Mathematical methods for Image Processing

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Plan

Introduction to image processing

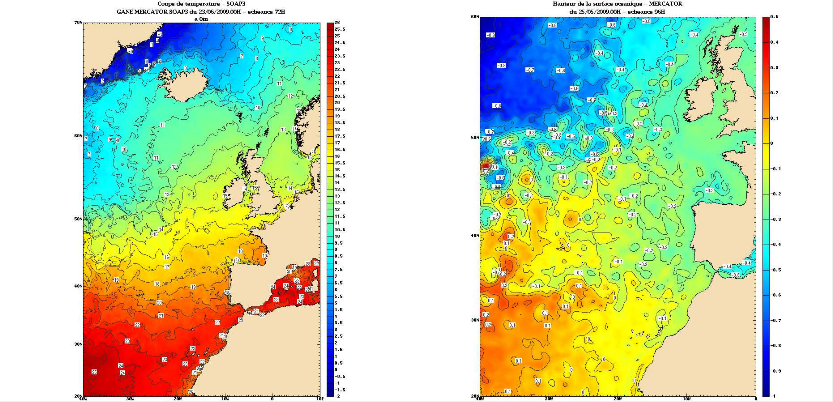
Applications:

- Pictures and movies
- medical imaging (CT, TEP, MRI. . .)
- Biological image (microscopy)
- Earth observation (security, climat. . .)
- Surveillance, safety (problem detection. . .)
- Astronomy, astrophysics

Tools:

- Compression
- Restoration
- Segmentation
- Registration
- Indexation
- Editing

Taylored Image compression



Surface temperature and surface elevation

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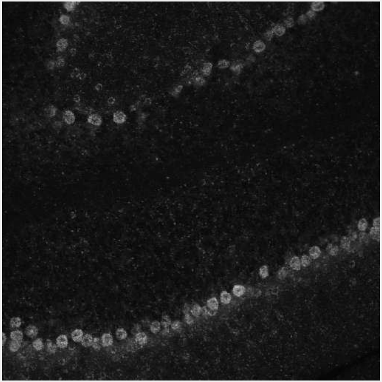
Taylored Image compression and visualisation

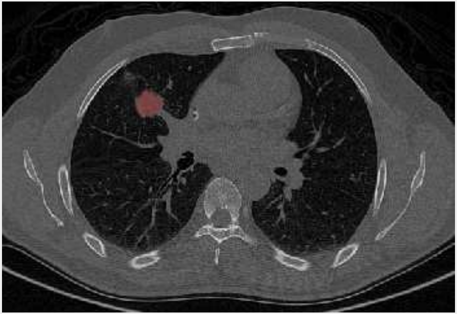


Cell nuclei in a mouse cerebellum

Image segmentation

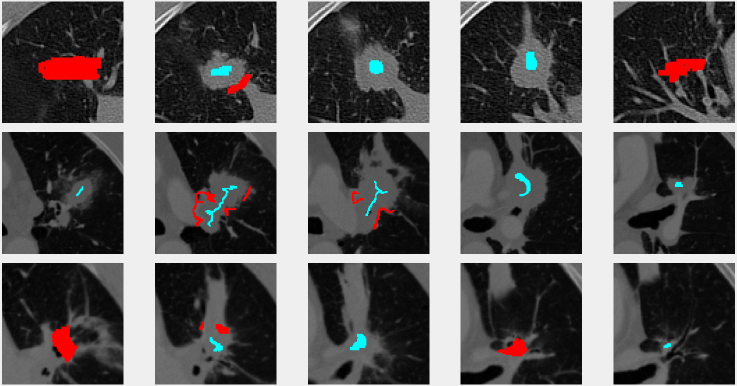
Original image and seed points; Computed segmentation

