geq 0$,

$$\begin{aligned} u &= \sum_{f\in F} \alpha_f \varphi_f + \lambda \left(\varphi_j - \sum_{f\in F} \alpha_f^j \varphi_f \right) \\ &= \sum_{f\in F} (\alpha_f - \lambda \alpha_f^j) \varphi_f + \lambda \varphi_j. \end{aligned}$$

Moreover, since $\alpha_f > 0$, for all $f \in F$, we have, for any sufficiently small $\lambda > 0$,

$$(\alpha_i')_{i\in J} \in \mathcal{C}(u),$$

where

$$\alpha_i' = \begin{cases} \alpha_i - \lambda \alpha_i^j & \text{, if } i \in F \\ \lambda & \text{, if } i = j \\ 0 & \text{, otherwise.} \end{cases}$$

Hence,

$$\begin{aligned} \sum_{f\in F} \alpha_f &\leq \sum_{j\in J} \alpha_j' \\ &\leq \sum_{f\in F} \alpha_f + \lambda \left(1 - \sum_{f\in F} \alpha_f^j \right). \end{aligned}$$

Finally, for any $j \in J \setminus F$,

$$\sum_{f\in F} \alpha_f^j \leq 1. \tag{4}$$

Let us consider $(\alpha_f^v)_{f\in F}$ such that $\alpha_f^v > 0$, for all $f \in F$, and

$$v = \sum_{f\in F} \alpha_f^v \varphi_f.$$

For any $(\beta_j)_{j\in J} \in \mathcal{C}(v)$, it is not difficult to see that, for any $f \in F$,

$$\alpha_f^v = \sum_{j\in J} \beta_j \alpha_f^j.$$

(Recall, that, since F is a face, $(\varphi_f)_{f\in F}$ is a basis.)

So, since (4) holds,

$$\begin{aligned}\sum_{f\in F} \alpha_f^v &= \sum_{j\in J} \beta_j \sum_{f\in F} \alpha_f^j, \\ &\leq \sum_{j\in J} \beta_j.\end{aligned}$$

Therefore

$$E(v) = \sum_{f\in F} \alpha_f^v = \langle v, \psi_F \rangle.$$

. □

Theorem 2 *For any $u \in \mathbb{R}^N$,*

$$E(u) = max_{F\in\mathcal{F}} \langle u, \psi_F \rangle.$$

Proof. Let us first consider $u \in \mathbb{R}^N$ such that u cannot be expressed as a linear combination of strictly fewer than N elements of $\mathcal{B}$.

By Proposition 1 and Definition 1, there exists $F \in \mathcal{F}$ such that

$$E(u) = \langle u, \psi_F \rangle.$$

It follows that

$$E(u) \leq max_{F'\in\mathcal{F}} \langle u, \psi_{F'} \rangle.$$

In order to prove the converse statement, let us consider a face $F' \in \mathcal{F}$. Since it is a face, there exists $v \in \mathbb{R}^N$ such that

$$E(v) = \langle v, \psi_{F'} \rangle$$

and v cannot be expressed as a linear combination of strictly fewer than N elements of $\mathcal{B}$.

Let us consider, for $\lambda \in [0, 1]$,

$$u^\lambda = \lambda u + (1 - \lambda) v.$$

Proposition 2 guarantees that for any sufficiently small $\lambda > 0$

$$E(u^\lambda) = \langle u^\lambda, \psi_{F'} \rangle.$$

So, for such a λ,

$$\begin{aligned}
\lambda \langle u, \psi_{F'} \rangle + (1-\lambda)\langle v, \psi_{F'} \rangle &= \langle u^\lambda, \psi_{F'} \rangle \\
&= E(\lambda u + (1-\lambda) v) \\
&\leq \lambda E(u) + (1-\lambda) E(v) \\
&\leq \lambda \langle u, \psi_F \rangle + (1-\lambda) \langle v, \psi_{F'} \rangle.
\end{aligned}$$

This guarantees that

$$\langle u, \psi_{F'} \rangle \leq \langle u, \psi_F \rangle.$$

It follows that, for any $u \in \mathbb{R}^N$ which cannot be expressed as a linear combination of strictly fewer than N elements of $\mathcal{B}$,

$$E(u) = max_{F \in \mathcal{F}} \langle u, \psi_F \rangle.$$

The result follows from the fact that

$$\Big\{ u \in \mathbb{R}^N, \; u \text{ cannot be expressed as a linear combination of} \\ \text{strictly fewer than } N \text{ elements of } \mathcal{B} \Big\}$$

is dense in $\mathbb{R}^N$ and the fact that both E and $max_{F \in \mathcal{F}} \langle ., \psi_F \rangle$ are continuous. □

