Mathematical methods for Image Processing

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Plan

Image Segmentation: Examples

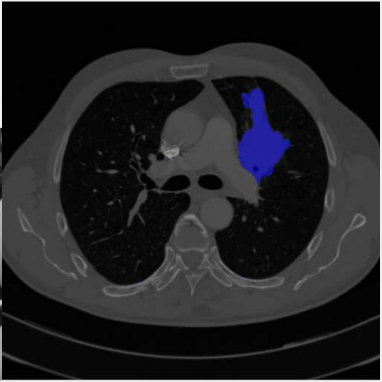


Image Segmentation: Principle

We look for a shape/set

$$\Omega^* \subset \{1, \ldots, N\}^2.$$

We solve

$$\Omega^* \in \text{Argmin}_{\Omega \subset \{1,\ldots,N\}^2} E(\Omega) + R(\Omega)$$

- E enforces a prior on the segmented object.
  Ex: If u is the image

$$E(\Omega) = \mu_1 \sum_{(m,n)\in\Omega} |u_{m,n} - c_1|^2 + \mu_2 \sum_{(m,n)\notin\Omega} |u_{m,n} - c_2|^2$$

- R enforces a prior on the shape.
  Ex: $R(\Omega)$ is the perimeter of Ω

Image Segmentation: The perimeter

We denote $\mathcal{P} = \{1, \dots, N\}^2$. Let

$$\sigma : \mathcal{P} \longrightarrow P(\mathcal{P})$$

be a notion of neighborhood. $P(\mathcal{P})$ is the power set of $\mathcal{P}$:

$$\text{For any } (m,n) \in \mathcal{P}, \qquad \sigma(m,n) \subset \mathcal{P}.$$

Examples :

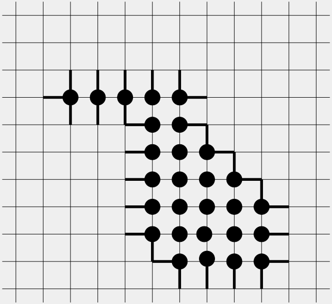
- Local neighborhoods:
  - 4 connectivity : $\sigma(m,n) = \{(m+1,n),(m-1,n),(m,n-1),(m,n+1)\}$
  - 8 connectivity : $\sigma(m,n) = \{(m+1,n+1),(m+1,n),...,(m-1,n-1)\}$
- Non-local neighborhood: If $U_{m,n}$ is a small neighborhood surrounding (m,n)

$$\sigma(m,n) = \{(m',n') \in \mathcal{P} | \ \|U_{m,n} - U_{m',n'}\| \leq \tau\}$$

Image Segmentation: The perimeter

Let $\Omega \subset \mathcal{P}$, we define

$$\partial\Omega = \Big\{((m,n),(m',n')) \in \mathcal{P}^2 | (m',n') \in \sigma(m,n) \text{ and } (m,n) \in \Omega \text{ and } (m',n') \notin \Omega\Big\}.$$



● : Pixels in Ω

| — : Connected neighbors in $\partial\Omega$

Image Segmentation: The perimeter

Let $\Omega \subset \mathcal{P}$, we define

$$\partial\Omega = \Big\{((m,n),(m',n')) \in \mathcal{P}^2 | (m',n') \in \sigma(m,n) \\ \text{and } (m,n) \in \Omega \text{ and } (m',n') \notin \Omega\Big\}.$$

$$\mathbf{P}(\Omega) = \sum_{((m,n),(m',n'))\in\partial\Omega} dl\left((m,n),(m',n')\right),$$

where $dl((m,n),(m',n')) \geq 0$ is some notion of length.
For simplicity, we assume for all $((m,n),(m',n')) \in \mathcal{P}^2$

$$dl((m,n),(m',n')) = dl((m',n'),(m,n))$$

and extend

$$\text{if} \quad (m',n') \notin \sigma(m,n) \quad \text{then} \quad dl((m,n),(m',n')) = 0.$$

