Uniqueness results by partial Cauchy data on arbitrary subboundary for 2-dimensional elliptic equations

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Contents

Our Main Achievements

Global Uniqueness by DN map on arbitrary subboundary or disjoint input and output subbnoundaries

Our results: the best possible among the existing results within smooth coefficients

1. Introduction

$\Omega \subset \mathbb{R}^n$: bounded domain

$$\begin{cases} \text{div}(\gamma \nabla u) & = & 0 \text{ in } \Omega \\ u|_{\partial\Omega} & = & f \end{cases}$$

$$\Lambda_\gamma : f \in H^{\frac{1}{2}}(\partial\Omega) \longrightarrow \gamma \frac{\partial u}{\partial \nu}|_{\partial\Omega} \in H^{-\frac{1}{2}}(\partial\Omega)$$

Electrical Impedance Tomography:

Find γ from Λ_γ

Reduction to potential determination

Electrical Impedance Tomography: $\mathrm{div}(\gamma\nabla u) = 0$

$\Longrightarrow$

Determination of potential in

$$\Delta w + qw = 0, \quad q = -\frac{\Delta\sqrt{\gamma}}{\sqrt{\gamma}}, w = \sqrt{\gamma}u$$

Works on uniqueness for whole Cauchy data

Incomplete references

spatial dimension $n \geq 3$:

- J. Sylvester - G. Uhlmann (1987): $\gamma \in C^2$
- L. Päivärinta - A. Panchenko - G. Uhlmann (2003): Lipschitz continuous γ

Uniqueness: whole Cauchy data

$n = 2$:

- A. Nachman (1996): $\gamma \in C^2$
- R. Brown - G. Uhlmann (1997): less regular γ
- K. Astala - L. Päivärinta (2006): $\gamma \in L^\infty$
- A. Bukhgeim (2008)

Partial Cauchy data

$$\begin{cases} \text{div}(\gamma\nabla u) &= 0 \text{ in } \Omega \\ u|_{\partial\Omega} &= f \end{cases}$$

$\Gamma, \widetilde{\Gamma} \subset \partial\Omega$: subboundary

$$\widetilde{\Lambda}_\gamma : \{f \in H^{\frac{1}{2}}(\partial\Omega) | \, \text{supp} f \subset \Gamma\} \longrightarrow \gamma \frac{\partial u}{\partial \nu}|_{\widetilde{\Gamma}}$$

uniqueness by partial Cauchy data

Γ: input subboundary, $\widetilde{\Gamma}$: output subboundary

$n \geq 3$:

- V.Isakov (2006): $\Gamma, \widetilde{\Gamma} \subset$ plane or sphere
- A. Bukhgeim - G. Uhlmann (2002):
  $\Gamma = \partial\Omega$, $\widetilde{\Gamma} \supset\supset$ half of $\partial\Omega$, $\gamma \in C^2$
- K. Knudsen (2006): $\gamma \in C^{\alpha+\frac{3}{2}}$, $\alpha > 0$
- H. Heck - J.-N. Wang (2006): stability
- C.Dos Santos Ferreira - C.Kenig - J.Sjöstrand -
  G. Uhlmann (2007): magnetic Schrödinger

- C. Kenig - J. Sjöstrand - G. Uhlmann (2007):

  $\Gamma \subset \partial\Omega$: arbitrary,

  $\widetilde{\Gamma} \supset\supset \partial\Omega \setminus \Gamma$

Open Problem

Open problem:

How is the uniqueness for Cauchy data on arbitrary subboundary Γ?

Subboundaries for Dirichlet inputs and Neumann outputs should be as small as possible.

Our Result $\Rightarrow$ One Answer

2. Determination of potentials

Joint work with

O. Imanuvilov (Colorado State University)

G. Uhlmann (University of California Irvine)

Ref: J. Amer. Math. Soc. (2010)

$\widetilde{\Gamma} \subset \partial\Omega$: arbitrary,

$q_j \in C^{2+\alpha}(\overline{\Omega})$ with $\alpha > 0$, complex-valued

$$C_{q_j} = \left\{ \left(u|_{\widetilde{\Gamma}}, \frac{\partial u}{\partial \nu}|_{\widetilde{\Gamma}} \right) \,\middle|\, (\Delta + q_j)u = 0 \text{ in } \Omega,\ u|_{\partial\Omega \setminus \widetilde{\Gamma}} = 0, \right.$$

$$\left. u \in H^1(\Omega) \right\}$$

(if 0 is eigenvalue, C_{q_j} contains the graph)

Theorem 1: $C_{q_1} = C_{q_2} \implies q_1 \equiv q_2$

Main Result II

Theorem 2

Assume: $\gamma_j > 0, \in C^{4+\alpha}(\overline{\Omega})$, $j = 1, 2$ with $\alpha > 0$.

$$\Lambda_{\gamma_1} u = \Lambda_{\gamma_2} u \text{ on } \widetilde{\Gamma} \text{ for all } u \in H^{\frac{1}{2}}(\Gamma), \ \operatorname{supp} u \subset \widetilde{\Gamma}.$$

$$\Longrightarrow \gamma_1 = \gamma_2.$$

Corollary 1

Γ: arbitrary subboundary of $\partial\Omega$ such that

$\Gamma_1 \supset \overline{\partial\Omega \setminus \Gamma}$.

$$\widehat{C}_{q_j} = \{(u|_{\Gamma}, \partial_\nu u|_{\Gamma_1}); \ (\Delta + q_j)u = 0, \quad u|_{\partial\Omega\setminus\Gamma} = 0\}.$$

Then: $\widehat{C}_{q_1} = \widehat{C}_{q_2}$ implies $q_1 \equiv q_2$ in Ω.

Remark on Corollary 1

corresponding 3D result:

C. Kenig, J. Sjöstrand and G. Uhlmann, Ann. of Math. (2007)

For Conductivity equation: Same uniqueness

Corollary 2: in domain with hole

$\Omega, D \in \mathbb{R}^2$: smooth boundary domain such that

$\overline{D} \subset \Omega$.

$V \subset \partial\Omega$: open set.

Let $q_j \in C^{2+\alpha}(\overline{\Omega \setminus D})$ for some $\alpha > 0$, $j = 1, 2$.

$$\widetilde{C}_{q_j} := \{(u|_V, \partial_\nu u|_V); (\Delta + q_j)u = 0 \text{ in } \Omega \setminus \overline{D},$$

$$\text{supp } f \subset V,\ u \in H^1(\Omega \setminus \overline{D})\}$$

Then $\widetilde{C}_{q_1} = \widetilde{C}_{q_2} \Longrightarrow q_1 \equiv q_2$.

