Mathematical methods for Image Processing

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Plan

Image Restoration

We look for a perfect image $v \in \mathbb{R}^{N^2}$ (or $\mathbb{R}^{N\times N}$) from noise corrupted linear measurements $u \in \mathbb{R}^P$:

$$u = Hv + b$$

where

$$H : \mathbb{R}^{N^2} \longrightarrow \mathbb{R}^P$$

and $b \in \mathbb{R}^P$ is some error (e.g. Gaussian noise of standard deviation σ).

- **Examples:** H is the identity (denoising), a convolution, a sampling operator (in a transformed space or not), ...
- **Applications :** Camera, Remote sensing imaging, Medical imaging (CT, IRM...), Biological imaging (many new microscopy acquisition devices), ...

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Ill-posed problems by the example

Assume

$$H : \mathbb{R}^{N \times N} \longrightarrow \mathbb{R}^{N \times N}$$

is a convolution with $h \in^{N \times N}$.

$u = h * v + b$ becomes, in Fourier domain,

$$\hat{u}_{k,l} = \hat{h}_{k,l} \hat{v}_{k,l} + \hat{b}_{k,l} \qquad , \forall k, l = 0..N-1.$$

- If $\hat{h}_{k,l} = 0$,

$$\hat{v}_{k,l} \text{ is lost}$$

- If $\hat{h}_{k,l} \neq 0$,

$$\hat{v}_{k,l} = \frac{\hat{u}_{k,l}}{\hat{h}_{k,l}} - \frac{\hat{b}_{k,l}}{\hat{h}_{k,l}}$$

  The noise is amplified by a factor $\frac{1}{\hat{h}_{k,l}}$

Regularize the solution

Choose R that is

- small for the typical data you are restoring
- large for "H^{-1} of the error/noise"

and solve

$$\begin{cases} w^* = \text{Argmin}_w \;\; R(w) \\ \text{under the constraint } \|Hw - u\|_2 \leq \tau \end{cases}$$

or an unconstrained version:

$$w^* = \text{Argmin}_{w \in \mathbb{R}^{N^2}} \;\; R(w) + \lambda \|Hw - u\|_2^2.$$

- These problem can be obtained as MAP estimates (Bayesian framework)
- Can lead to bounds on the error $\|w^* - v\| \leq ...$ ("Compressed sensing" framework, developed Wednesday and Thursday)

Bayesian Modeling

Assume images are distributed according to the prior

$$\mathbb{P}(w) \propto e^{-\mu R(w)}.$$

Assume b is i.i.d. centered Gaussian noise of standard deviation $\sigma > 0$:

$$\mathbb{P}(b) \propto e^{-\frac{\|b\|_2^2}{2\sigma^2}}.$$

We have, for the data u obtained from w

$$\mathbb{P}(u|w) \propto e^{-\frac{\|u-Hw\|_2^2}{2\sigma^2}}.$$

Applying *Bayes* law, we get the posterior distribution

$$\begin{aligned}
\mathbb{P}(w|u) &= \frac{\mathbb{P}(u|w)\mathbb{P}(w)}{\mathbb{P}(u)}, \\
&\propto \mathbb{P}(u|w)\mathbb{P}(w), \\
&\propto e^{-\frac{\|u-Hw\|_2^2}{2\sigma^2}} e^{-\mu R(w)}.
\end{aligned}$$

Bayesian Modeling

The Maximum A Posteriori (MAP) estimate

$$\text{maximize } \mathbb{P}(w|u)$$

equivalently

$$\text{minimize } -\log\left(\mathbb{P}(w|u)\right)$$

This leads to

$$w^{MAP} = \text{Argmin}_{w \in W} \frac{\|u - Hw\|_2^2}{2\sigma^2} + \mu R(w).$$

Identical to

$$w^* = \text{Argmin}_{w \in \mathbb{R}^{N^2}} \; R(w) + \lambda \|Hw - u\|_2^2,$$

when $\lambda = 2\sigma^2\mu$.

Bayesian Modeling: Comments

- Assumption on R is often very wrong
- Differ from the "compressed sensing" approach because R is built independently of the noise and H.
- Powerful for designing models when many variables interact
- Leads to well founded strategies to tune the parameters (λ, in the example): See Expectation-Maximization algorithms.