Image Segmentation: perimeter and total variation

With 4-connectivity and anisotropic TV

$$
\begin{aligned}
TV\left(\mathbf{1}_{|\Omega}\right) &= \sum_{m,n=1}^{N} |\nabla \mathbf{1}_{|\Omega}(m,n)| \\
&= \sum_{m,n=1}^{N} |\mathbf{1}_{|\Omega}(m+1,n) - \mathbf{1}_{|\Omega}(m,n)| + |\mathbf{1}_{|\Omega}(m,n+1) - \mathbf{1}_{|\Omega}(m,n)| \\
&= \sum_{m,n=1}^{N} \Big(\mathbf{1}_{|(m,n)\in\Omega \text{ and } (m+1,n)\notin\Omega} + \mathbf{1}_{|(m,n)\notin\Omega \text{ and } (m+1,n)\in\Omega} \\
&\qquad + \mathbf{1}_{|(m,n)\in\Omega \text{ and } (m,n+1)\notin\Omega} + \mathbf{1}_{|(m,n)\notin\Omega \text{ and } (m,n+1)\in\Omega} \Big) \\
&= \sum_{((m,n),(m',n'))\in\partial\Omega} 1,
\end{aligned}
$$

Image Segmentation: Mumford-Shah model

Given an image u, we look for

$$(\Omega^*, w^*) \in \text{Argmin}\, E(\Omega, w)$$

where for $\Omega \subset \mathcal{P}$ and $w \in \mathbb{R}^{N^2}$

$$E(\Omega, w) = \mathbf{P}(\Omega) \; + \; \lambda \sum_{\substack{m,n=1 \\ ((m,n),(m+1,n))\notin\partial\Omega \\ ((m,n),(m,n+1))\notin\partial\Omega}}^{N} |\nabla w_{m,n}|^2 \; + \; \mu \, \|w-u\|_2^2,$$

where $\lambda \geq 0$, $\mu \geq 0$ are parameters. The model favors

- u is smooth inside Ω
- u has a large gradient on $\partial\Omega$
- Ω is smooth

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Image Segmentation: Chan-Vese model

Given an image u, we look for

$$(\Omega^*, c_1^*, c_2^*) \in \text{Argmin}\, E(\Omega, c_1, c_2)$$

where for $\Omega \subset \mathcal{P}$, c_1 and $c_2 \in \mathbb{R}$

$$E(\Omega, c_1, c_2) = \mathbf{P}(\Omega) + \mu_1 \sum_{(m,n)\in\Omega} |u_{m,n} - c_1|^2 + \mu_2 \sum_{(m,n)\notin\Omega} |u_{m,n} - c_2|^2,$$

where $\mu_1 \geq 0$, $\mu_2 \geq 0$ are parameters. The model favors

- u has a constant gray-level inside Ω; a constant gray-level outside Ω
- Ω is smooth
- $\partial\Omega$ is not particularly driven toward high contrast pixels

Image Segmentation: Boykov-Jolly model

Given an image u, we look for

$$\Omega^* \in \text{Argmin}\, E(\Omega)$$

where for $\Omega \subset \mathcal{P}$

$$E(\Omega) = \mathbf{P}(\Omega) + \sum_{(m,n)\in\Omega} -\log\left(\mathbb{P}(u_{m,n}|(m,n)\in\mathcal{O})\right) + \sum_{(m,n)\notin\Omega} -\log\left(\mathbb{P}(u_{m,n}|(m,n)\in\mathcal{F})\right),$$

where

- $\mathbb{P}(c|(m,n)\in\mathcal{O})$ is the probability density function of the probability that a pixels of $\mathcal{O}$ has the gray level c.
- $\mathbb{P}(c|(m,n)\in\mathcal{F})$, same for the background $\mathcal{F}$

and, for $\tau > 0$

$$dl\left((m,n),(m',n')\right) = \frac{e^{-\frac{(u_{m,n}-u_{m',n'})^2}{2\tau^2}}}{\sqrt{(m-m')^2+(n-n')^2}}.$$