Remark

For Conductivity equation: Same uniqueness

Anisotropic conductivity

$\sigma = \{\sigma_{ij}\}_{1\le i,j\le 2}$: positive definite symmetric matrix

$$\sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}\left(\sigma_{ij}\frac{\partial u}{\partial x_j}\right) = 0 \quad \text{in } \Omega, \qquad u|_{\partial\Omega} = g.$$

Dirichlet-to-Neumann map:

$$\Lambda_\sigma(g) = \sum_{i,j=1}^{2} \sigma_{ij}\nu_i \frac{\partial u}{\partial x_j}|_{\partial\Omega}.$$

Uniqueness modulo transforms of x

Theorem 3 $\sigma_k = \{\sigma_{ij}^{(k)}\}_{1\le i,j\le 2} \in C^{7+\alpha}(\overline{\Omega})$ for $k = 1, 2$ with some constant $\alpha > 0$.

$\widetilde{\Gamma} \subset \partial\Omega$: arbitrary relatively open subset.

$\Lambda_{\sigma_1}(g)|_{\widetilde{\Gamma}} = \Lambda_{\sigma_2}(g)|_{\widetilde{\Gamma}}$ for all $g \in H^{\frac{1}{2}}(\partial\Omega)$, supp $g \subset \widetilde{\Gamma}$.

Then $\exists F : \overline{\Omega} \to \overline{\Omega}$: diffeomorphism, $F|_{\widetilde{\Gamma}} = I$,

$F \in C^{7+\alpha}(\overline{\Omega})$, $F_*\sigma_1 = \sigma_2$.

Here $F_*\sigma = \frac{(DF)\cdot\sigma\cdot(DF)^T\cdot F^{-1}}{|detDF|}$, DF: the differential of F.

3. General 2D second-order elliptic systems

Joint with

O. Imanuvilov (Colorado State University)

$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}(\partial_{x_1} + i\partial_{x_2}),\ \frac{\partial}{\partial z} = \frac{1}{2}(\partial_{x_1} - i\partial_{x_2})$

Ω: smooth bounded domain in $\mathcal{R}^2$

$\widetilde{\Gamma} \subset \partial\Omega$: arbitrary

$$L(x, D)u = \Delta u + 2A\partial_z u + 2B\partial_{\bar{z}} u + Qu = 0 \quad \text{in } \Omega$$

Here $u = (u_1, \ldots, u_N)$ and A, B, Q: $N \times N$ matrices

$C_{A,B,Q} =$

$$\left\{ \left(u, \frac{\partial u}{\partial \nu}\right)\Big|_{\tilde{\Gamma}} ;\ L(x, D)u = 0,\ u|_{\partial\Omega\setminus\widetilde{\Gamma}} = 0, \quad u|_{\widetilde{\Gamma}} = f \right\}$$

Theorem 4 Let $A_j, B_j, Q_j \in C^{6+\alpha}(\bar{\Omega})$, $j = 1, 2$. Suppose that $C_{A_1,B_1,Q_1} = C_{A_2,B_2,Q_2}$. Then

$$A_1 = A_2 \quad \text{and } B_1 = B_2 \quad \text{on } \tilde{\Gamma},$$

$$2\partial_z(A_1 - A_2) + B_2(A_1 - A_2) + (B_1 - B_2)A_1 - (Q_1 - Q_2) = 0$$

$$2\partial_z(B_1 - B_2) + A_2(B_1 - B_2) + (A_1 - A_2)B_2 - (Q_1 - Q_2) = 0$$

in Ω.

Corollary

Corollary 4-1 Let $Q_j \in C^{4+\alpha}(\overline{\Omega})$ and $(A_j, B_j) \in C^{5+\alpha}(\overline{\Omega})$. Then $C_{A_1,B_1,Q_1} = C_{A_2,B_2,Q_2}$ implies $A_1 = A_2$, $B_1 = B_2$, $Q_1 = Q_2$ in Ω provided that either $A_1 = A_2$ or $B_1 = B_2$ or $Q_1 = Q_2$ holds.

Partial Cauchy data uniquely determines any two of A, B, Q in

$$\Delta u + 2A\partial_z u + 2B\partial_{\bar{z}} u + Qu$$

Special Case: single equation

Joint with

O. Imanuvilov (Colorado State University)

G. Uhlmann (University of California Irvine)

Corollary 3

Let Ω: simply connected.

$C_{(A_1,B_1,q_1)} = C_{(A_2,B_2,q_2)} \Longleftrightarrow$

$\exists \eta \in C^{5+\alpha}(\overline{\Omega})$ such that $\eta = \frac{\partial \eta}{\partial \nu} = 0$ on $\widetilde{\Gamma}$,

$L_1(x, D) = e^{-\eta} L_2(x, D) e^{\eta}$.

Here $L_j(x, D) = \Delta + 2A_j \frac{\partial}{\partial z} + 2B_j \frac{\partial}{\partial \overline{z}} + q_j$

Magnetic Schrödinger equation

Let $\widetilde{A} = (\widetilde{A}_1, \widetilde{A}_2)$, $\widetilde{A}^{(j)} = (\widetilde{A}_1^{(j)}, \widetilde{A}_2^{(j)})$.

$$\mathcal{L}_{\widetilde{A},\widetilde{q}} = \sum_{k=1}^{2} \left(\frac{1}{i} \frac{\partial}{\partial x_k} + \widetilde{A}_k \right)^2 + \widetilde{q}.$$

Set of partial Cauchy data

$$\widetilde{C}_{\widetilde{A}^{(j)},\widetilde{q}^{(j)}} = \left\{ (u|_{\widetilde{\Gamma}}, \frac{\partial u}{\partial \nu}|_{\widetilde{\Gamma}}) | \mathcal{L}_{\widetilde{A}^{(j)},\widetilde{q}^{(j)}}(x, D)u = 0 \text{ in } \Omega, \right.$$

$$\left. u|_{\partial\Omega\setminus\widetilde{\Gamma}} = 0, u \in H^1(\Omega) \right\}.$$

Corollary 4

real-valued vector fields $\widetilde{A}^{(1)}, \widetilde{A}^{(2)} \in C^{5+\alpha}(\overline{\Omega})$

complex-valued potentials $\widetilde{q}^{(1)}, \widetilde{q}^{(2)} \in C^{4+\alpha}(\overline{\Omega})$.