One of the consequences of Theorem 2 is that, for all $j \in J$ and all $F \in \mathcal{F}$,

$$\langle \varphi_j, \psi_F \rangle \leq 1.$$

We can now reformulate, for $u \in \mathbb{R}^N$, the definition of $\text{rk}(u)$ for the energy E. We have

$$\text{rk}(u) = \dim \Big(\text{Span}(\psi_F)_{F \in \mathcal{F}(u)} \Big),$$

for

$$\mathcal{F}(u) = \{ F \in \mathcal{F}, \langle u, \psi_F \rangle = E(u) \}.$$

Let us also adapt the general definition given in the introduction to the current context. We write, for $u \in \mathbb{R}^N$,

$$\overline{u} = \{ v \in \mathbb{R}^N, E(v) = E(u) \text{ and } \forall F \in \mathcal{F}, F \in \mathcal{F}(u) \Leftrightarrow \langle v, \psi_F \rangle = E(v) \}.$$

As remarked in [5], $\overline{u}$ is an open polyhedron of an affine manifold whose dimension is $N - \mathrm{rk}(u)$.

Finally, let us write, for all $u \in \mathbb{R}^N$,

$$I(u) = \{j \in J, \forall F \in \mathcal{F}(u), \langle \varphi_j, \psi_F \rangle = 1\}.$$

The following proposition holds:

Proposition 3 *For any $u \in \mathbb{R}^N$,*

$$\overline{u} \subset \left\{ \sum_{i \in I(u)} \alpha_i \varphi_i, \forall i \in I(u), \alpha_i \geq 0 \text{ and } \sum_{i \in I(u)} \alpha_i = E(u) \right\} \tag{5}$$

and

$$\left\{ \sum_{i \in I(u)} \alpha_i \varphi_i, \forall i \in I(u), \alpha_i > 0, \text{ and } \sum_{i \in I(u)} \alpha_i = E(u) \right\} \subset \overline{u}.$$

Proof. In order to prove the first inclusion, we consider $v \in \overline{u}$. Proposition 1 guarantees that there exists $I \subset J$ such that $(\varphi_i)_{i \in I}$ is a linearly independent system of vectors and there exists $(\alpha_i)_{i \in I} \in \mathcal{C}(v)$ such that, for all $i \in I$, $\alpha_i > 0$ and

$$E(v) = \sum_{i \in I} \alpha_i.$$

Our goal is to prove that $I \subset I(u)$; in order to do so, we assume there exists $i_0 \in I \setminus I(u)$. Under this hypothesis, there exists $F \in \mathcal{F}(u)$ such that

$$\langle \varphi_{i_0}, \psi_F \rangle < 1.$$

Therefore, since $\alpha_i > 0$, for all $i \in I$,

$$\begin{aligned} \langle v, \psi_F \rangle &= \sum_{i \in I} \alpha_i \langle \varphi_i, \psi_F \rangle \\ &< \sum_{i \in I} \alpha_i = E(v) \end{aligned}$$

This contradicts the hypotheses that $v \in \overline{u}$. So $I \subset I(u)$, and the first inclusion is proved.

In order to prove the second inclusion, we consider $(\alpha_i)_{i \in I(u)} \in \mathbb{R}^{I(u)}$ such that $\alpha_i > 0$, for all $i \in I(u)$, and

$$\sum_{i \in I(u)} \alpha_i = E(u).$$

We write

$$v = \sum_{i \in I(u)} \alpha_i \varphi_i. \tag{6}$$

For any $F \in \mathcal{F}(u)$,

$$\begin{aligned} \langle v, \psi_F \rangle &= \sum_{i \in I(u)} \alpha_i \langle \varphi_i, \psi_F \rangle \\ &= E(u). \end{aligned}$$

Consider $F \notin \mathcal{F}(u)$. We require to prove that

$$\langle v, \psi_F \rangle < E(u).$$

We know that

$$\langle u, \psi_F \rangle < E(u). \tag{7}$$

Now, u obviously belong to $\overline{u}$, hence by (5) there exists $(\alpha'_i)_{i \in I(u)}$, such that, for all $i \in I(u)$, $\alpha'_i \geq 0$,

$$u = \sum_{i \in I(u)} \alpha'_i \varphi_i,$$

and

$$E(u) = \sum_{i \in I(u)} \alpha'_i.$$

So, (7) becomes

$$\sum_{i \in I(u)} \alpha'_i \langle \varphi_i, \psi_F \rangle < \sum_{i \in I(u)} \alpha'_i,$$

which guarantees that there exists $i_0 \in I(u)$ such that

$$\langle \varphi_{i_0}, \psi_F \rangle < 1.$$

Finally, we know that, for any $F \notin \mathcal{F}(u)$, there exists $i_0 \in I(u)$ such that

$$\langle \varphi_{i_0}, \psi_F \rangle < 1.$$

Since, in (6), for all $i \in I(u)$, $\alpha_i > 0$,

$$\begin{aligned} \langle v, \psi_F \rangle &= \sum_{i \in I(u)} \alpha_i \langle \varphi_i, \psi_F \rangle \\ &< \sum_{i \in I(u)} \alpha_i \\ &< E(u). \end{aligned}$$