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Image Segmentation: Introduction to level sets

We consider an image $\varphi \in \mathbb{R}^{\mathcal{P}}$ and let

$$\Omega = \left\{(m,n) \in \{1,\ldots,N\}^2, \varphi_{m,n} \geq 0\right\}.$$

Instead of manipulating Ω, we manipulate φ because it is in Euclidean space. We let

$$\mathbf{1}_{|.\geq 0}(t) = \begin{cases} 1 & \text{, if } t \geq 0, \\ 0 & \text{, if } t < 0. \end{cases}$$

$$\sum_{(m,n)\in\Omega} a_{m,n} \quad \text{becomes} \quad \sum_{m,n=0}^{N} a_{m,n} \mathbf{1}_{|.\geq 0}(\varphi_{m,n})$$

$$\sum_{(m,n)\notin\Omega} b_{m,n} \quad \text{becomes} \quad \sum_{m,n=0}^{N} b_{m,n} \left(1 - \mathbf{1}_{|.\geq 0}(\varphi_{m,n})\right)$$

Image Segmentation: Introduction to level sets

Remember

$$\partial\Omega = \Big\{((m,n),(m',n')) \in \mathcal{P}^2 | (m',n') \in \sigma(m,n)$$
$$\text{and } (m,n) \in \Omega \text{ and } (m',n') \notin \Omega\Big\}$$

and

$$dl\,((m,n),(m',n')) \qquad , \text{ if } (m',n') \notin \sigma(m,n)$$

Therefore

$$\sum_{((m,n),(m',n'))\in\partial\Omega} dl\,((m,n),(m',n'))$$

becomes

$$\sum_{m,n=0}^{N} \sum_{m',n'=0}^{N} dl\,((m,n),(m',n'))\, \mathbf{1}_{|.\geq 0}(\varphi_{m,n}) \left(1 - \mathbf{1}_{|.\geq 0}(\varphi_{m',n'})\right).$$

Image Segmentation: Introduction to level sets

The function

$$\mathbf{1}_{|.\geq 0}$$

is

- either not differentiable (if $t = 0$)
- or has a null gradient (if $t \neq 0$)

So, we approximate

$$\mathbf{1}_{|.\geq 0}(\varphi_{m,n})$$

by

$$H_\varepsilon(\varphi_{m,n}),$$

with

$$H_\varepsilon(t) = \begin{cases} 1 & \text{, if } t \geq \varepsilon, \\ \frac{1}{2}\left(1 + \frac{t}{\varepsilon} + \frac{1}{\pi}\sin\left(\frac{\pi t}{\varepsilon}\right)\right) & \text{, if } -\varepsilon \leq t \leq \varepsilon, \\ 0 & \text{, if } t \leq -\varepsilon. \end{cases}$$

Image Segmentation: Introduction to level sets

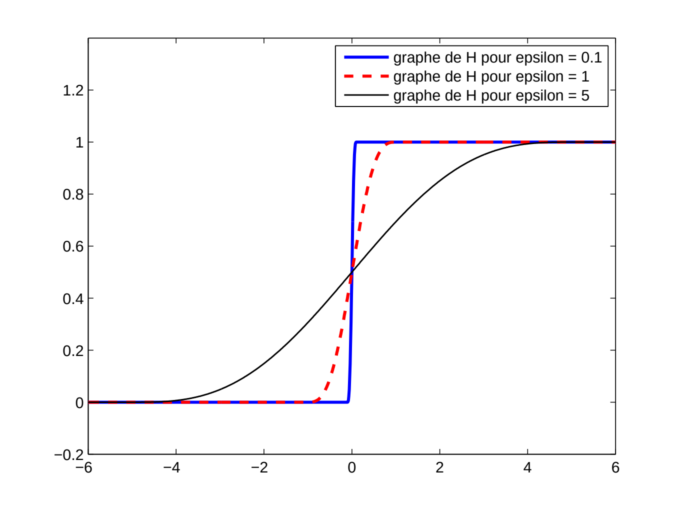

Figure: Function H_ε for $\varepsilon = 0.1,\ 1,\ 5$.