Then

$$\widetilde{C}_{\widetilde{A}^{(1)},\widetilde{q}^{(1)}} = \widetilde{C}_{\widetilde{A}^{(2)},\widetilde{q}^{(2)}}$$

$$\Longrightarrow \widetilde{q}^{(1)} = \widetilde{q}^{(2)}, \quad \mathrm{rot}\,\widetilde{A}^{(1)} = \mathrm{rot}\,\widetilde{A}^{(2)}$$

Here: $\widetilde{A} = (\widetilde{A_1}, \widetilde{A_2})$, $\mathrm{rot}\,\widetilde{A} = \frac{\partial \widetilde{A}_2}{\partial x_1} - \frac{\partial \widetilde{A}_1}{\partial x_2}$

References on uniqueness

2D case

- Z.Sun (1993): with full Cauchy data for small coefficient
- H.Kang-G.Uhlmann (2004): with full Cauchy data for special magnetic Schrödinger equations

3D case

- G. Nakamura-Z.Sun-G. Uhlmann (1995):
  full Cauchy data
- D. Dos Santos Ferreira-C. Kenig
  -J.Sjöstrand-G.Uhlmann (2007):
  partial Cauchy data

Diffusion-convection equation

partial Cauchy data

$$\widetilde{C}_{a^{(j)},b^{(j)}} = \left\{ \left(u|_{\widetilde{\Gamma}}, \frac{\partial u}{\partial \nu}|_{\widetilde{\Gamma}}\right) \middle| \Delta u + a^{(j)}(x)\frac{\partial u}{\partial x_1} + b^{(j)}(x)\frac{\partial u}{\partial x_2} \right.$$

$$\left. + q(x)u = 0 \text{ in } \Omega,\ u|_{\partial\Omega\setminus\widetilde{\Gamma}} = 0, u \in H^1(\Omega) \right\}.$$

Corollary 5. Let $\alpha > 0$, q given.

complex-valued $(a^{(j)}, b^{(j)}) \in C^{5+\alpha}(\overline{\Omega}) \times C^{5+\alpha}(\overline{\Omega})$

Then $\widetilde{C}_{a^{(1)},b^{(1)}} = \widetilde{C}_{a^{(2)},b^{(2)}} \Longrightarrow (a^{(1)}, b^{(1)}) \equiv (a^{(2)}, b^{(2)})$

References on uniqueness:

2D case: J. Cheng-M. Yamamoto (2004): full Cauhcy data

3D case: J. Cheng-G. Nakamura-E. Somersalo (2001): full data

General Second Order Operator

$$P(x,D)u = \Delta_g + A\frac{\partial u}{\partial z} + B\frac{\partial u}{\partial \bar{z}} + q(x)u = 0.$$

Here Δ_g denotes the Laplace-Beltrami operator associated to the Riemannian metric $g \in C^{7+\alpha}(\overline{\Omega})$.

Let: g be positive definite in Ω and

$$\Delta_g = \frac{1}{\sqrt{\det g}}\sum_{j,k=1}^{2}\frac{\partial}{\partial x_k}\left(\sqrt{\det g}\, g^{jk}\frac{\partial u}{\partial x_j}\right),$$

where $g^{ij} = g^{-1}$.

Assume that $g \in C^{7+\alpha}(\bar{\Omega})$, $(A, B, q), (A_j, B_j, q_j) \in C^{5+\alpha}(\overline{\Omega}) \times C^{5+\alpha}(\overline{\Omega}) \times C^{4+\alpha}(\overline{\Omega})$, $j = 1, 2$ for some $\alpha > 0$ are complex-valued functions.

Let $\widetilde{\Gamma} \subset \partial\Omega$ be a fixed non-empty open subset.

We define the set of Cauchy data

$$C_{g,A,B,q} = \left\{ \left(u|_{\tilde{\Gamma}}, \frac{\partial u}{\partial \nu}\bigg|_{\tilde{\Gamma}} \right) \mid \left(\Delta_g + 2A\frac{\partial}{\partial z} + 2B\frac{\partial}{\partial \bar{z}} + q\right)u = 0 \right.$$

$$\left. \text{in } \Omega, \quad u \in H^1(\Omega), u|_{\partial\Omega \setminus \widetilde{\Gamma}} = 0 \right\}$$

Uniqueness for single general equation

Theorem 5

Let Ω be a simply connected domain.

Then $C_{(g,A,B,q)} = \widetilde{C}_{(\tilde{g},\tilde{A},\tilde{B},\tilde{q})} \Longleftrightarrow$

P can be obtained from $\widetilde{P}$ by change of varables

$F : \Omega \to \Omega$ which is a diffeomorphism such that

$F|_{\widetilde{\Gamma}} = Identity$ and the gauge transformation

$\widetilde{P} \to e^{-\eta}\widetilde{P}e^{\eta}$ where $\eta \in C^{6+\alpha}(\overline{\Omega})$, $\eta|_{\widetilde{\Gamma}} = \frac{\partial\eta}{\partial\nu}|_{\widetilde{\Gamma}} = 0.$