The total variation

Definition

Let $w \in L^1\left([0,N]^2\right)$, if the following supremum is finite, we say w is of bounded variation, denote $w \in BV\left([0,N]^2\right)$ and define

$$TV(w) = \sup\left\{\int_{[0,N]^2} w \ \operatorname{div}\varphi \ dxdy \quad , \quad \varphi \in C^1\left(]0,N[^2, \mathbb{R}^2\right) \ \ et\ |\varphi| \leq 1\right\} < +\infty$$

Above $C^1\left(]0,N[^2,\mathbb{R}^2\right)$ contains C^1 functions from $]0,N[^2$ into $\mathbb{R}^2$, $|.|$ is the Euclidean norm in $\mathbb{R}^2$ and $|\varphi| \leq 1$ means

$$\begin{array}{lll} & |(\varphi_1(x,y),\varphi_2(x,y))| \leq 1 & ,\forall (x,y) \in]0,N[^2. \\ \text{i.e} & \varphi_1(x,y)^2 + \varphi_2(x,y)^2 \leq 1 & ,\forall (x,y) \in]0,N[^2. \end{array}$$

The total variation, Example 1 : when w is C^1

For any $\varphi = (\varphi_1, \varphi_2) \in C^1\left(]0,N[^2, \mathbb{R}^2\right)$, tel que $|\varphi| \leq 1$

$$\begin{aligned}
\int_{[0,N]^2} w \ \operatorname{div} \varphi \ dxdy &= \int_{[0,N]^2} w \left(\frac{\partial \varphi_1}{\partial x} + \frac{\partial \varphi_2}{\partial y} \right) \ dxdy, \\
&= -\int_{[0,N]^2} \frac{\partial w}{\partial x} \ \varphi_1 + \frac{\partial w}{\partial y} \ \varphi_2 \ dxdy, \\
&= -\int_{[0,N]^2} \nabla w \ . \ \varphi \ dxdy, \\
&\leq \int_{[0,N]^2} |\nabla w| \ dxdy. \qquad (1)
\end{aligned}$$

where $\nabla w \ . \ \varphi$ is the inner product in $\mathbb{R}^2$.

The total variation, Example 1 : when w is C^1

We have, for all $\varphi \in C^1\left(]0,N[^2, \mathbb{R}^2\right)$ such that $|\varphi| \leq 1$

$$\int_{[0,N]^2} w \ \operatorname{div}\varphi \ dxdy \leq \int_{[0,N]^2} |\nabla w| \ dxdy \qquad ,$$

and therefore

$$TV(w) \leq \int_{[0,N]^2} |\nabla w| \ dxdy.$$

In fact, we can prove the converse inequality and state:

Proposition

If $w \in BV\left([0,N]^2\right)$ and w is C^1, we have

$$TV(w) = \int_{[0,N]^2} |\nabla w| \ dxdy.$$

The total variation, Example 2: Characteristic function of a set

Theorem

If $E \subset [0,N]^2$ is an open set with a smooth boundary (For instance Lipschitz) and if $w = \mathbf{1}_{|E}$ then $w \in BV\left([0,N]^2\right)$ and

$$TV(w) = \mathcal{H}^1(\partial E),$$

where $\mathcal{H}^1$ is the Hausdorff[a] measure of dimension 1.

[a] The Haussdorf measure of dimension 1 of a set is the lenght of this set, if the set is "1D"; it is 0, if the dimension of the set is strictily smaller than 1; it is $+\infty$, if it is of dimension strictly larger than 1.

The total variation, Co-area formula

For a function $w \in L^1\left([0,N]^2\right)$ and $t \in \mathbb{R}$, we denote the t level set of w by:

$$\mathcal{L}_w(t) = \{(x,y) \in [0,N]^2, w(x,y) \geq t\}.$$

Theorem (Co-area formula)

If $w \in BV\left([0,N]^2\right)$, then

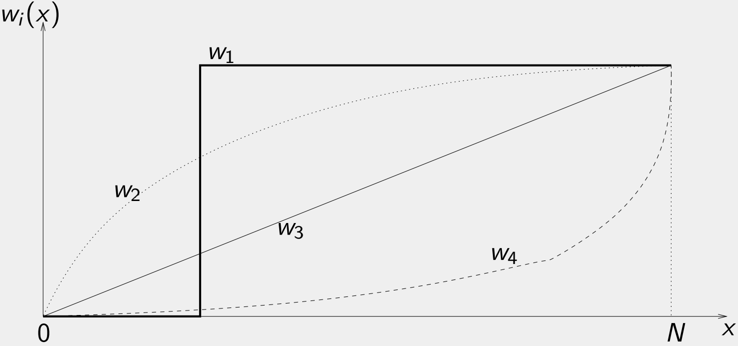
- *for almost every $t \in \mathbb{R}$,*
$$TV\left(\mathbf{1}_{\mathcal{L}_w(t)}\right) < \infty;$$
- *the function*
$$t \longmapsto TV\left(\mathbf{1}_{\mathcal{L}_w(t)}\right) \qquad \text{is measurable;}$$
- *and*
$$TV(w) = \int_{-\infty}^{\infty} TV\left(\mathbf{1}_{\mathcal{L}_w(t)}\right) dt.$$