It follows that $v \in \overline{u}$, as required. □

As a consequence, for any $u \in \mathbb{R}^N$, we have the following quality

$$\begin{aligned} N - \mathrm{rk}(u) &= \dim(\overline{u}) \\ &= \dim\left(\mathrm{Span}(\varphi_i)_{i \in I(u)}\right) - 1 \end{aligned} \tag{8}$$

Corollary 1 *For any $u \in \mathbb{R}^N$, there exist $I \subset J$ and $(\alpha_i)_{i\in I} \in \mathcal{C}(u)$ such that*

$$\#I \leq N - \mathrm{rk}(u) + 1.$$

Moreover, for $\mathbb{H}_{N-\mathrm{rk}(u)}$ almost every $v \in \overline{u}$

$$N - \mathrm{rk}(v) + 1 = \min_{(\alpha_j)_{j\in J} \in \mathcal{C}(v)} \#\{j \in J, \text{ such that } \alpha_j \neq 0\},$$

where $\mathbb{H}_{N-\mathrm{rk}(u)}$ stands for the Hausdorff measure of dimension $N - \mathrm{rk}(u)$.

Proof. The first statement is a direct consequence of Proposition 3, Proposition 1 and (8).

The second statement follows from the observation that, if $v \in \overline{u}$ is such that

$$\min_{(\alpha_j)_{j\in J} \in \mathcal{C}(v)} \#\{j \in J, \text{ such that } \alpha_j \neq 0\} \leq N - \mathrm{rk}(v),$$

then v belongs to the intersection of an affine manifold of dimension $N - \mathrm{rk}(v)$ which does not contain 0 (the affine manifold containing $\overline{u}$) with a finite union of affine manifolds of dimension smaller than $N - \mathrm{rk}(v)$ containing 0 (all the affine manifolds with $N - \mathrm{rk}(v)$ non-zero coordinates in $\mathcal{B}$).

It follows that v belongs to a subset of $\overline{u}$ whose $\mathbb{H}_{N-\mathrm{rk}(u)}$ measure is zero. □

The above Corollary tell us that, in the context of the Basis Pursuit norm, the rank is closely related to the notion of sparsity. If the rank is large, there exists a sparse representation of u, in $\mathcal{B}$.

In order to adapt Theorem 1 to the Basis Pursuit norm, we first define for $L \in \{1, \ldots, N\}$, $\tau > 0$ and $u \in \partial\mathcal{L}_E(\tau)$ the event:

$$P_L(u) \Longleftrightarrow L = \min_{(\alpha_j)_{j\in J} \in \mathcal{C}(u)} \#\{j \in J, \text{ such that } \alpha_j \neq 0\}.$$

In words, the condition $P_L(u)$ means: the sparsest decomposition (with positive coordinates) of u in $\mathcal{B}$ has L non-zero coordinates. We also consider the sets:

$$\mathcal{S}_L = \{u \in \partial\mathcal{L}_E(\tau), \text{ such that } P_L(u) \text{ holds}\},$$

for $L \in \{1, \ldots, N\}$ and $\tau > 0$.

For $u \in \partial\mathcal{L}_E(\tau)$ and $L \in \{1, \dots, N\}$ three different situations might occur, depending on the value of $N - \text{rk}(u) + 1$ with respect to L.

- If $N - \text{rk}(u) + 1 < L$: we can deduce, from the first statement of Corollary 1, that
$$\overline{u} \cap \mathcal{S}_L = \emptyset.$$
- If $N - \text{rk}(u) + 1 = L$: the second statement of Corollary 1 guarantees that there exists $\mathcal{Z}_{\overline{u}}$ such that
$$\mathbb{H}_{N-\text{rk}(u)}\left(\mathcal{Z}_{\overline{u}}\right) = 0$$
and
$$\overline{u} \cap \mathcal{S}_L = \overline{u} \setminus \mathcal{Z}_{\overline{u}}.$$
- If $N - \text{rk}(u) + 1 > L$: we trivially have
$$\mathbb{H}_{N-\text{rk}(u)}\left(\overline{u} \cap \mathcal{S}_L\right) = 0$$

We deduce that:

$$\mathcal{S}_L = \left(\cup_{\text{rk}(u)=N-L+1}(\overline{u} \setminus \mathcal{Z}_{\overline{u}})\right) \cup \left(\cup_{\text{rk}(u)>N-L+1}\mathcal{Z}'_{\overline{u}}\right) \tag{9}$$

where, for all u such that $\text{rk}(u) > N - L + 1$, $\mathbb{H}_{N-\text{rk}(u)}\left(\mathcal{Z}'_{\overline{u}}\right) = 0$.