4. Disjoint input and output subboundaries

Joint with

O. Imanuvilov (Colorado State University)

G. Uhlmann (University of California Irvine)

Ref: Inverse Problems 2011

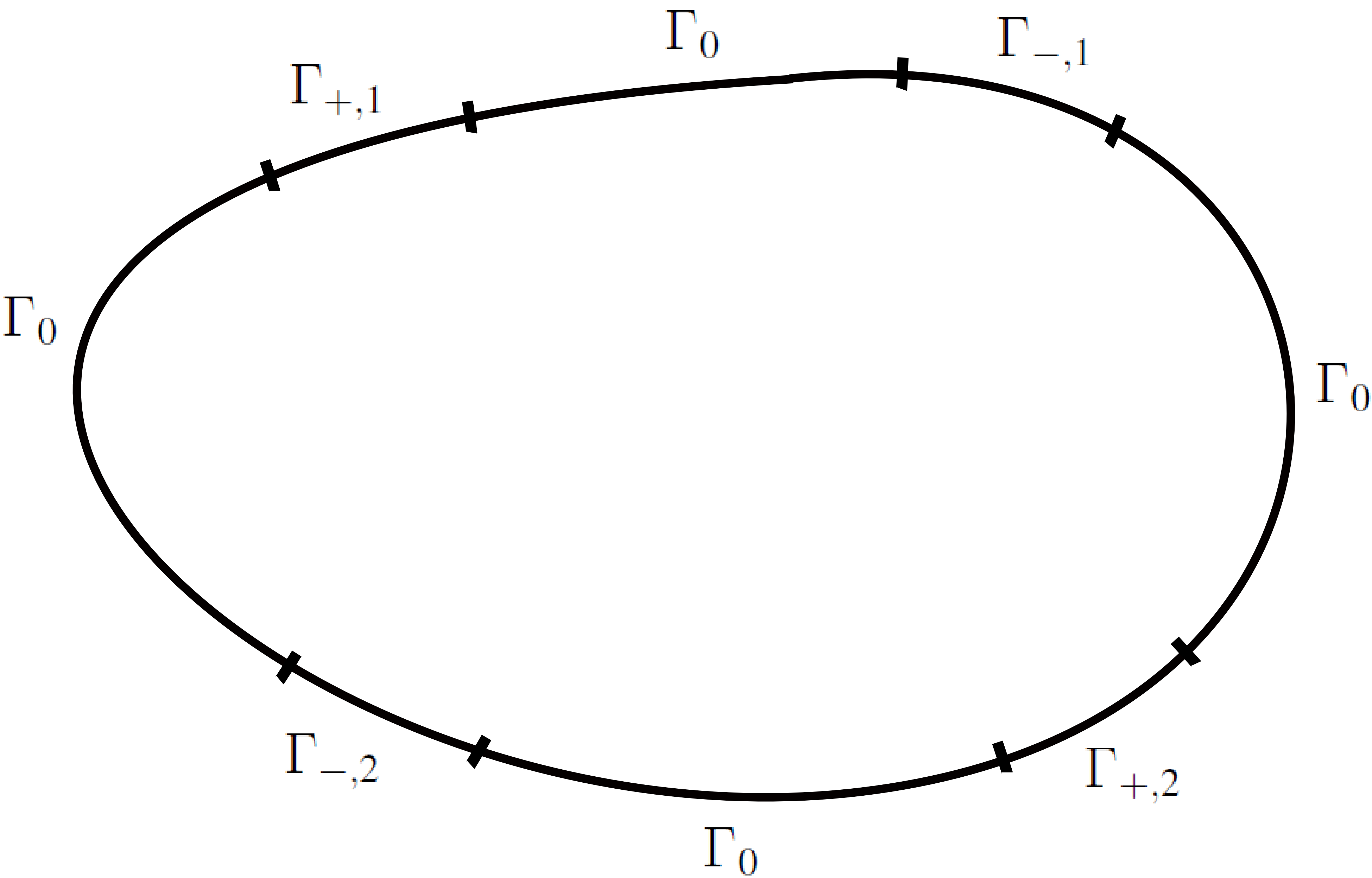
$$C_q = \{(u|_{\Gamma_+}, \partial_\nu u|_{\Gamma_-});$$
$$\Delta u + q(x)u = 0, u|_{\partial\Omega\setminus\Gamma_+} = 0\}$$

Theorem 6

Let $\Gamma_+, \Gamma_-, \Gamma_0 \subset \partial\Omega \neq \emptyset$ be relatively open s.t.

- $\partial\Omega = \overline{\Gamma_+ \cup \Gamma_- \cup \Gamma_0}$
- $\Gamma_+ \cap \Gamma_- = \Gamma_+ \cap \Gamma_0 = \Gamma_- \cap \Gamma_0 = \emptyset$
- $\Gamma_+ = \cup_{j=1}^{2} \Gamma_{+,j}$, $\Gamma_- = \cup_{j=1}^{2} \Gamma_{-,j}$, $\Gamma_0 = \cup_{k=1}^{4} \Gamma_{0,k}$
- $\Gamma_{\pm,j}, \Gamma_{0,k}, \neq \emptyset$, are relatively open connected subsets of $\partial\Omega$ and mutually disjoint.
- $\Gamma_{-,1}, \Gamma_{0,2}, \Gamma_{+,1}, \Gamma_{0,3}, \Gamma_{-,2}, \Gamma_{0,4}, \Gamma_{+,2}$ in clockwise

Then $C_{q_1}(\Gamma, \widetilde{\Gamma}) = C_{q_2}(\Gamma, \widetilde{\Gamma})$ for $q_1, q_2 \in C^{2+\alpha}(\overline{\Omega})$ implies $q_1 = q_2$.



Γ_0
$\Gamma_{+,1}$
$\Gamma_{-,1}$
Γ_0
Γ_0
$\Gamma_{-,2}$
$\Gamma_{+,2}$
Γ_0

5. Less regular potentials

Joint with

O. Imanuvilov (Colorado State University)

SIAM Math. Anal.

Schrödinger equation with potential q:

$$(\Delta + q)u = 0 \quad \text{in } \Omega.$$

Let $\widetilde{\Gamma} \subset \partial\Omega$: arbitrary relatively open subset,

$$C_q = \left\{ \left(u, \frac{\partial u}{\partial \nu} \right) |_{\widetilde{\Gamma}};\ (\Delta + q)u = 0, \right.$$

$$\left. u|_{\partial\Omega \setminus \widetilde{\Gamma}} = 0, u|_{\widetilde{\Gamma}} = f \right\}.$$

Theorem 7

Let $q_1, q_2 \in C^{\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1)$ if $\widetilde{\Gamma} = \partial\Omega$

Let $q_1, q_2 \in W_p^1(\Omega)$ for some $p > 2$ if $\widetilde{\Gamma}$ is arbitrary.

Then $C_{q_1} = C_{q_2}$ implies $q_1 = q_2$

References in 2D case

$\widetilde{\Gamma}$; arbitrary

- Imanuvilov, Uhlmann and Yamamoto (2010): for $q \in C^{2+\alpha}(\overline{\Omega})$
- Guillarmou and Tzou (2011): for $C^{2+\alpha}(\overline{\Omega})$.

$\widetilde{\Gamma} = \partial\Omega$

- Astala and Päivärinta (2006): L^{∞}-conductivity
- Blasten (2011): potential in piecewise $W_p^1(\Omega)$, $p > 2$
- Bukhgeim (2008)

6. Lamé equations

Joint with

O. Imanuvilov (Colorado State University)

Ref: J. Inverse and Ill-posed Problems (2011)

Isotropic Lamé system:

$$\mathcal{L}_{\lambda,\mu}u \equiv \left(\sum_{j,k,l=1}^{2} \frac{\partial}{\partial x_j}\left(C_{1jkl}\frac{\partial u_k}{\partial x_l}\right), \sum_{j,k,l=1}^{2} \frac{\partial}{\partial x_j}\left(C_{2jkl}\frac{\partial u_k}{\partial x_l}\right)\right)$$

in 2D domain Ω. Here:

$$C_{ijkl} = \lambda(x)\delta_{ij}\delta_{kl} + \mu(x)(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}),$$

$1 \leq i, j, k, l \leq 2$, δ_{ij}: Kronecker delta.

