Mathematical methods for Image Processing

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Plan

Image Restoration: Linear inverse problems

Let

$$u = Hv + b$$

where

$$H : \mathbb{R}^{N^2} \longrightarrow \mathbb{R}^P$$

We know u and H. We look for an estimate of v.
We minimize the Rudin-Osher-Fatemi (ROF) model

$$\min_{w \in \mathbb{R}^{N^2}} \quad E(w),$$

where

$$E(w) \;=\; TV_\varepsilon(w) + \lambda \|Hw - u\|_2^2.$$

To do so, we implement a gradient descent algorithm and need to calculate

$$\nabla E(w).$$

Image Restoration: Linear inverse problems

Let w, $w' \in \mathbb{R}^{N^2}$, we have

$$\begin{aligned}\|H(w+w')-u\|_2^2 &= \langle (Hw-u) \ + \ Hw' \ , \ (Hw-u) \ + \ Hw' \rangle \\ &= \langle Hw-u, Hw-u\rangle + 2\langle Hw', Hw-u\rangle \\ &\qquad + \langle Hw', Hw'\rangle \\ &= \|Hw-u\|_2^2 + \langle w', 2H^*(Hw-u)\rangle + o(\|w'\|)\end{aligned}$$

where H^* is the adjoint of H.
We therefore have[1]

$$\nabla E(w) = \nabla TV_\varepsilon(w) + 2\lambda H^*(Hw-u).$$

It is easy to compute as soon as we can compute H and H^*.

[1] Remember

$$E(w) \ = \ TV_\varepsilon(w) + \lambda\|Hw-u\|_2^2.$$

Image Restoration, denoising : $H = H^* = Id$

Top: noisy images; Bottom: denoised images .

Image Restoration, denoising : $H = H^* = Id$

Figure: Denoising : $\sigma = 20$ and $\lambda = +\infty$, 0.5, 0.05, 0.017 and 0.005. Clean image.

Image Restoration, deblurring : H is a convolution

We have for all $w \in \mathbb{R}^{N^2}$

$$(Hw)_{m,n} = (h * w)_{m,n} = \sum_{m',n'=0}^{N-1} h_{m-m',n-n'} w_{m',n'}$$

for known convolution kernel $h \in \mathbb{R}^{N^2}$.
There are many algorithms to compute H.

Image Restoration, deblurring : H is a convolution

For w and $w' \in \mathbb{R}^{N^2}$, we have

$$\begin{aligned}
\langle Hw, w' \rangle &= \sum_{m,n=0}^{N-1} (Hw)_{m,n} w'_{m,n} \\
&= \sum_{m,n=0}^{N-1} \sum_{m',n'=0}^{N-1} h_{m-m',n-n'} w_{m',n'} w'_{m,n} \\
&= \sum_{m',n'=0}^{N-1} w_{m',n'} \sum_{m,n=0}^{N-1} h_{m-m',n-n'} w'_{m,n} \\
&= \sum_{m',n'=0}^{N-1} w_{m',n'} (\tilde{h} * w')_{m',n'}
\end{aligned}$$

where $\tilde{h}_{x,y} = h_{-x,-y}$, for all $x, y \in \mathbb{Z}$.
So

$$(H^* w')_{m,n} = (\tilde{h} * w')_{m,n} \qquad , \forall (m,n) \in \{0, \dots, N-1\}.$$

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Image Restoration, deblurring : H is a convolution

Top: blurred image; Bottom: restored and ideal image.

Image Restoration, inpainting : H multiplies by a mask

We know an image in a region $\mathcal{C} \subset \{0, \ldots, N-1\}^2$. We set for all $w \in \mathbb{R}^{N^2}$

$$(Hw)_{m,n} = M_{m,n} \; . \; w_{m,n}$$

for known mask

$$M_{m,n} = \begin{cases} 1 & \text{, if } (m,n) \in \mathcal{C} \\ 0 & \text{, otherwise} \end{cases}$$

Computing Hw is straightforward.

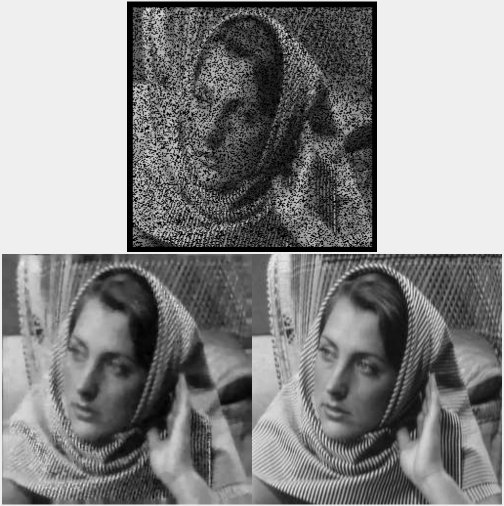
Image Restoration, inpainting : H multiplies by a mask

