Mathematical methods for Image Processing

Hands-on session
4

Image zooming using the total variation regularization

In this Hands-on session, we will consider an optimization method for zooming in images. We will consider a simple sampling model : Given an initial image u, of size $KN \times KN$, the under-sampled image (in a ratio K) v is defined by

$$v_{i,j} = \frac{1}{K^2} \sum_{k,l=0}^{K-1} u_{Ki+k,Kj+l} \qquad , \forall (i,j) \in \{0, \dots, N-1\}^2.$$

We denote the (linear) sampling operator defined in such a manner Q. We therefore have $v = Q(u)$.

We denote

$$\mathcal{C} = \{w \in I\!R^{(KN)^2}, Q(w) = v\}.$$

We know that $u \in \mathcal{C}$, this is only information we are sure of concerning u.

In order to restore u we consider the optimization problem defined by

$$(P) : \begin{cases} \text{minimize } E(w) \\ \text{under thet constraint } w \in \mathcal{C}, \end{cases}$$

for a well chosen regularization term E.

In this Hands-on session, we consider the smooth approximation of the total variation defined using

$$\nabla w_{i,j} = \begin{pmatrix} (D_x w)_{i,j} \\ (D_y w)_{i,j} \end{pmatrix} = \begin{pmatrix} w_{i+1,j} - w_{i,j} \\ w_{i,j+1} - w_{i,j} \end{pmatrix}$$

for all $(i,j) \in \{0, \dots, KN-1\}^2$ (we assume w periodized outside $\{0, \dots, KN-1\}^2$).

We approximate the Euclidean norm in $I\!R^2$ by

$$\varphi_\beta(|\nabla w_{i,j}|^2),$$

with

$$\varphi_\beta(t) = \sqrt{t + \beta},$$

for some parameter $\beta > 0$. In practice, we take $\beta = 0.01$. The approximation of the total variation is then defined by

$$E_\beta(w) = \sum_{i,j=0}^{KN-1} \varphi_\beta\left(|\nabla w_{i,j}|^2\right).$$

Exercice 1.

(1) Check that $\mathcal{C}$ is a closed affine space. Characterize the vector space $\mathcal{C}'$ defining its direction.

(2) Check that the operator that maps any $w \in I\!R^{(KN)^2}$ to $P_{\mathcal{C}}(w) \in I\!R^{(KN)^2}$, as defined by

$$P_{\mathcal{C}}(w)_{i,j} = w_{i,j} - Q(w)_{[\frac{i}{K}],[\frac{j}{K}]} + v_{[\frac{i}{K}],[\frac{j}{K}]},$$

where $[t]$ is integer floor function applied to t, corresponds to the orthogonal projection onto $\mathcal{C}$.

(3) Check that all the elements of $\mathcal{C}$ have the same mean. Conclude that (P) has a minimizer. Propose an example in which (P) has several solutions.

(4) Check that
$$\nabla E(w) = 2\left(D_x^*(X) + D_y^*(Y)\right),$$
where D_x^* et D_y^* are as in the Hands-on session 1, and
$$X_{i,j} = \varphi_\beta'(|\nabla w_{i,j}|^2) D_x w_{i,j},$$
$$Y_{i,j} = \varphi_\beta'(|\nabla w_{i,j}|^2) D_y w_{i,j},$$
for $(i,j) \in \{0,\ldots,KN-1\}^2$, and where $\varphi_\beta'(t)$ is the derivative of φ_β at the point t.

(5) Detail a projected gradient algorithm solving (P).

(6)
- Go to the web-page
  http ://www.math.univ-toulouse.fr/~fmalgouy/index.html
  navigate to reach the web-page of the lecture.
- Download and unzip the archive *tp3.zip*.

(7) Read the program *echantillonne.sci* and the program *zoom_ TV.sci*. Detail what they compute.

(8) Launch the script and comment on the results obtained.

(9) Modify the function *zoom_ TV* to use a constant step-size.

(10) Run the function *zoom_ TV* for several values of (constant) step-size and for the Armijo step-size. Conclude concerning the number of iteration required by the algorithm to converge and on the step-size.

Exercice 2.

(1) Write a program implementing thegradient descent algorithm with constant step-size and the Armijo step-size, to minimize the penalized cost function whose solution permits to approximate a solution of (P). For that, you'll consider the functional :
$$F(w) = E(w) + \lambda \sum_{i,j=0}^{N-1} (Q(w)_{i,j} - v_{i,j})^2,$$
for a large value of λ.

(2) Adapt the program to use the steppest descent step-size rule.

(3) Compare the convergence speeds for the different step-size rule.

(4) Compare the qualitative results with those obtained during the Hands-on session 1-2 in the denoising context.

Exercice 3.

(1) Write a function that permits to compute the convolution operator
$$\begin{array}{rcl} H : I\!R^{N^2} & \longrightarrow & I\!R^{N^2} \\ w & \longmapsto & Hw \end{array}$$
where, for all $(m,n) \in \{1,\ldots,N\}^2$,
$$(Hw)_{m,n} = \sum_{m',n'=-1}^{1} w_{m+m',n+n'}$$
(The images are assumed periodized.)

(2) Write a program implementing a gradient descent algorithm with constant step-size and the Armijo step-size for minimizing
$$F(w) = TV_\varepsilon(w) + \lambda\|Hw - v\|^2,$$
for parameter ε and $\lambda > 0$.

(3) Test different values for the parameters ε and λ when deconvolving
$$v = Hu + b$$
where u is an ideal image and b is a Gaussian white noise of standard deviation 2.