Assume that

$$\mu > 0,\ \lambda + \mu > 0 \quad \text{on } \overline{\Omega}$$

Set:

$$\Lambda_{\lambda,\mu}(f) = \left(\sum_{j,k,l=1}^{2} \nu_j C_{1jkl} \frac{\partial u_k}{\partial x_l}, \sum_{j,k,l=1}^{2} \nu_j C_{2jkl} \frac{\partial u_k}{\partial x_l} \right),$$

with $\nu = (\nu_1, \nu_2)$: the outward unit normal vector to $\partial\Omega$.

Partial Cauchy data:

$$C_{\lambda,\mu} = \{(u, \Lambda_{\lambda,\mu}(f))|_{\tilde{\Gamma}};\ \mathcal{L}_{\lambda,\mu} u = 0 \quad \text{in } \Omega,$$
$$u|_{\partial\Omega} = f,\ \text{supp}\, f \subset \widetilde{\Gamma}\}$$

Theorem 8

Let $\boldsymbol{\Omega}$: simply connected smooth domain

$\boldsymbol{\mu_1, \mu_2}$ be positive constants and $\boldsymbol{\lambda_1, \lambda_2 \in C^3(\overline{\Omega})}$.

Then $\boldsymbol{C_{\lambda_1,\mu_1} = C_{\lambda_2,\mu_2}}$ implies $\boldsymbol{(\lambda_1, \mu_1) = (\lambda_2, \mu_2)}$

References: incomplete

- Akamatsu-Nakamura-Steinberg (1991):
  2D, uniqueness of derivatives on $\partial\Omega$
- Nakamura-Uhlmann (1993):
  2D, uniqueness of λ and μ by full Cauchy data
  near constants
- Eskin and Ralston (2002): 3D, uniqueness of
  λ, μ by full Cauchy data near constant μ

7. Nonlinear elliptic equations

Joint with

O. Imanuvilov (Colorado State University)

$$P(x,D)u := -\Delta u + q(x)u + f(x,u) = 0 \text{ in } \Omega$$

Partial Cauchy data:

$$C_{q,f} = \{(u, \partial_\nu u)|_{\tilde{\Gamma}};\ -\Delta u + qu + f(x,u) = 0 \text{ in } \Omega,$$
$$u|_{\partial\Omega \setminus \tilde{\Gamma}} = 0,\ \ u \in H^1(\Omega)\}$$

Assumptions on nonlinear terms:

- $f \in C^2(\overline{\Omega} \times \mathbb{R}),\ f(x,0) \equiv f_y(x,0) \equiv 0$
- $\exists p > 1, C_1, C_2, \delta$ such that

  $|f(x,y)| \leq C_1|y|^p,\ (x,y) \in \Omega \times [-\delta, \delta]$

  $f(x,y)y \geq C_1|y|^{p+1} - C_2,\ (x,y) \in \Omega \times \mathbb{R}$
- $\exists p_1 > 0, p_2 > 0, C_1 > 0$ such that

$$|f_y(x,y)| \leq C_1(1 + |y|^{p_1}),$$

$$|f_{yy}(x,y)| \leq C_1(1 + |y|^{p_2}),\ (x,y) \in \Omega \times \mathbb{R}$$

Theorem 9 (uniqueness of potentials)

Let $q_1, q_2 \in C^{2+\alpha}(\overline{\Omega})$.

Then $C_{q_1,f_1} = C_{q_2,f_2}$ implies

$q_1 = q_2$ in Ω.

Theorem 10 (uniqueness of nonlinear terms)

Let $\widetilde{C}_{0,f_1} = \widetilde{C}_{0,f_2}$. Then $(f_1 - f_2)|_{\mathcal{O}} = 0$

where

$$\mathcal{O} = \{(x, u(x,t));\ 0 \le t \le 1,\ u(x,t) \in H^2(\Omega),\ \ t \in [0,1],$$

$$u(x,0) \equiv 0,$$

$$t \to u(\cdot,t) : [0,1] \to H^2(\Omega) \text{ is continuous},$$

$$-\Delta u(\cdot,t) + f_1(x,u) = 0 \text{ in } \Omega,$$

$$u(\cdot,t)|_{\partial\Omega\setminus\widetilde{\Gamma}} = 0, \quad 0 \le t \le 1\}$$

8. Navier-Stokes equations

Joint with

O. Imanuvilov (Colorado State University)

$$P_\mu(u,p) \equiv \Big(\sum_{j=1}^{2}(-2\partial_j(\mu(x)\epsilon_{1j}(u)) + u_j\partial_j u_1 + \partial_1 p,$$

$$\sum_{j=1}^{2}(-2\partial_j(\mu(x)\epsilon_{2j}(u)) + u_j\partial_j u_2 + \partial_2 p\Big)$$

Here $\epsilon_{ij}(u) = \frac{1}{2}(\partial_j u_i + \partial_i u_j)$,

$\mu \in C^4(\overline{\Omega})$, $\mu > 0$ on $\overline{\Omega}$

Fulll Cauchy data

$$C_\mu = \{(u, \partial_\nu u, p)|_{\partial\Omega};\ P_\mu(x, D)u = 0 \quad \text{in } \Omega,$$
$$\text{div}\, u = 0,\ u \in H^2(\Omega)\}$$

Theorem 11

If $C_{\mu_1} = C_{\mu_2}$, then $\mu_1 = \mu_2$ in Ω.

9. Key to Proof

- $u_1 = u_1(\tau)$: τ - depending solution to
  $\Delta u_1 + q_1 u_1 = 0, \quad u_1|_{\partial\Omega\setminus\widetilde{\Gamma}} = 0$
- $u_2 : \Delta u_2 + q_2 u_2 = 0, \quad u_2|_{\partial\Omega} = u_1|_{\partial\Omega}$
  DN maps are equal $\Rightarrow \nabla u_1 = \nabla u_2$ on $\widetilde{\Gamma}$

$$u = u_1 - u_2 \Longrightarrow \Delta u + q_2 u = -(q_1 - q_2)u_1$$

$$u|_{\partial\Omega} = 0, \quad \frac{\partial u}{\partial \nu}|_{\widetilde{\Gamma}} = 0$$

- $v = v(\tau)$: sol. $\Delta v + q_2 v = 0$, $v|_{\partial\Omega\setminus\widetilde{\Gamma}} = 0$

$$0 = \int_\Omega v(\Delta u + q_2 u)dx = -\int_\Omega (q_1 - q_2) v u_1 dx$$

$\Longrightarrow \int_\Omega (q_1 - q_2)(x) v(\tau)(x) u_1(\tau)(x) dx = 0$ for all $\tau > 0$

$\Longrightarrow q_1 = q_2$?