We can now state:

Theorem 3 *We assume that E and D satisfy the hypotheses given right after the definition of (P) (see Section 2).*

Let f be a norm on $\mathbb{R}^N$ and U be a random variable whose distribution law is uniform in a set $\mathcal{A}$ satisfying

$$\mathcal{L}_f(\tau_1) \subset \mathcal{A} \subset \mathcal{L}_f(\tau_2),$$

for $\tau_2 \geq \tau_1 > 0$.

There exists non-negative real numbers $(C_L)_{1\leq L\leq N}$ such that, for all $L \in \{1, \dots, N\}$,

$$C_L \tau^{L-1} \frac{\tau_1^{N-L+1}}{\mathbb{L}(\mathcal{A})} + \circ(\tau^{L-1}) \leq \mathbb{P}(P_L(U_s)) \leq C_L \tau^{L-1} \frac{\tau_2^{N-L+1}}{\mathbb{L}(\mathcal{A})} + \circ(\tau^{L-1})$$

as τ goes to 0, where $\mathbb{L}(\mathcal{A})$ is the Lebesgue measure of $\mathcal{A}$ and U_s denotes the solution to (P) when u_0 is a realization of U.

Proof. The theorem follows from Theorem 1 and the last statement of Theorem 3 in [5]. The latter implies that, under the hypotheses of the above theorem, the probability that U_s belongs to any of the sets $\mathcal{Z}_{\overline{u}}$ or $\mathcal{Z}'_{\overline{u}}$ (in (9)) is zero. □

Theorem 3 tells us that, as τ goes to 0, the solution to (P) is more and more likely to be sparse. Again, if one wants to build a model (P) favoring sparse solutions, the goal is to design $\mathcal{B}$ and D such that, given the data distribution, the constants C_L are large for L small.

Again, the only way to go beyond the probabilities given in the above theorem is by violating the hypotheses made on (P).

We would like to conclude with a short remark regarding the following question: Writing u_s for the solution to (P), what is the likelihood that any $(\alpha_j)_{j\in J} \in \mathcal{C}(u_s)$ such that

$$\sum_{j\in J} \alpha_j = \inf_{(\beta_j)_{j\in J}\in\mathcal{C}(u)} \sum_{j\in J} \beta_j$$

is also the sparsest decomposition of u_s, when u_0 is random in a large set? This remark takes the form of the theorem:

Theorem 4 *Let f be a norm on $\mathbb{R}^N$ and U be a random variable whose distribution law is uniform in a set $\mathcal{A}$ satisfying*

$$\mathcal{L}_f(\tau_1) \subset \mathcal{A} \subset \mathcal{L}_f(\tau_2),$$

for $\tau_2 \geq \tau_1 > 0$. Let $\tau > 0$ be such that $\mathcal{L}_E(\tau) \subset \mathcal{A}$.

Suppose that, for every $u \in \mathbb{R}^N$, the infimum of

$$\inf_{(\alpha_j)_{j\in J}\in\mathcal{C}(u)} \sum_{j\in J} \alpha_j$$

is reached at a unique element of $\mathcal{C}(u)$. Write U_s for the random variable solving (P), when u_0 is a realisation of U.

With probability 1, the $(\alpha_j)_{j\in J}$ provided by the computation of $E(U_s)$ is also the sparsest decomposition, with positive coordinates, of U_s in $\mathcal{B}$.

Proof. First, observe that, if $u_0 \in \mathcal{L}_E(\tau)$, the solution to (P) is u_0. But, for almost every $u_0 \in \mathcal{L}_E(\tau)$, the sparsest decomposition, with positive coordinates, of u_s in $\mathcal{B}$, has N non zero coordinates. Proposition 1 and the uniqueness assumption guarantee that the $(\alpha_j)_{j\in J}$ provided by the computation of $E(u_0)$ also has N non zero coordinates. It is therefore the sparsest decomposition, with positive coordinates, of $u_s = u_0$ in $\mathcal{B}$, for almost every $u_0 \in \mathcal{L}_E(\tau)$. In the following, we will therefore focus on the case $u_0 \notin \mathcal{L}_E(\tau)$.

Let $L \in \{1, \ldots, N\}$, using (9) and the fact that the probability that U_s belongs to $\mathcal{Z}_{\overline{u}}$ (for u such that $N-L+1 = \text{rk}(u)$) or $\mathcal{Z}'_{\overline{u}}$ (for u such that $N-L+1 < \text{rk}(u)$) is 0 (see the last statement of Theorem 3 in [5]), we know that: with probability 1,

$$\text{rk}(U_s) = N + 1 - \min_{(\alpha_j)_{j\in J}\in\mathcal{C}(U_s)} \#\{j \in J, \alpha_j \neq 0\}.$$

Using (8), we obtain with probability 1

$$\dim\Big(\text{Span}(\varphi_i)_{i\in I(U_s)}\Big) = \min_{(\alpha_j)_{j\in J}\in\mathcal{C}(U_s)} \#\{j \in J, \alpha_j \neq 0\}. \tag{10}$$

Finally, we deduce from (5) and the uniqueness assumption that, for any realisation u_s of U_s, the unique set of coordinates provided by the computation of $E(u_s)$ takes the form $(\alpha_i)_{i\in I(u_s)}$.

