Mathematical methods for Image Processing

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Plan

1. Hands-on session with examples: Image zooming using the TV regularization

Exercise 1, question 1

Check that $\mathcal{C}$ is a closed affine space. Characterize the vector space $\mathcal{C}'$ defining its direction.

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We have $\mathcal{C} = u + \ker(Q)$. It is a affine space and $\mathcal{C}' = \ker(Q)$.

- If $w \in \mathcal{C}$, then $Q(w-u) = Q(w) - Q(u) = v - v = 0$. Therefore, $w \in u + \ker(Q)$.
- If $w \in u + \ker(Q)$, let $w' \in \ker(Q)$ such that $w = u + w'$. We have

  $$Q(w) = Q(u) + Q(w') = v.$$

  Therefore, $w \in \mathcal{C}$.

Exercise 1, question 2

Check that the operator that maps any $w \in \mathbb{R}^{(KN)^2}$ to $P_{\mathcal{C}}(w) \in \mathbb{R}^{(KN)^2}$, as defined by

$$P_{\mathcal{C}}(w)_{i,j} = w_{i,j} - Q(w)_{[\frac{i}{K}],[\frac{j}{K}]} + v_{[\frac{i}{K}],[\frac{j}{K}]},$$

where $[t]$ is integer floor function applied to t, corresponds to the orthogonal projection onto $\mathcal{C}$.

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We want to prove that

$$P_{\mathcal{C}}(w) = \text{Argmin}_{w' \in \mathcal{C}} \|w' - w\|^2$$

i.e. we want to prove that

- $P_{\mathcal{C}}(w) \in \mathcal{C}$
- $\forall w' \in \mathcal{C}, \|w - w'\|^2 \geq \|w - P_{\mathcal{C}}(w)\|^2$

Exercise 1, question 2

We have, for all $i, j = 0 \ldots N-1$ (Notice $[\frac{Ki+k}{K}] = i$ and $[\frac{Kj+l}{K}] = j$)

$$
\begin{aligned}
Q(P_{\mathcal{C}}(w))_{i,j} &= \frac{1}{K^2} \sum_{k,l=0}^{K-1} P_{\mathcal{C}}(w)_{Ki+k,Kj+l} \\
&= \frac{1}{K^2} \sum_{k,l=0}^{K-1} w_{Ki+k,Kj+l} + \frac{1}{K^2} \sum_{k,l=0}^{K-1} Q(w)_{i,j} + \frac{1}{K^2} \sum_{k,l=0}^{K-1} v_{i,j} \\
&= Q(w)_{i,j} - Q(w)_{i,j} + v_{i,j} = v_{i,j}
\end{aligned}
$$

Therefore,

$$P_{\mathcal{C}}(w) \in \mathcal{C}$$

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Exercise 1, question 2

Let $w' \in \mathcal{C}$, we have

$$
\begin{aligned}
& \langle w - P_{\mathcal{C}}(w), w' - P_{\mathcal{C}}(w) \rangle \\
= & \sum_{i,j=0}^{N-1} \sum_{k,l=0}^{K-1} (w_{Ki+k,Kj+l} - P_{\mathcal{C}}(w)_{Ki+k,Kj+l})(w'_{Ki+k,Kj+l} - P_{\mathcal{C}}(w)_{Ki+k,Kj+l}) \\
= & \sum_{i,j=0}^{N-1} \sum_{k,l=0}^{K-1} (Q(w)_{i,j} - v_{i,j})(w'_{Ki+k,Kj+l} - w_{Ki+k,Kj+l} + Q(w)_{i,j} - v_{i,j}) \\
= & \sum_{i,j=0}^{N-1} (Q(w)_{i,j} - v_{i,j}) \left(\sum_{k,l=0}^{K-1} (w'_{Ki+k,Kj+l} - w_{Ki+k,Kj+l} + Q(w)_{i,j} - v_{i,j}) \right)
\end{aligned}
$$

Exercise 1, question 2

and

$$\sum_{k,l=0}^{K-1} (w'_{Ki+k,Kj+l} - w_{Ki+k,Kj+l} + Q(w)_{i,j} - v_{i,j})$$
$$= \quad K^2 Q(w')_{i,j} - K^2 Q(w)_{i,j} + K^2 Q(w)_{i,j} - K^2 v_{i,j} = 0$$

Therefore, finally for any $w' \in \mathcal{C}$

$$\langle w - P_{\mathcal{C}}(w), w' - P_{\mathcal{C}}(w) \rangle = 0$$

and

$$\begin{aligned} \|w - w'\|^2 &= \|w - P_{\mathcal{C}}(w)\|^2 + \|w' - P_{\mathcal{C}}(w)\|^2 + 2\langle w - P_{\mathcal{C}}(w), w' - P_{\mathcal{C}}(w) \rangle \\ &= \|w - P_{\mathcal{C}}(w)\|^2 + \|w' - P_{\mathcal{C}}(w)\|^2 \\ &\geq \|w - P_{\mathcal{C}}(w)\|^2 \end{aligned}$$

Exercise 1, question 3

Check that all the elements of $\mathcal{C}$ have the same mean. Conclude that (P) has a minimizer. Propose an example in which (P) has several solutions.

Exercise 1, question 3

For any $w \in \mathcal{C}$,

$$\sum_{i,j=0}^{N-1} \sum_{k,l=0}^{K-1} w_{Ki+k,Kj+l} = \sum_{i,j=0}^{N-1} K^2 Q(w)_{i,j}$$
$$= K^2 \sum_{i,j=0}^{N-1} v_{i,j}$$

Therefore, they all have the same mean. Rewriting

$$(P) : \begin{cases} \text{minimize } E(u+w) \\ \text{under thet constraint } w \in \ker(Q), \end{cases}$$

E is convex, finite and coercive on $\ker(Q) \subset \{w | \text{mean of } w \text{ is } 0\}$.
(P) has a solution.
Non-uniqueness: Take a 1D example, in which u is an increasing function.

Exercise 1, question 4

Check that

$$\nabla E(w) = 2\left(D_x^*(X) + D_y^*(Y)\right),$$

where D_x^* et D_y^* are as in the Hands-on session 1, and

$$X_{i,j} = \varphi_\beta'(|\nabla w_{i,j}|^2) D_x w_{i,j},$$

$$Y_{i,j} = \varphi_\beta'(|\nabla w_{i,j}|^2) D_y w_{i,j},$$

for $(i,j) \in \{0,\ldots,KN-1\}^2$, and where $\varphi_\beta'(t)$ is the derivative of φ_β at the point t.

See lecture on total variation.

Exercise 1, question 5

Detail a projected gradient algorithm solving (P).

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Algorithm 2 Proximal gradient algorithm

Entry: β, v

Output: Approximation of a solution of (P): w^*

Initialize w

Set $L = \frac{8}{\sqrt{\beta}}$ (see lecture on TV)

While Not converged **Do**

Compute $d = \nabla E(w)$ (see question 4)

Update : $w \leftarrow P_{\mathcal{C}}(w - \frac{1}{L} d)$ (see question 2)

End while