How to choose $u_1(\tau)$, $v(\tau)$? $\Leftarrow$

complex geometrical optics solutions $\rightarrow$

which choices?

Key to proof

complex geometric optics solution by

Carleman estimate with suitable weight

- C. Kenig - J. Sjöstrand - G. Uhlmann (2007)
- O. Imanuvilov - G. Uhlmann - M. Yamamoto (2008)

Preliminaries

$i = \sqrt{-1}$, $z = x_1 + ix_2$, $z \leftrightarrow x = (x_1, x_2)$,

$\partial_z = \frac{1}{2}(\partial_{x_1} - i\partial_{x_2})$, $\partial_{\bar{z}} = \frac{1}{2}(\partial_{x_1} + i\partial_{x_2})$,

$\Phi(z) = \varphi(z) + i\psi(z) \in C^2(\overline{\Omega})$: holomorphic in Ω, Im $\Phi|_{\partial\Omega\setminus\widetilde{\Gamma}} = 0$ ($\Rightarrow \nabla\varphi \cdot \nu = 0$ on $\partial\Omega \setminus \widetilde{\Gamma}$)

$$\mathcal{H} = \{z \in \overline{\Omega} | \partial_z \Phi(z) = 0\}$$

Assume: $\mathcal{H} \cap \partial\Omega \subset \partial\Omega \setminus \overline{\widetilde{\Gamma}}$, $\partial_z^2 \Phi(z) \neq 0$ $(z \in \mathcal{H})$

$(\Longrightarrow \mathcal{H} = \{\widetilde{x}_1, \ldots, \widetilde{x}_\ell\})$

First Step: Carleman estimate

Proposition 1

$u \in H_0^1(\Omega)$: real valued. Then for all large $|\tau|$:

$$
|\tau| \, \| u e^{\tau\varphi} \|^2_{L^2(\Omega)} + \| u e^{\tau\varphi} \|^2_{H^1(\Omega)} + \left\| \frac{\partial u}{\partial \nu} e^{\tau\varphi} \right\|^2_{L^2(\partial\Omega \setminus \widetilde{\Gamma})}
$$

$$
+ \tau^2 \left\| \left| \frac{\partial \Phi}{\partial z} \right| u e^{\tau\varphi} \right\|^2_{L^2(\Omega)}
$$

$$
\le C \left(\| (\Delta u) e^{s\varphi} \|^2_{L^2(\Omega)} + |\tau| \int_{\widetilde{\Gamma}} \left| \frac{\partial u}{\partial \nu} \right|^2 e^{2\tau\varphi} d\sigma \right)
$$

Carleman estimate

Proposition 1 is:

- Carleman estimate with degenerate weight
- essential for constructing complex geometrical optics solutions
- by Imanuvilov

Application of Carleman estimate

$\Longrightarrow$ Existence of τ-depending solutions to Schrödinger equation with bounds:

$$\Delta u + qu = f \text{ in } \Omega, \quad u|_{\partial\Omega\setminus\widetilde{\Gamma}} = g$$

Proposition 2 Let $\partial\Omega \setminus \widetilde{\Gamma} \subset \{x \in \partial\Omega;\ (\nu \cdot \nabla\varphi) = 0\}$.

$\exists\ \tau_0 > 0$ such that for all $|\tau| > \tau_0$ there exists a solution such that

$$\|ue^{-\tau\varphi}\|_{L^2(\Omega)} \le C\left(\frac{\|fe^{-\tau\varphi}\|_{L^2(\Omega)}}{|\tau|^{\frac{1}{2}}} + \|ge^{-\tau\varphi}\|_{L^2(\partial\Omega\setminus\widetilde{\Gamma})}\right)$$

Second Step: construction of complex geometrical optics solutions

$$\partial_{\bar{z}}^{-1} g(z) = -\frac{1}{\pi}\int_{\Omega} \frac{g(\xi_1, \xi_2)}{\xi_1 + i\xi_2 - z} d\xi_1 d\xi_2, \ \partial_z^{-1} g = \overline{\partial_{\bar{z}}^{-1} \overline{g}}$$

$$R_{\Phi,\tau} g := e^{\tau(\overline{\Phi(z)} - \Phi(z))} \partial_{\bar{z}}^{-1}(g e^{\tau(\Phi(z) - \overline{\Phi(z)})})$$

$$\widetilde{R}_{\Phi,\tau} g := e^{\tau(\overline{\Phi(z)} - \Phi(z))} \partial_z^{-1}(g e^{\tau(\Phi(z) - \overline{\Phi(z)})})$$

Complex geometrical optics solutions

$$\Delta u_1 + q_1 u_1 = 0 \quad \text{in } \Omega$$

$$u_1|_{\partial\Omega\setminus\widetilde{\Gamma}} = 0$$

$\Longrightarrow$ Find

$$u_1(x) = e^{\tau\Phi(z)}\left(a(z) + a_0(z)/\tau\right) + \overline{e^{\tau\Phi(z)}}\,\overline{\left(a(z) + a_1(z)/\tau\right)} + e^{\tau\varphi}u_{11} + e^{\tau\varphi}u_{12}$$

Let polynomials $M_1(z)$ and $M_3(z)$ satisfy

$$\partial_z^j(\partial_{\overline{z}}^{-1}(aq_1) - M_1(z)) = 0, \quad z \in \mathcal{H}, \ j = 0, 1, 2$$

$$\partial_{\overline{z}}^j(\partial_z^{-1}(\overline{a}q_1)(z) - M_3(\overline{z})) = 0, \quad z \in \mathcal{H}, \ j = 0, 1, 2$$

$e_1, e_2 \in C^\infty(\Omega)$: $e_1 + e_2 \equiv 1$ on $\overline{\Omega}$, e_2 vanishes in a neighborhood of $\mathcal{H} \setminus \partial\Omega$, e_1 vanishes in a neighborhood of $\partial\Omega$.