Moreover, Proposition 1 and the uniqueness assumption guarantee that

$$\begin{aligned}\dim\Big(\text{Span}(\varphi_i)_{i\in I(U_s)}\Big) &\geq \dim\Big(\text{Span}(\varphi_i)_{i\in I(U_s) \text{ and } \alpha_i\neq 0}\Big)\\ &\geq \#(\varphi_i)_{i\in I(U_s) \text{ and } \alpha_i\neq 0}\\ &\geq \#\{i \in J, \alpha_i \neq 0\}\end{aligned}$$

Together with (10), this yields the statement of the theorem. □

4 The total variation

This section is mostly independent of the previous one. We focus on an approximation of the total variation (TV) defined, for $u \in \mathbb{R}^{N^2}$ (note that the ambient space is now of dimension N^2), by

$$TV(u) = \sum_{i,j=1}^{N} g(\nabla u_{i,j}), \tag{11}$$

where ∇u stands for a discrete analogue of the gradient of u, and g is a norm in $\mathbb{R}^2$. We will also assume that g is a norm taking the form

$$f(d) = max_{\psi\in\mathcal{D}}\langle d, \psi\rangle_2, \forall d \in \mathbb{R}^2, \tag{12}$$

for a dictionary of elements of $\mathbb{R}^2$. Here and throughout the section $\langle ., .\rangle_2$ denotes the usual scalar product in $\mathbb{R}^2$. The notation $\langle ., .\rangle$ stands for the usual scalar product in $\mathbb{R}^{N^2}$.

As we shall see, such a functional almost satisfies the hypotheses of Theorem 1 and can approximate the standard TV without significantly changing the results. (Recall that TV is usually defined with g equal to the Euclidean norm.)

One motivation for considering TV comes from its extensive use in image processing (see [13–16,2], for some examples in image restoration, and [17–19], for examples in texture discrimination). In these context, we would like to mention the well known staircasing artifact (see [15,14,20–23]); this artifact consists in the creation of large homogeneous zones (where $\nabla u = (0,0)$) in the solution to (P), when E equals TV. The link between staircasing and a notion similar to our rank has already been established in [14,24]. However, Theorem 1 does not apply in this context and we need to reexamine the link between rank and staircasing in the framework of Theorem 1.

Another motivation comes from the fact that TV consists of a combination of two linear analyses (the x and y derivatives); such a combination might be useful in practice. For instance, the modification of the results we present here to energies which take the form of a sum of l^1 norms in different bases is straightforward. It corresponds to the situation where g is the l^1 norm in $\mathbb{R}^2$.

To relate this to Theorem 1, the problem is to establish a link between the rank of an element $u \in \mathbb{R}^{N^2}$ and some properties of u. As we will see in the next section, the results are not as straightforward as in the previous section. We are however able to conclude that the rank of u is large when u contains staircasing. Since the constant C_K in Theorem 1 is a sum over all the $\overline{u}$ of rank K, this shows that solutions with large staircasing will be over-represented when solving (P) with $E = TV$. However, we are not able to adapt Theorem 1 completely (as was done for the Basis Pursuit norm) to the current context. We also mention that, for the 1D total variation, the notions of staircasing and rank coincide.

4.1 Unit ball of the total variation and polyhedrons

We consider the TV defined by

$$TV(u) = \sum_{i,j=1}^{N} g(\nabla u_{i,j}),$$

where $u \in \mathbb{R}^{N^2}$ is an image, ∇u stands for a discrete analogue of the gradient of u, and g is a norm of the form (12) in $\mathbb{R}^2$.

We have, for any $u \in \mathbb{R}^{N^2}$,

$$\begin{aligned} TV(u) &= \sum_{i,j=1}^{N} max_{\psi\in\mathcal{D}} \langle \nabla u_{i,j}, \psi \rangle_2 \\ &= max_{(\psi_{i,j})_{1\leq i,j\leq N} \in \mathcal{D}^{N^2}} \sum_{i,j=1}^{N} \langle \nabla u_{i,j}, \psi_{i,j} \rangle_2 \\ &= max_{(\psi_{i,j})_{1\leq i,j\leq N} \in \mathcal{D}^{N^2}} \sum_{i,j=1}^{N} u_{i,j}\, div(\psi)_{i,j}. \end{aligned} \tag{13}$$