For w and $w' \in \mathbb{R}^{N^2}$, we have

$$\begin{aligned}
\langle Hw, w' \rangle &= \sum_{m,n=0}^{N-1} (Hw)_{m,n} w'_{m,n} \\
&= \sum_{m,n=0}^{N-1} \big(M_{m,n} w_{m,n}\big) w'_{m,n} \\
&= \sum_{m,n=0}^{N-1} w_{m,n} \big(M_{m,n} w'_{m,n}\big) \\
&= \langle w, Hw' \rangle
\end{aligned}$$

So $H^* = H$.

Image Restoration, inpainting : H multiplies by a mask



Top: Image with missing pixels; Bottom: restored and ideal image

Image Restoration: Zooming in

We consider the composition

$$H = S \circ C$$

of a convolution $C : \mathbb{R}^{(KN)^2} \longrightarrow \mathbb{R}^{(KN)^2}$ and sampling $S : \mathbb{R}^{(KN)^2} \longrightarrow \mathbb{R}^{N^2}$ defined by

$$(Sw)_{m,n} = w_{Km,Kn} \qquad , \text{ for all } m, n = 0..N-1.$$

We have

$$(S \circ C)^* = C^* \circ S^*.$$

We only need to calculate $S^* : \mathbb{R}^{N^2} \longrightarrow \mathbb{R}^{(KN)^2}$.

Image Restoration: Zooming in

For $w \in \mathbb{R}^{(KN)^2}$ and $w' \in \mathbb{R}^{N^2}$, we have

$$\begin{aligned}
\langle Sw, w' \rangle &= \sum_{m,n=0}^{N-1} (Sw)_{m,n} w'_{m,n} \\
&= \sum_{m,n=0}^{N-1} w_{Km,Kn} w'_{m,n} \\
&= \sum_{m,n=0}^{N-1} \sum_{k,l=0}^{K-1} w_{Km+k,Kn+l} (S^* w')_{Km+k,Kn+l} \\
&= \langle w, S^* w' \rangle
\end{aligned}$$

with

$$(S^* w')_{Km+k,Kn+l} = \begin{cases} w'_{m,n} & \text{, if } (k,l) = (0,0) \\ 0 & \text{, otherwise} \end{cases}$$

S^* zooms the image by interlacing 0. The result $S^* w'$ is a kind of "Dirac comb".

Image Restoration: Zooming in

Un-zoomed; zoomed (x4) by two methods

Image Restoration: De-compression

The lossy part of the compression of an image u consists in

- Compute Wu, where W is a unitary matrix or transform (JPEG 2000 ; wavelet, JPG : local cosine)
- Quantize every entry
- Encode the quantization interval

Therefore for any entry i of Wu we only know that

$$\tau_i^- \leq (Wu)_i < \tau_i^+$$

where τ_i^- and τ_i^+ are the known bounds of the quantization interval.
We estimate u by solving

$$\left\{ \begin{array}{l} \min_w TV_\varepsilon(w) \\ \text{such that } \tau_i^- \leq (Ww)_i \leq \tau_i^+ \quad , \forall i. \end{array} \right.$$

Solved by a projected gradient descent algorithm (i.e. proximal gradient algorithm).

Image Restoration: De-compression

We denote

$$\mathcal{C} = \{w \in \mathbb{R}^{N^2} | \forall i, \tau_i^- \leq (Ww)_i \leq \tau_i^+\}$$

We denote $P_{\mathcal{C}}(w)$ the projection of any $w \in \mathbb{R}^{N^2}$ onto $\mathcal{C}$ and have

$$\begin{aligned} P_{\mathcal{C}}(w) &= \text{Argmin}_{w' \in \mathcal{C}} \|w - w'\|^2 \\ &= \text{Argmin}_{\forall i, \tau_i^- \leq (Ww')_i \leq \tau_i^+} \sum_i ((Ww')_i - (Ww)_i)^2 \end{aligned}$$

We can prove that

$$(WP_{\mathcal{C}}(w))_i = \text{Argmin}_{\tau_i^- \leq t \leq \tau_i^+} (t - (Ww)_i)^2$$

and therefore

$$(WP_{\mathcal{C}}(w))_i = \begin{cases} \tau_i^+ & \text{, if } \tau_i^+ \leq (Ww)_i \\ (Ww)_i & \text{, if } \tau_i^- \leq (Ww)_i \leq \tau_i^+ \\ \tau_i^- & \text{, if } (Ww)_i \leq \tau_i^- \end{cases}$$

Image Restoration: De-compression

Top: compressed images (for differents compression level). Bottom: restored images