Mathematical methods for Image Processing

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Plan

1. Hands-on session : Image dequantization using the H^1 regularization

Exercise 1, question 2.a

Calculate the derivatives of

$$\varphi_\tau(t) = \left(sup\left(|t| - \frac{\tau}{2}, 0 \right) \right)^2.$$

(You may distinguish the three cases: $t \leq -\frac{\tau}{2}$, $-\frac{\tau}{2} < t < \frac{\tau}{2}$ and $\frac{\tau}{2} \leq t$.)

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- If $t < -\frac{\tau}{2}$, then $\varphi_\tau(t) = (-t - \frac{\tau}{2})^2$ and $\varphi'_\tau(t) = 2(t + \frac{\tau}{2})$
- If $-\frac{\tau}{2} < t < \frac{\tau}{2}$, then $\varphi_\tau(t) = 0$ and $\varphi'_\tau(t) = 0$
- If $\frac{\tau}{2} < t$, then $\varphi_\tau(t) = (t - \frac{\tau}{2})^2$ and $\varphi'_\tau(t) = 2(t - \frac{\tau}{2})$

Therefore

$$\varphi'_\tau(t) = \begin{cases} 2(t + \frac{\tau}{2}) & \text{, if } t < -\frac{\tau}{2} \\ 0 & \text{, if } -\frac{\tau}{2} \leq t \leq \frac{\tau}{2} \\ 2(t - \frac{\tau}{2}) & \text{, otherwise} \end{cases}$$

Exercise 1, question 2.b

Deduce from the previous question and the Hands-on session 1-2, the gradient of

$$F_\lambda(w) = E(w) + \lambda \sum_{i,j=0}^{N-1} \varphi_\tau\left(w_{i,j} - v_{i,j}\right).$$

Exercise 1, question 2.b

Deduce from the previous question and the Hands-on session 1-2, the gradient of

$$F_\lambda(w) = E(w) + \lambda \sum_{i,j=0}^{N-1} \varphi_\tau\left(w_{i,j} - v_{i,j}\right).$$

For w and $w' \in \mathbb{R}^{N^2}$, we have

$$\varphi_\tau\left(w_{i,j} + w'_{i,j} - v_{i,j}\right) = \varphi_\tau\left(w_{i,j} - v_{i,j}\right) + \varphi'_\tau\left(w_{i,j} - v_{i,j}\right) w'_{i,j} + o(|w'_{i,j}|)$$

Therefore, we compute

$$X = \left(\varphi'_\tau\left(w_{i,j} - v_{i,j}\right)\right)_{0 \leq i,j < N}$$

and have

$$\nabla F_\lambda(w) = D_x^* D_x w + D_y^* D_y w + \lambda X$$

Exercise 2, question 1

Let $w \in \mathbb{R}^{N^2}$. What are the coordinates of $\Pi(w)$ (the projection of w onto $\overline{\mathbb{C}}$). (Begin with a proof arguing that we can compute N^2 projections: We project independently every $w_{i,j}$ onto the interval $[v_{i,j} - \frac{\tau}{2}, v_{i,j} + \frac{\tau}{2}]$.)

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Exercise 2, question 1

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Writing $\tau_{i,j}^- = v_{i,j} - \frac{\tau}{2}$ and $\tau_{i,j}^+ = v_{i,j} + \frac{\tau}{2}$, we have

$$\Pi(w) = \text{Argmin}_{w' : \forall i,j \tau_{i,j}^- \leq w'_{i,j} \leq \tau_{i,j}^+} \|w' - w\|^2$$

Let

$$w_{i,j}^* = \text{Argmin}_{\tau_{i,j}^- \leq t \leq \tau_{i,j}^+} (t - w_{i,j})^2$$

and $w^* = (w_{i,j}^*)_{0 \leq i,j < N}$.

We have $w^* \in \overline{\mathcal{C}}$. Moreover, for any $w' \in \overline{\mathcal{C}}$, and any i, j

$$(w_{i,j}^* - w_{i,j})^2 \leq (w'_{i,j} - w_{i,j})^2$$

and therefore

$$\|w^* - w\|^2 = \sum_{i,j=0}^{N-1} (w_{i,j}^* - w_{i,j})^2 \leq \sum_{i,j=0}^{N-1} (w'_{i,j} - w_{i,j})^2 = \|w' - w\|^2$$

Exercise 2, question 1

Therefore

$$\Pi(w)_{i,j} = \begin{cases} \tau_{i,j}^{+} & \text{, if } w_{i,j} \geq \tau_{i,j}^{+} \\ w_{i,j} & \text{, if } \tau_{i,j}^{+} \geq w_{i,j} \geq \tau_{i,j}^{-} \\ \tau_{i,j}^{-} & \text{, if } \tau_{i,j}^{-} \geq w_{i,j} \end{cases}$$

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Thank you for your attention !

I wish you the best !