Image Segmentation: Introduction to level sets

For $\varphi = (\varphi_{m,n})_{1\leq m,n\leq N} \in \mathbb{R}^{N\times N}$

$$E_1(\varphi) = \sum_{m,n=0}^{N} a_{m,n} H_\varepsilon\left(\varphi_{m,n}\right),$$

We have for φ and $h \in \mathbb{R}^{N\times N}$

$$\begin{aligned}
E_1(\varphi + h) - E_1(\varphi) &= \sum_{m,n=0}^{N} a_{m,n}\left(H_\varepsilon\left(\varphi_{m,n} + h_{m,n}\right) - H_\varepsilon\left(\varphi_{m,n}\right)\right), \\
&= \sum_{m,n=0}^{N} a_{m,n}\left(H'_\varepsilon\left(\varphi_{m,n}\right) h_{m,n} + o\left(|h_{m,n}|\right)\right), \\
&= \sum_{m,n=0}^{N} a_{m,n} H'_\varepsilon\left(\varphi_{m,n}\right) h_{m,n} + o\left(\|h\|_2\right), \\
&= \langle \nabla E_1(\varphi), h\rangle + o\left(\|h\|_2\right)
\end{aligned}$$

with

$$\nabla E_1(\varphi) = \left(a_{m,n} H'_\varepsilon\left(\varphi_{m,n}\right)\right)_{1\leq m,n\leq N}.$$

Image Segmentation: Introduction to level sets

We therefore have for

$$E_2(\varphi) = \sum_{m,n=0}^{N} b_{m,n} \left(1 - H_\varepsilon \left(\varphi_{m,n}\right)\right),$$

$$\nabla E_2(\varphi) = \left(-b_{m,n} H'_\varepsilon \left(\varphi_{m,n}\right)\right)_{1 \leq m,n \leq N}.$$

Let

$$E_3(\varphi) = \sum_{m,n=0}^{N} \sum_{m',n'=0}^{N} dl\left((m,n),(m',n')\right) H_\varepsilon \left(\varphi_{m,n}\right) \left(1 - H_\varepsilon \left(\varphi_{m',n'}\right)\right).$$

We have

Image Segmentation: Introduction to level sets

$$
\begin{aligned}
& E_3(\varphi + h) \\
= & \sum_{m,n=0}^{N} \sum_{m',n'=0}^{N} dI\left((m,n),(m',n')\right) H_\varepsilon\left(\varphi_{m,n} + h_{m,n}\right) \\
& \quad \left(1 - H_\varepsilon\left(\varphi_{m',n'} + h_{m',n'}\right)\right), \\
= & \sum_{m,n=0}^{N} \sum_{m',n'=0}^{N} dI\left((m,n),(m',n')\right) \left[H_\varepsilon\left(\varphi_{m,n}\right) + H'_\varepsilon\left(\varphi_{m,n}\right) h_{m,n} + o\left(|h_{m,n}|\right)\right] \\
& \quad \left[1 - H_\varepsilon\left(\varphi_{m',n'}\right) - H'_\varepsilon\left(\varphi_{m',n'}\right) h_{m',n'} + o\left(|h_{m',n'}|\right)\right], \\
= & E_3(\varphi) + o\left(\|h\|_2\right) + \sum_{m,n=0}^{N} \sum_{m',n'=0}^{N} dI\left((m,n),(m',n')\right) \\
& \quad \left[-H_\varepsilon\left(\varphi_{m,n}\right) H'_\varepsilon\left(\varphi_{m',n'}\right) h_{m',n'} + H'_\varepsilon\left(\varphi_{m,n}\right) h_{m,n}\left(1 - H_\varepsilon\left(\varphi_{m',n'}\right)\right)\right].
\end{aligned}
$$