The *div* operator is simply the adjoint to the ∇ operator (an example of a *div* operator can be found in [25]); it maps $(\mathbb{R}^2)^{N^2}$ onto $\mathbb{R}^{N^2}$. We obtain

$$TV(u) = max_{\varphi\in\mathcal{D}^{TV}} \langle u, \varphi \rangle_{N^2},$$

with

$$\mathcal{D}^{TV} = \left\{ div(\psi), \text{ for } \psi = (\psi_{i,j})_{1\leq i,j\leq N} \in \mathcal{D}^{N^2} \right\}.$$

Of course, since $\mathcal{D}$ is finite, $\mathcal{D}^{TV}$ is finite and $\mathcal{L}_{TV}(1)$ is a polyhedron. The fact that TV is not a norm will be addressed once the results of the next section have been established.

4.2 A link between rank and staircasing

For a given $u \in \mathbb{R}^{N^2}$, such that $TV(u) \neq 0$, we define the following sets of indexes:

$$I_0 = \{(i,j), f \text{ is differentiable at } \nabla u_{i,j}\},$$

$$I_1 = \{(i,j), f \text{ is not differentiable at } \nabla u_{i,j} \text{ and } \nabla u_{i,j} \neq (0,0)\},$$

and

$$I_2 = \{(i,j), \nabla u_{i,j} = (0,0)\}.$$

Of course, we have

$$I_0 \cup I_1 \cup I_2 = \{1, \ldots, N\}^2,$$

and

$$I_0 \cap I_1 = I_0 \cap I_2 = I_1 \cap I_2 = \emptyset. \tag{14}$$

Proposition 4 *Let TV be defined by (11) and (12). For any $u \in \mathbb{R}^{N^2}$, such that $TV(u) \neq 0$,*

$$\#I_2 - \#I_0 \leq \mathrm{rk}(u) \leq 2\#I_2 + \#I_1,$$

where # denotes the number of elements of a set.

Of course, the total variation, as defined by (11) and (12) does not satisfy the hypotheses of Theorem 1. Indeed, it fails to be a norm. As a consequence, in order to apply Theorem 1, we need to add some elements to $\mathcal{D}^{TV}$. We claim that adding the elements $\varepsilon\mathbb{1}$ and $-\varepsilon\mathbb{1}$, where ε is a small non-negative number and $\mathbb{1}_{i,j} = 1$, for all $(i,j) \in \{1, \ldots, N\}^2$, would not significantly modify the rank. Depending on $u \in \mathbb{R}^{N^2}$, the rank is either unchanged or increased by one unit (as is the set of active constraints). This would not significantly modify the point which we wish to address.

Proof. First, given (12), it is not difficult to see that

$$I_0 = \{(i,j), \exists! \psi^1_{i,j} \in \mathcal{D}, f(\nabla u_{i,j}) = \langle \nabla u_{i,j}, \psi^1_{i,j} \rangle_2\}, \quad (15)$$

where $\exists!$ stands for "there exists a unique". Also,

$$\begin{aligned} I_1 = \{(i,j), \exists! (\psi^1_{i,j}, \psi^2_{i,j}) \in \mathcal{D}^2, \psi^1_{i,j} \neq \psi^2_{i,j} \\ \text{and } f(\nabla u_{i,j}) = \langle \nabla u_{i,j}, \psi^1_{i,j} \rangle_2 = \langle \nabla u_{i,j}, \psi^2_{i,j} \rangle_2\}, \quad (16) \end{aligned}$$

and

$$I_2 = \{(i,j), \forall \psi \in \mathcal{D}, f(\nabla u_{i,j}) = \langle \nabla u_{i,j}, \psi \rangle_2\}. \quad (17)$$

For any, $(i,j) \in I_2$, we chose two linearly independent elements $(\psi^1_{i,j}, \psi^2_{i,j}) \in \mathcal{D}^2$.

In order to obtain the upper bound, we consider

$$V \triangleq \{\psi \in \mathcal{D}^{N^2}, TV(u) = \langle u, div(\psi) \rangle_{N^2}\}.$$

Looking at the definition of the rank, we know that

$$\begin{aligned} \mathrm{rk}(u) &= \dim\left(\mathrm{Span}\{div\psi, \psi \in V\}\right) \\ &= \dim\left(div\left(\mathrm{Span}(V)\right)\right). \end{aligned}$$

It is not difficult to see that

$$V = \{\psi \in \mathcal{D}^{N^2}, \forall (i,j) \in \{1, \dots N\}^2, \langle \nabla u_{i,j}, \psi_{i,j} \rangle = f(\nabla u_{i,j})\},$$

which can be written (using (15), (16) and (17)) as

$$\begin{aligned} V = \{\psi \in \mathcal{D}^{N^2} &, \forall (i,j) \in I_0, \psi_{i,j} = \psi^1_{i,j} \\ &, \forall (i,j) \in I_1, \psi_{i,j} = \psi^1_{i,j} \text{ or } \psi_{i,j} = \psi^2_{i,j} \\ &, \forall (i,j) \in I_2, \psi_{i,j} \in \mathcal{D}\} \end{aligned}$$