The total variation, Example 3: if w is a non-decreasing signal

If $w \in BV([0,N])$ is non-decreasing and C^1,

$$\begin{aligned} TV(w) &= \int_0^N |w'| \, dx \\ &= \int_0^N w' \, dx \\ &= w(N) - w(0). \end{aligned}$$

The total variation, Example 3: if w is a non-decreasing signal



$$TV(w_1) = TV(w_2) = TV(w_3) = TV(w_4)$$

The total variation, on the grid

For any $w \in \mathbb{R}^{N^2}$, we set

$$TV(w) = \sum_{m,n=1}^{N} |\nabla w_{m,n}|,$$

where $|.|$ is the Euclidean norm in $\mathbb{R}^2$,

$$\nabla w_{m,n} = \begin{pmatrix} \partial_m w_{m,n} \\ \partial_n w_{m,n} \end{pmatrix} = \begin{pmatrix} w_{m+1,n} - w_{m,n} \\ w_{m,n+1} - w_{m,n}, \end{pmatrix}$$

(We assume w periodic.)

The total variation, on the grid

$$TV(w) = \sum_{m,n=1}^{N} |\nabla w_{m,n}|,$$

- **Good properties :** continuous, convexe, a semi-norm over $\mathbb{R}^{N^2}$, a norm over $\{w \in \mathbb{R}^{N^2}, \sum_{m,n=1}^{N} w_{m,n} = 0\}$
- **Bad properties :**
  - **Not coercive** over $\mathbb{R}^{N^2}$ $(TV(w+c) = TV(w))$: Not a problem as long as the data fidelity term does not make the image mean diverge.
  - **Not differentiable**, as soon as there is $(m,n) \in \{1,\ldots,N\}^2$ such that

    $$|\nabla w_{m,n}| = 0.$$

    Many ways to avoid this problem. In this lecture, we smooth TV

    $$TV_\varepsilon(w) = \sum_{m,n=1}^{N} \varphi_\varepsilon(|\nabla w_{m,n}|^2),$$

    where $\varepsilon > 0$ is small and $\varphi_\varepsilon(t) = \sqrt{t+\varepsilon}$, for $t > 0$.

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The total variation, on the grid: Computing $\nabla TV_\varepsilon(w)$

For

$$\varphi_\varepsilon(t) = \sqrt{t+\varepsilon} \quad ,\forall t > 0,$$

we have

$$\varphi'_\varepsilon(t) = \frac{1}{2\sqrt{t+\varepsilon}} \quad ,\forall t > 0. \tag{1}$$

We denote the finite difference operators :

$$\begin{array}{rcl} \partial_m : \mathbb{R}^{N^2} & \longrightarrow & \mathbb{R}^{N^2}, \\ (w_{m,n})_{1\leq m,n\leq N} & \longmapsto & (w_{m+1,n} - w_{m,n})_{1\leq m,n\leq N}, \end{array}$$

and

$$\begin{array}{rcl} \partial_n : \mathbb{R}^{N^2} & \longrightarrow & \mathbb{R}^{N^2}, \\ (w_{m,n})_{1\leq m,n\leq N} & \longmapsto & (w_{m,n+1} - w_{m,n})_{1\leq m,n\leq N}. \end{array}$$

The total variation, on the grid: Computing $\nabla TV_\varepsilon(w)$

For any w, $w' \in \mathbb{R}^{N^2}$ we have

$$\begin{aligned}
\langle w, \partial_m w' \rangle &= \sum_{m,n=1}^{N} w_{m,n} \left(w'_{m+1,n} - w'_{m,n} \right), \\
&= \sum_{m,n=1}^{N} w_{m,n} w'_{m+1,n} - \sum_{m,n=1}^{N} w_{m,n} w'_{m,n}, \\
&= \sum_{m,n=1}^{N} w_{m-1,n} w'_{m,n} - \sum_{m,n=1}^{N} w_{m,n} w'_{m,n}, \\
&= \sum_{m,n=1}^{N} \left(w_{m-1,n} - w_{m,n} \right) w'_{m,n}, \\
&= \langle \partial_m^* w, w' \rangle.
\end{aligned}$$