Set $\Phi = \varphi + i\psi$

Choice of a, a_0, a_1:

$$a, a_0, a_1 \in C^2(\overline{\Omega}), \quad \partial_{\bar z} a = \partial_{\bar z} a_0 = \partial_{\bar z} a_1 \equiv 0$$

$$\text{Re}\, a|_{\partial\Omega\setminus\widetilde{\Gamma}} = 0, \; a = \partial_z a = 0 \text{ on } \mathcal{H} \cap \partial\Omega$$

$$(a_0(z) + \overline{a_1(z)})|_{\partial\Omega\setminus\widetilde{\Gamma}} = \frac{(\partial_{\bar z}^{-1}(a q_1) - M_1(z))}{4\partial_z \Phi} + \frac{(\partial_z^{-1}(\overline{a(z)} q_1) - M_3(\bar z))}{4\overline{\partial_z \Phi}}$$

Choice of u_{11}:

$$u_{11} = -\frac{1}{4}e^{i\tau\psi}\widetilde{R}_{\Phi,\tau}(e_1(\partial_{\bar{z}}^{-1}(aq_1) - M_1(z)))$$

$$-\frac{1}{4}e^{-i\tau\psi}R_{\Phi,-\tau}(e_1(\partial_z^{-1}(\overline{a(z)}q_1) - M_3(\bar{z})))$$

$$-\frac{e^{i\tau\psi}}{\tau}\frac{e_2(\partial_{\bar{z}}^{-1}(aq_1) - M_1(z))}{4\partial_z\Phi}$$

$$-\frac{e^{-i\tau\psi}}{\tau}\frac{e_2(\partial_z^{-1}(\overline{a(z)}q_1) - M_3(\bar{z}))}{4\overline{\partial_z\Phi}}$$

Apply Proposition 2.

We have to verify

$$(\nabla \varphi \cdot \nu) = 0 \quad \text{on } \partial\Omega \setminus \widetilde{\Gamma}$$

$\Leftarrow$ Im $\Phi = 0$ on $\partial\Omega \setminus \widetilde{\Gamma}$ and C-R equations

Therefore $\widetilde{\Gamma}$ can be arbitrary!

Proposition 2 $\Longrightarrow$ Find u_{12} such that

$$\Delta(u_{12}e^{\tau\varphi}) + q_1 u_{12} e^{\tau\varphi} = -q_1 u_{11} e^{\tau\varphi} + h_1 e^{\tau\varphi} \quad \text{in } \Omega,$$

$$u_{12}|_{\partial\Omega\setminus\widetilde{\Gamma}} = \frac{1}{4}\widetilde{R}_{\Phi,\tau}(e_1(\partial_{\bar{z}}^{-1}(a(z)q_1) - M_1(z)))$$
$$+ \frac{1}{4}R_{\Phi,-\tau}(e_1(\partial_z^{-1}(\overline{a(z)}q_1) - M_3(\bar{z})))$$

$$\|u_{12}\|_{L^2(\Omega)} = o\left(\frac{1}{\tau}\right), \quad \tau \to \infty$$

Here:

$$h_1 = e^{\tau i\psi}\Delta\left(\frac{e_2(\partial_{\bar{z}}^{-1}(a(z)q_1) - M_1(z))}{4\tau\partial_z\Phi}\right)$$

$$+ e^{-\tau i\psi}\Delta\left(\frac{e_2(\partial_z^{-1}(\overline{a(z)}q_1) - M_3(\bar{z}))}{\overline{4\tau\partial_z\Phi}}\right)$$

$$-\frac{a_0 q_1}{\tau}e^{i\tau\psi} - \frac{\overline{a_1}q_1}{\tau}e^{-i\tau\psi}$$

Similarly:

$$\Delta v + q_2 v = 0 \quad \text{in } \Omega, \qquad v|_{\partial\Omega\setminus\widetilde{\Gamma}} = 0$$

Construct solution v of the form

$$v(x) = e^{-\tau\Phi(z)}(a(z) + b_0(z)/\tau)$$
$$+ e^{-\tau\overline{\Phi(z)}}\overline{(a(z) + b_1(z)/\tau)} + e^{-\tau\varphi}v_{11} + e^{-\tau\varphi}v_{12}$$

$$v_{11} = -\frac{1}{4}e^{-i\tau\psi}\widetilde{R}_{\Phi,-\tau}(e_1(\partial_{\bar{z}}^{-1}(q_2 a(z)) - M_2(z)))$$

$$- \frac{1}{4}e^{i\tau\psi}R_{\Phi,\tau}(e_1(\partial_z^{-1}(q_2\overline{a(z)}) - M_4(\bar{z})))$$

$$+ \frac{e^{-i\tau\psi}}{\tau}\frac{e_2(\partial_{\bar{z}}^{-1}(aq_2) - M_2(z))}{4\partial_z\Phi}$$

$$+ \frac{e^{i\tau\psi}}{\tau}\frac{e_2(\partial_z^{-1}(\overline{a(z)}q_2) - M_4(\bar{z}))}{4\overline{\partial_z\Phi}}$$

Here

$$\partial_z^j(\partial_{\bar{z}}^{-1}(aq_2) - M_2(z)) = 0, \quad z \in \mathcal{H}, \;\; j = 0, 1, 2,$$

$$\partial_{\bar{z}}^{j}(\partial_{z}^{-1}(\bar{a}q_2)(z) - M_4(\bar{z})) = 0, \quad z \in \mathcal{H}, \; j = 0, 1, 2$$

b_0, b_1: holomorphic functions such that

$$(b_0 + \bar{b}_1)|_{\partial\Omega\backslash\tilde{\Gamma}} = -\frac{(\partial_{\bar{z}}^{-1}(aq_2) - M_2(z))}{4\partial_z\Phi}$$

$$-\frac{(\partial_{z}^{-1}(\overline{a(z)}q_2) - M_4(\bar{z}))}{4\overline{\partial_z\Phi}}$$

v_{12}: solution to

$$\Delta(v_{12}e^{-\tau\varphi}) + q_2 v_{12} e^{-\tau\varphi} = -q_2 v_{11} e^{-\tau\varphi} - h_2 e^{-\tau\varphi} \quad \text{in } \Omega$$

$$v_{12}|_{\partial\Omega\setminus\widetilde{\Gamma}} = \frac{1}{4}\widetilde{R}_{\Phi,-\tau}(e_1(\partial_{\bar{z}}^{-1}(a(z)q_2) - M_2(z)))$$
$$+ \frac{1}{4}R_{\Phi,\tau}(e_1(\partial_z^{-1}(\overline{a(z)}q_2) - M_4(\bar{z})))$$

$$\|v_{12}\|_{L^2(\Omega)} = o\left(\frac{1}{\tau}\right):$$

$$h_2 = e^{-\tau i\psi}\Delta\left(\frac{e_2(\partial_{\bar{z}}^{-1}(a(z)q_2) - M_2(z))}{4\tau\partial_z\Phi}\right)$$

$$+ e^{\tau i\psi}\Delta\left(\frac{e_2(\partial_z^{-1}(\overline{a(z)}q_2) - M_4(\bar{z}))}{4\tau\overline{\partial_z\Phi}}\right)$$

$$- \frac{b_0(z)q_2}{\tau}e^{-i\tau\psi} - \frac{\overline{b_1(z)}q_2}{\tau}e^{i\tau\psi}$$