We write

$$\psi^1 = (\psi^1_{i,j})_{1 \leq i,j \leq N},$$

$\psi^{2,i,j} \in (\mathbb{R}^2)^{N^2}$, for $(i,j) \in I_1 \cup I_2$, such that

$$\psi^{2,i,j}_{i',j'} = \begin{cases} (0,0) & \text{, if } (i',j') \neq (i,j), \\ \psi^2_{i,j} - \psi^1_{i,j} & \text{, if } (i',j') = (i,j), \end{cases}$$

and $\psi^{1,i,j} \in (\mathbb{R}^2)^{N^2}$, for $(i,j) \in I_2$, such that

$$\psi^{1,i,j}_{i',j'} = \begin{cases} (0,0) & \text{, if } (i',j') \neq (i,j), \\ \psi^1_{i,j} & \text{, if } (i',j') = (i,j). \end{cases}$$

We have

$$V \subset \text{Span}\left(\{\psi^1\} \cup \{\psi^{2,i,j}\}_{(i,j)\in I_1 \cup I_2} \cup \{\psi^{1,i,j}\}_{(i,j)\in I_2}\right).$$

So

$$div\,(\text{Span}(V)) \subset \text{Span}\left(\{div\left(\psi^1\right)\} \cup \{div\left(\psi^{2,i,j}\right)\}_{(i,j)\in I_1\cup I_2} \cup \{div\left(\psi^{1,i,j}\right)\}_{(i,j)\in I_2}\right),$$

which, since $div\,(\psi^1) \equiv 0$, implies that

$$\text{rk}(u) \leq 2\#I_2 + \#I_1.$$

In order to obtain the lower bound, we write

$$\mathcal{F}(u) \triangleq \{\varphi \in \mathcal{D}^{TV}, TV(u) = \langle u, \varphi\rangle_{N^2}\}$$

and

$$\overline{u} \triangleq \{v \in \mathbb{R}^{N^2}, \mathcal{F}(v) = \mathcal{F}(u) \text{ and } TV(v) = TV(u)\}.$$

Let us also consider

$$\begin{aligned} W &\triangleq \{\lambda v, \text{ with } \lambda > 0 \text{ and } v \in \overline{u}\}, \\ &= \{v \in \mathbb{R}^{N^2}, \mathcal{F}(v) = \mathcal{F}(u)\}. \end{aligned}$$

We can easily deduce from the definition of $\mathcal{F}(u)$ that $\overline{u}$ is an open polyhedron in an affine manifold of dimension $N^2 - \text{rk}(u)$. (Indeed, $\overline{u}$ is defined by $\text{rk}(u)$ independent equalities and a consistent system of strict inequalities.) Since $TV(u) \neq 0$, it is not difficult to see that W is contained in an affine manifold of dimension $N^2 - \text{rk}(u) + 1$.

Using again (15), (16), (17), we see that

$$\begin{aligned} W \subset \{v, &\forall (i,j) \in I_0, \langle \nabla v_{i,j}, \psi^1_{i,j}\rangle_2 = f(\nabla v_{i,j}) \\ &\forall (i,j) \in I_1, \exists \lambda_{i,j} > 0, \nabla v_{i,j} = \lambda_{i,j} \nabla u_{i,j} \\ &\forall (i,j) \in I_2, \nabla v_{i,j} = (0,0)\} \end{aligned}$$

So,

$$W \subset W' \triangleq \{v, (\nabla v_{i,j})_{1\leq i,j\leq N} \in W''\},$$

with

$$W'' = \{\varphi \in (\mathbb{R}^2)^{N^2}, \forall (i,j) \in I_2, \varphi_{i,j} = (0,0) \text{ and } \\ \forall (i,j) \in I_1, \exists \lambda_{i,j} > 0, \varphi_{i,j} = \lambda_{i,j} \nabla u_{i,j}\}.$$

Notice that $\dim(W'') = 2N^2 - 2\#I_2 - \#I_1$ and that $\dim(W'') = \dim(W') + 1$. So,

$$\begin{aligned} N^2 - \mathrm{rk}(u) + 1 &= \dim(W) \\ &\leq \dim(W') + 1 \\ &\leq 2N^2 - 2\#I_2 - \#I_1 + 1. \end{aligned}$$

Using the fact that $N^2 = \#I_0 + \#I_1 + \#I_2$, we finally obtain

$$\#I_2 - \#I_0 \leq \mathrm{rk}(u),$$

which finishes the proof. □

Proposition 4 tells us that $\mathrm{rk}(u)$ is large, when u is such that $\#I_2$ is large (strong staircasing). Indeed, when $\#I_2$ is large, since

$$\#I_0 + \#I_1 + \#I_2 = N^2,$$

$\#I_0$ is small.