Therefore

$$\partial_m^* w_{m,n} = w_{m-1,n} - w_{m,n} \quad , \forall (m,n) \in \{1, \ldots, N\}^2.$$

The total variation, on the grid: Computing $\nabla TV_\varepsilon(w)$

$$
\begin{aligned}
& TV_\varepsilon(w+w') - TV_\varepsilon(w) \\
= & \sum_{m,n=1}^{N} \varphi_\varepsilon\Big(|\nabla(w+w')_{m,n}|^2\Big) - \varphi_\varepsilon\Big(|\nabla w_{m,n}|^2\Big), \\
= & \sum_{m,n=1}^{N} \varphi_\varepsilon\Big(|\nabla w_{m,n}|^2 + 2(\partial_m w_{m,n}\partial_m w'_{m,n} + \partial_n w_{m,n}\partial_n w'_{m,n}) + o(|\nabla w'_{m,n}|)\Big) \\
& \qquad -\varphi_\varepsilon\Big(|\nabla w_{m,n}|^2\Big), \\
= & \sum_{m,n=1}^{N} 2\ \varphi'_\varepsilon\Big(|\nabla w_{m,n}|^2\Big)(\partial_m w_{m,n}\partial_m w'_{m,n} + \partial_n w_{m,n}\partial_n w'_{m,n}) + o(|\nabla w'_{m,n}|).
\end{aligned}
$$

We denote $X \in \mathbb{R}^{N^2}$ and $Y \in \mathbb{R}^{N^2}$ such that

$$
X_{m,n} = 2\ \varphi'_\varepsilon\Big(|\nabla w_{m,n}|^2\Big)\ \partial_m w_{m,n} \qquad \text{and} \qquad Y_{m,n} = 2\ \varphi'_\varepsilon\Big(|\nabla w_{m,n}|^2\Big)\ \partial_n w_{m,n},
$$

The total variation, on the grid: Computing $\nabla TV_\varepsilon(w)$

We get

$$\begin{aligned} TV_\varepsilon(w+w') - TV_\varepsilon(w) &= \sum_{m,n=1}^{N} X_{m,n}\partial_m w'_{m,n} + \sum_{m,n=1}^{N} Y_{m,n}\partial_n w'_{m,n} \\ &\quad + \sum_{m,n=1}^{N} o(|\nabla w'_{m,n}|) \\ &= \langle X, \partial_m w'\rangle + \langle Y, \partial_n w'\rangle + o(\|\nabla w'\|_1), \\ &= \langle \partial_m^* X + \partial_n^* Y, w'\rangle + o(\|w'\|_2). \end{aligned}$$

since

$$\sum_{m,n=1}^{N} |\nabla w'_{m,n}| \leq \sum_{m,n=1}^{N} |\nabla w'_{m,n}|_1 \leq 4 \sum_{m,n=1}^{N} |w'_{m,n}| \leq 4N\|w'\|_2.$$

The total variation, on the grid: Computing $\nabla TV_\varepsilon(w)$

Finally,

$$\nabla TV_\varepsilon(w) = \partial_m^* X + \partial_n^* Y,$$

with

$$X_{m,n} = 2\,\varphi'_\varepsilon\left(|\nabla w_{m,n}|^2\right)\,\partial_m w_{m,n} \qquad \text{and} \qquad Y_{m,n} = 2\,\varphi'_\varepsilon\left(|\nabla w_{m,n}|^2\right)\,\partial_n w_{m,n},$$

We also have (admitted), for all w and w'

$$\|\nabla TV_\varepsilon(w') - \nabla TV_\varepsilon(w)\| \leq \frac{8}{\sqrt{\varepsilon}}\|w' - w\|$$

The total variation : To go further

- Non-local Total Variation: see Li-Malgouyres-Zeng
- Numerical methods: Based on Graph cuts (see Chambolle and Darbon), primal-dual approach (see Chambolle, Chambolle-Pock)
- Staircase effect: see Nikolova
- Theoretical Justification: Compressed sensing with Co-sparse/Analysis prior (see Gribonval)