Third Step: Stationary Phase

Key Theorem: Assume: $F \in C^0(\mathbb{R}^2)$,

$\{y \in \mathrm{supp}\, F \mid \nabla\varphi(x) = 0\} = \{y_1, \ldots, y_\ell\}$,

$\det\,(\partial_i\partial_j\varphi)(y_k) \neq 0$

$$\Longrightarrow \int_{\mathbb{R}^2} e^{i\tau\varphi(y)} F(y)\,dy$$

$$= \frac{2\pi}{\tau} \sum_{k=1}^{\ell} \frac{e^{i\tau\varphi(y_k)} F(y_k)}{|\det \partial_i\partial_j\varphi(y_k)|^{\frac{1}{2}}} \times e^{\frac{i\pi}{4}\mathrm{sgn}\,(\partial_i\partial_j\varphi(y_k))}$$

$$+\, o\left(\frac{1}{\tau}\right)$$

Here:

$$\operatorname{sgn} A := {}^{\#}\,[\text{positive eigenvalues of } A] - {}^{\#}[\text{negative eigenvalues of } A]$$

Result by stationary phase

Proposition 3: Let $\{\widetilde{x}_1, \dots, \widetilde{x}_\ell\}$ be the set of critical points of the function $Im\Phi$. Then for any potentials $q_1, q_2 \in C^{2+\alpha}(\overline{\Omega})$, $\alpha > 0$ with the same Dirichlet-to-Neumann maps and for any holomorphic function a, we have

$$\sum_{k=1}^{\ell} \frac{(q_1 - q_2)|a|^2(\widetilde{x}_k)\mathrm{Re}\, e^{2i\tau \mathrm{Im}\Phi(\widetilde{x}_k)}}{|(\det \mathrm{Im}\Phi'')(\widetilde{x}_k)|^{\frac{1}{2}}} + R = 0, \ \tau > 0$$

Here R is independent of τ:

$$R = \int_\Omega (q_1 - q_2)(a(a_0 + b_0) + \overline{a}(\overline{a_1} + \overline{b_1}))dx$$

$$+\frac{1}{4}\int_\Omega (q_1 - q_2)\left(a\frac{\partial_{\overline{z}}^{-1}(aq_2) - M_2(z)}{\partial_z \Phi} + \overline{a}\frac{\partial_z^{-1}(q_2\overline{a}) - M_4(\overline{z})}{\overline{\partial_z \Phi}}\right)dx$$

$$-\frac{1}{4}\int_\Omega (q_1 - q_2)\left(a\frac{(\partial_{\overline{z}}^{-1}(aq_1) - M_1(z))}{\partial_z \Phi} + \overline{a}\frac{(\partial_z^{-1}(\overline{a}q_1) - M_3(\overline{z}))}{\overline{\partial_z \Phi}}\right)dx$$

Sketch of Proof of Proposition 3

- Take geometric optics solution u_1 to

  $\Delta u_1 + q_1 u_1 = 0, \quad u_1|_{\partial\Omega\setminus\widetilde{\Gamma}} = 0$

- $u_2 : \Delta u_2 + q_2 u_2 = 0, \quad u_2|_{\partial\Omega} = u_1|_{\partial\Omega}$

  DN maps are equal $\Rightarrow \nabla u_1 = \nabla u_2$ on $\widetilde{\Gamma}$

$$u = u_1 - u_2 \Longrightarrow \Delta u + q_2 u = -(q_1 - q_2) u_1$$

$$u|_{\partial\Omega} = 0, \quad \frac{\partial u}{\partial \nu}|_{\widetilde{\Gamma}} = 0$$

- Take geometric optics solution v to

  $\Delta v + q_2 v = 0, \quad v|_{\partial\Omega\setminus\widetilde{\Gamma}} = 0$

$$0 = \int_\Omega v(\Delta u + q_2 u)dx = -\int_\Omega (q_1 - q_2) v u_1 dx :$$

Stationary phase + estimates for $u_{12} \Longrightarrow$

$$2\sum_{k=1}^{\ell} \frac{\pi((q_1 - q_2)|a|^2)(\widetilde{x_k})\mathrm{Re}\, e^{2i\tau \mathrm{Im}\Phi(\widetilde{x_k})}}{|(\det \mathrm{Im}\Phi'')(\widetilde{x_k})|^{\frac{1}{2}}}$$

$$+ R = o(1) \quad \text{as } \tau \to \infty$$

[left side] = almost periodic function in τ

Bohr's theorem implies [left side] = 0 for all τ

Hint to Completion of Theorem 1

We can choose Φ such that

$$\operatorname{Im}\Phi(\widetilde{x_k}) \neq \operatorname{Im}\Phi(\widetilde{x_j}),\ j \neq k$$

Let $a(\widetilde{x_k}) \neq 0$

Then Proposition 3 implies

$$q_1(\widetilde{x_k}) = q_2(\widetilde{x_k})$$

10. Conclusion and Further Topics

Our proof:

Pointwise uniqueness by choice of $\Phi \Longrightarrow$

Given one pair of data yields uniqueness at one point

$\Longrightarrow$ reconstruction

Future Topics

- Stability
- Completion for Lamé equations
- Sharp uniqueness
- $\Gamma, \widetilde{\Gamma}$: arbitrary without $\partial\Omega \setminus \Gamma \subset \widetilde{\Gamma}$,

$$C_q = \left\{ \left(u|_{\Gamma}, \frac{\partial u}{\partial \nu}|_{\widetilde{\Gamma}} \right) \Big| (\Delta + q)u = 0 \text{ in } \Omega, \right.$$

$$\left. u|_{\Omega \setminus \Gamma} = 0,\ u \in H^1(\Omega) \right\}$$

Thank you very much!