Image segmentation



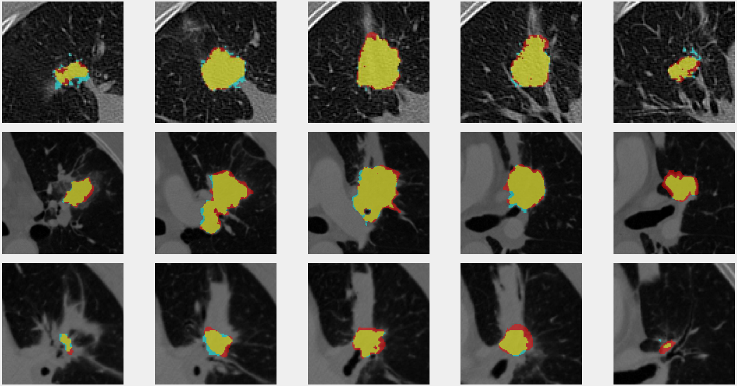
Segmentation of a lung tumor

Image segmentation



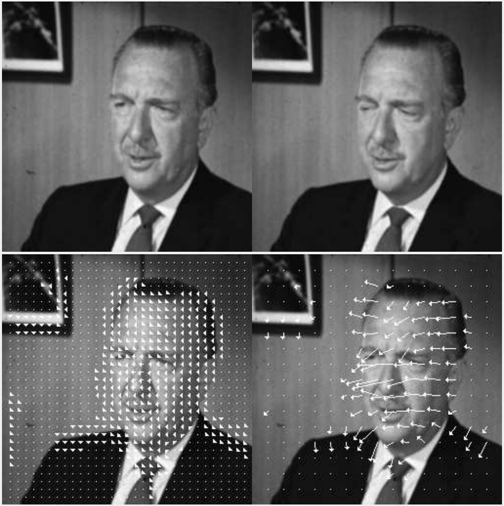
Seeds for segmenting lung tumors (lines contain several slice of 3D CT image).

Image segmentation



Segmentation: Correct (yellow), expert only (red), system only (blue)

Image registration



Registration of consecutive images in a movie

Image restoration : denoising

Top: noisy images; Bottom: denoised images .

Image restoration : deblurring

Top: blurred image; Bottom: restored and ideal image.

Image restoration : inpainting

Top: Image with missing pixels; Bottom: restored and ideal image

Image restoration : zooming

Un-zoomed; zoomed (x4) by two methods

Restoration of compressed images

Top: compressed images (for differents compression level). Bottom: restored images

Introduction to optimization

Let W be a Euclidean space (usually $W = \mathbb{R}^{N\times N}$), we denote

- $\langle w, w' \rangle$: the inner product between w and $w' \in W$
- $\|w\|$: the norm of w

Let

$$E : W \longrightarrow \mathbb{R}$$

we want to find w^* such that

$$E(w^*) \leq E(w) \qquad \text{, for all } w \in W.$$

We write: $w^* \in \text{Argmin}_{w\in W} \; E(w)$.

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Introduction to optimization: Examples

Below ∇ is a finite difference operator and T is a "sparsifying transform"

- H^1 regularization

$$E(w) = \sum_{m,n=0}^{N-1} |\nabla w_{m,n}|^2 + \lambda \|w - u\|^2$$

- Total variation regularization

$$E(w) = \sum_{m,n=0}^{N-1} |\nabla w_{m,n}| + \lambda \|w - u\|^2$$

- ℓ^1 minimisation

$$E(w) = \|w\|_1 + \lambda \|Tw - u\|^2$$

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Introduction to optimization

The design of

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is crucial. We want:

- Guarantee that w^* is close to the targeted ideal image
  - statistics, compressed sensing

Introduction to optimization

The design of

$$E : W \longrightarrow \mathbb{R}$$

is crucial. We want:

- Guarantee that w^* is close to the targeted ideal image
  - statistics, compressed sensing
- w^* exists and can be numericaly approximated in a reasonable amount of time
  - Mathematical optimization

Introduction to optimization: basic properties of functions

Definition

Let $E : W \longrightarrow \mathbb{R}$

- E *is* **proper** *iif*
  - $\forall w \in W$, *we have* $E(w) > -\infty$
  - *there exists* $w \in W$ *such that* $E(w) < +\infty$
- E *is* **lower-semicontinuous** *iif* $\forall w \in W, \forall \varepsilon > 0$ *there is a neighborhood of* w *such that* $\forall w' \in U, E(w') \geq E(w) - \varepsilon$
- E *is* **coercive** *iif* $\lim_{\|w\| \to +\infty} E(w) = +\infty$