Image Segmentation: Introduction to level sets

In first term, we switch variable names

$$\sum_{m,n=0}^{N}\sum_{m',n'=0}^{N} -dI\left((m,n),(m',n')\right) H_\varepsilon\left(\varphi_{m,n}\right) H'_\varepsilon\left(\varphi_{m',n'}\right) h_{m',n'} =$$

$$\sum_{m',n'=0}^{N}\sum_{m,n=0}^{N} -dI\left((m',n'),(m,n)\right) H_\varepsilon\left(\varphi_{m',n'}\right) H'_\varepsilon\left(\varphi_{m,n}\right) h_{m,n}.$$

switch sums and use the symetry of *dI* and get

$$\sum_{m,n=0}^{N}\sum_{m',n'=0}^{N} -dI\left((m,n),(m',n')\right) H_\varepsilon\left(\varphi_{m,n}\right) H'_\varepsilon\left(\varphi_{m',n'}\right) h_{m',n'} =$$

$$\sum_{m,n=0}^{N}\sum_{m',n'=0}^{N} -dI\left((m,n),(m',n')\right) H_\varepsilon\left(\varphi_{m',n'}\right) H'_\varepsilon\left(\varphi_{m,n}\right) h_{m,n}.$$

Image Segmentation: Introduction to level sets

Finally, we get

$$\langle \nabla E_3(\varphi), h\rangle = \sum_{m,n=0}^{N} \sum_{m',n'=0}^{N} \Big[-dI\left((m,n),(m',n')\right) H_\varepsilon\left(\varphi_{m',n'}\right) H'_\varepsilon\left(\varphi_{m,n}\right) h_{m,n}\Big] + \Big[dI\left((m,n),(m',n')\right) H'_\varepsilon\left(\varphi_{m,n}\right) h_{m,n}\Big(1 - H_\varepsilon\left(\varphi_{m',n'}\right)\Big)\Big]$$

and deduce

$$\begin{aligned}
\nabla E_3(\varphi)_{m,n} &= \sum_{m',n'=0}^{N} dI\left((m,n),(m',n')\right) \\
&\quad \Big(-H_\varepsilon\left(\varphi_{m',n'}\right) H'_\varepsilon\left(\varphi_{m,n}\right) + H'_\varepsilon\left(\varphi_{m,n}\right)\Big(1 - H_\varepsilon\left(\varphi_{m',n'}\right)\Big)\Big), \\
&= \sum_{m',n'=0}^{N} dI\left((m,n),(m',n')\right) H'_\varepsilon\left(\varphi_{m,n}\right)\left(1 - 2H_\varepsilon\left(\varphi_{m',n'}\right)\right).
\end{aligned}$$

Algorithm for Chan-Vese model

$$(\varphi^*, c_1^*, c_2^*) \in \text{Argmin}\, \mu_1 E_1(\varphi, c_1) + \mu_2 E_2(\varphi, c_2) + E_3(\varphi)$$

with

$$a_{m,n} = (u_{m,n} - c_1)^2 \qquad \text{and} \qquad b_{m,n} = (u_{m,n} - c_2)^2$$

Algorithm 1 Example for Chan-Vese model

Entry: u, μ_1, μ_2

Output: Approximate minimizer described by: φ^*

Initialize φ

While Not converged **Do**

Compute the optimal c_1 and $c_2 \in \mathbb{R}$

Compute the images a and b

Compute $d = \mu_1 \nabla E_1(\varphi, c_1) + \mu_2 \nabla E_2(\varphi, c_2) + \nabla E_3(\varphi)$

Compute a step-size $t \geq 0$

Update : $\varphi \leftarrow \varphi - t\, d$

End while

To go further

- Graph-cut technique provides an exact optimal solution (See "What energy function can be minimized via graph-cut", Kolmogorov, Zabih)
- "Fast Marching" algorithm are very fast.