Also, combining Theorem 1 and Proposition 4, we know that if the total variation of a solution u to a model of the form (P) is small, it is very likely that $\mathrm{rk}(u)$ is large. It is however not possible to interpret this property in terms of staircasing. The only information we have is that $\#I_1 + 2\#I_2$ is likely to be large too.

It is a classical problem in mathematical papers modelling the staircasing artifact in images. Our understanding of the staircasing artifact in dimension 2 is not as good as its counterpart for 1 D signals, see [14,20,22]. Most results, for images, only establish the stability of homogeneous regions, when the initial datum in the model (P) already contains staircasing, see [24]. This is of course not in contradiction with the result proved here. Moreover, when adapted to 1 D signals, Proposition 4 makes a clear link between staircasing (in 1 D) and the rank of an element. (The modification of the proof of Proposition 4 is straightforward.)

It seems that the information provided by the rank of an image is not sufficient to establish the high probability of obtaining a strong staircasing in models of the form (P). This phenomenon has been observed in many different and independent experiments and can probably be explained though. One way to do so might be to improve the analysis carried out in [5]. For instance, it could be useful to estimate, for a given rank $K > 0$, the relative contribution to C_K of the elements $\overline{u}$ for which $\#I_2$ is large and of elements for which $\#I_1$ is large.

However, the proposition guarantees that solutions with a strong staircasing will be over-represented among solutions to (P), when $E = TV$, since they contribute to C_K with K large.

Finally, if one wants to avoid solutions to (P) with a large rank (maybe in the expectation of reducing the staircasing artifact), the construction of the constant C_K shows that this can be achieved by improving the data fidelity term. However, in this context, Theorem 1 tells us that might be of limited interest. Of course, we can expect to outperform the results predicted by Theorem 1 when the data fidelity term fails to comply to its hypotheses. Some experiments on the model described in [2] (they have not yet been published) show the solutions to such models almost do not contain staircasing. To the authors knowledge, the impact of the data fidelity term on staircasing has not yet been exploited by authors working in that field (see [15,21,23] and references therein).

4.3 Remark

The modification of the proof of Proposition 4 to other energies is straightforward in certain cases. For instance, a similar statement holds for an energy of the form

$$E(u) = \sum_{k=1}^{N^2} |\langle u, \varphi_k \rangle_{N^2}|, \tag{18}$$

for $u \in \mathbb{R}^{N^2}$ and a basis $(\varphi_k)_{1 \leq k \leq N^2}$ of $\mathbb{R}^{N^2}$.

In this case, we write

$$I_0 = \{k, \langle u, \varphi_k \rangle_{N^2} = 0\}$$

and

$$I_1 = \{k, \langle u, \varphi_k \rangle_{N^2} \neq 0\}.$$

The proposition becomes:

Proposition 5 *Let E be defined by (18). For any $u \in \mathbb{R}^{N^2}$, such that $E(u) \neq 0$,*

$$\mathrm{rk}(u) = \#I_0 + 1$$

The proof is a straightforward modification of the proof of Proposition 4.

The modification of Theorem 1 to this context is straightforward and will not be stated here. Theorem 1 says that, under quite general assumptions on the data distribution law and for a quite general class of data fidelity terms: when τ is small, the solution u to (P), for E defined by (18), is very likely to be sparse in the basis $(\varphi_k)_{1\leq k\leq N^2}$.

Again, in order to increase the sparsity of the solution in this basis, one should improve the data fidelity term to get better C_K.

Also, this result suggests that it might be possible to extend some theoretical results which are usually stated for an energy of the form (18) to all the energies whose level sets are polyhedral. In such cases, the hypotheses on $\#I_0$ should be replaced by analogous hypotheses on the rank of u. (For instance results similar to those presented in [26] can be enriched this way.)

Finally, Proposition 5 is only a small improvement on Corollary 1, when $\mathcal{B} = \tilde{\mathcal{B}} \cup \{-\varphi, \varphi \in \tilde{\mathcal{B}}\}$, where $\tilde{\mathcal{B}} = (\tilde{\varphi}_k)_{1\leq k\leq N^2}$ is the basis such that, for all $u \in \mathbb{R}^{N^2}$,

$$u = \sum_{k=1}^{N^2} \langle u, \varphi_k \rangle_{N^2} \tilde{\varphi}_k.$$

Also observe that, in dimension 1, the total variation can be approximated by a functional of the form (18). We only require to modify the way TV deals with constant images; the modification of Theorem 1 gives a formal statement saying that, in 1 dimension, when τ is small and E is the total variation, the solution to (P) is very likely to contain staircasing.

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