Introduction to optimization: basic properties of functions

Definition

*Let $t \in \mathbb{R}$, we call t-**levelset** of E:*

$$\mathcal{L}_E(t) = \{w \in W | E(w) \leq t\}$$

Proposition

Let $E : W \longrightarrow \mathbb{R}$

- *If E is proper, lower-semicontinuous and coercive, then*

$$\text{for every } t \in \mathbb{R}, \mathcal{L}_E(t) \text{ is compact}$$

- *If E is proper, lower-semicontinuous and coercive then*

$$\text{Argmin}_{w \in W} \; E(w) \neq \emptyset.$$

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Introduction to optimization: basic properties of functions

Definition

We say that $C \subset W$ is convex iif

$$tw + (1-t)w' \in C \qquad , \forall w, w' \in C, \forall t \in [0,1]$$

Definition

Let $E : W \longrightarrow \mathbb{R}$

- *E is* **convex** *iif*

$$E(tw + (1-t)w') \leq tE(w) + (1-t)E(w') \qquad , \forall w, w' \in W, \forall t \in [0,1]$$

- *E is* **strictly convex** *iif*

$$E(tw + (1-t)w') < tE(w) + (1-t)E(w') \qquad , \forall w \neq w' \in W, \forall t \in (0,1)$$

- *If E is C^1, E is* **strongly convex** *(also called elliptic) of modulus $\alpha > 0$ iif*

$$\forall w, w' \in W, \qquad \langle \nabla E(w') - \nabla E(w), w' - w \rangle \geq \alpha \|w' - w\|^2$$

Introduction to optimization: basic properties of functions

Proposition

- *If E is convex then,*

$$\text{for all } t \in \mathbb{R}, \mathcal{L}_E(t) \text{ is convex.}$$

- *If E is convex then*

$$\text{Argmin}_{w \in W} \ E(w) \text{ is convex.}$$

- *If E is strictly convex and $\text{Argmin}_{w \in W} \ E(w) \neq \emptyset$ then*

$$\text{Argmin}_{w \in W} \ E(w) \text{ is reduced to a unique } w^*.$$

We write $w^* = \text{Argmin}_{w \in W} \ E(w)$.

Introduction to optimization: basic properties of functions

Proposition

If E is convex, then E is continuous on the interior of

$$\mathrm{Dom}\,(E) = \{w \in W | E(w) < +\infty\}.$$

Proposition

If E is C^1,

- *E is convex iif*

$$\forall w, w' \in W, \qquad E(w') \geq E(w) + \langle \nabla E(w), w' - w \rangle$$

- *E is strongly convex of modulus $\alpha > 0$ iif*

$$\forall w, w' \in W, \qquad E(w') \geq E(w) + \langle \nabla E(w), w' - w \rangle + \frac{\alpha}{2} \|w' - w\|^2$$

- *If E is strongly convex then E is strictly convex and coercive.*

Introduction to optimization: basic properties of functions

Definition

Let E be convex. For any $w \in W$ we call sub-gradient of E at w

$$\partial E(w) = \{g \in W | \forall w' \in W, E(w') \geq E(w) + \langle g, w' - w \rangle\}$$

Proposition

Let E be convex.

- *If E is C^1 then*

$$\partial E(w) = \{\nabla E(w)\}.$$

- $w^* \in \text{Argmin}_{w \in W} \; E(w)$ *iif* $0 \in \partial E(w^*)$.

Introduction to optimization: basic properties of functions

Definition

Let $L > 0$, we say that E has **Lipschitz gradient** *of parameter L iif*

$$\forall w, w' \in W, \qquad \|\nabla E(w') - \nabla E(w)\| \leq L\|w' - w\|$$

Proposition

If E is C^2 (its Hessian is denoted $\nabla^2 E(w)$):

- *E is strongly convex of modulus $\alpha > 0$ iif the smallest eigenvalue of $\nabla^2 E(w)$ is larger than α, for all $w \in W$.*
- *E has a L-Lipschitz gradient iif the largest eigenvalue of $\nabla^2 E(w)$ is smaller than L, for all $w \in W$.*

Introduction to optimization: To go further

- "Introductory lectures on convex optimization: A basic course", Yurii Nesterov
- "Convex Analysis", Ralph T. Rockafellar
- "Non linear programming", Dimitri Bertzekas