Best location of actuators for the stabilization of the Navier-Stokes equations

J. Weller-Calvo[1] , J.-M. Buchot[2], M. Fournié[2], J.-P. Raymond[2] et C. Airiau[1]

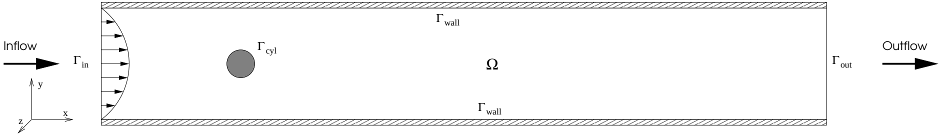
[1]Institut de Mécanique des Fluides de Toulouse
[2]Institut de Mathématiques de Toulouse

Control of Fluid-Structure Systems
and Inverse Problems
June 25, 2012

The physical context

Flows around vehicles, buildings, bridges ⇒ vibrations, noise ...
→ Control mechanisms: body profile, texture, mobile elements, actuators ...

A confined circular cylinder



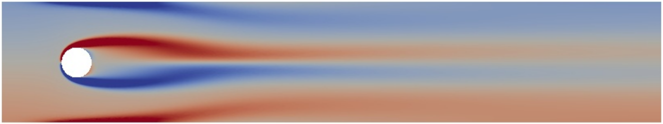
The incompressible Navier-Stokes equations (NS)

$\mathbf{w}(\mathbf{x}, t)$ = flow velocity, $q(\mathbf{x}, t)$ = pressure,
$\Omega^\infty = \Omega \times (0, \infty)$, $\Gamma_{loc}^\infty = \Gamma_{loc} \times (0, \infty)$

$$\frac{\partial \mathbf{w}}{\partial t} + (\mathbf{w} \cdot \nabla)\mathbf{w} - \nabla \cdot (\nu \nabla \mathbf{w} - pI) = 0, \quad \nabla \cdot \mathbf{w} = 0 \text{ in } \Omega^\infty$$
$$\mathbf{w} = \mathbf{w}_{in} \text{ on } \Gamma_{in}^\infty, \quad \mathbf{w} = 0 \text{ on } \Gamma_{wall}^\infty, \quad \mathbf{w} = M\mathbf{u} \text{ on } \Gamma_{cyl}^\infty$$
$$(\nu \nabla \mathbf{w} - pI) \cdot \mathbf{n} = 0 \text{ on } \Gamma_{out}^\infty$$
$$\mathbf{w}(\cdot, 0) = \mathbf{w}_0 \text{ in } \Omega$$

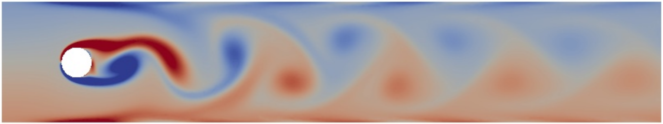
The physical context

- Steady inflow condition $\rightarrow$ (NS) admit a steady solution $(\mathbf{w}_s, q_s)$.



Steady solution for $Re_{cyl} = 150$ (isocontours of vorticity $\frac{\partial w_2}{\partial x_1} - \frac{\partial w_1}{\partial x_2}$)

- For Re_{cyl} large enough this solution is unstable. Perturbing the flow (inflow condition, initial condition)
  $\Rightarrow$ vortices in the cylinder wake

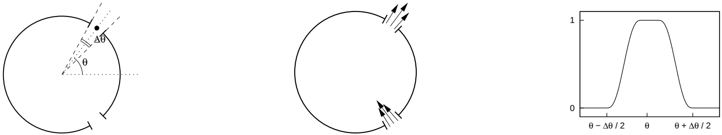


Vorticity at $t > t_{pert} + t_{transition}$,
following a perturbation at $t{=}t_{pert}$, for $Re_{cyl} = 150$

The physical context

- Aim: to minimize the vortical activity in the cylinder wake.

- Method: blowing/suction from the cylinder.



Position *Directions* *Profile of M*

- Feedback/Closed-loop control: Control is defined by a feedback operator K:

$$\mathbf{u}(\mathbf{x}, t) = K[\mathbf{x}](\mathbf{w}(\cdot, t) - \mathbf{w}_s) \qquad \mathbf{x} \in \Gamma_{cyl}$$

The physical context

Usual approach for control design in this context

- One considers the departure from steady state $\mathbf{z} = \mathbf{w} - \mathbf{w}_s,\ p = q - q_s$

- The (NS) equations are linearized about the steady state leaving:

$$\frac{\partial \mathbf{z}}{\partial t} + (\mathbf{z}\cdot\nabla)\mathbf{w}_s + (\mathbf{w}_s\cdot\nabla)\mathbf{z} - \nabla\cdot(\nu\nabla\mathbf{z} - pI)\cdot\mathbf{n} = 0, \quad \nabla\cdot\mathbf{z} = 0 \ \text{ in } \ \Omega^\infty$$
$$\mathbf{z} = 0 \ \text{ on } \ \Gamma^\infty_{in}\cup\Gamma^\infty_{wall}, \quad \mathbf{z} = M\mathbf{u} \ \text{ on } \ \Gamma^\infty_{cyl}$$
$$(\nu\nabla\mathbf{z} - pI)\cdot\mathbf{n} = 0 \ \text{ on } \ \Gamma^\infty_{out}$$
$$\mathbf{z}(0) = \mathbf{w}_0 - \mathbf{w}_s \ \text{ in } \ \Omega$$

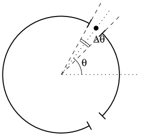
- The control law is determined for (LNS) then applied to (NS) $\rightarrow$
  Local stabilization result

The physical context

Idea of this work

→ Enlarging the stability domain could improve the actuation law.

- Two actuators, placed symmetrically on the cylinder, with $\Delta\theta = 10^o$.
  The position θ can vary.



- A stabilizing control law of minimum norm is determined for each possible location of the actuators: $\theta = 5^o, \cdots, 175^o$, using (LNS) Equations.

- The position corresponding to a minimal cost control is retained. Control laws are tested on (NS) Equations.

From the LNSE to a linear control system

A linear control system

$$\dot{\mathbf{z}} = A\mathbf{z} + B\mathbf{u}, \qquad \mathbf{z}(0) = \mathbf{z}_0$$

Two strategies

- In a theoretical context:
  - Step 1: Projection of the LNSE onto a divergence-free subspace (Leray projector) $\rightarrow$ Oseen operator.
  - Step 2: Lift the Dirichlet boundary condition for the control (properties of the Oseen operator are used.)
- In a numerical context:
  - Step 1: Variational formulation with Lagrange multiplier approach to account for the Dirichlet boundary condition, then discretization by finite elements.
  - Step 2: Projection to simultaneously deal with the divergence and boundary conditions (discrete equivalent of the Leray projector.)

Computing the control

The aim of this work:

- Given an actuator position, impose a certain decay rate to $\mathbf{z}$
- Find the best actuator location for doing this
  (*best* $=$ that minimizes the norm of $\mathbf{u}$.)

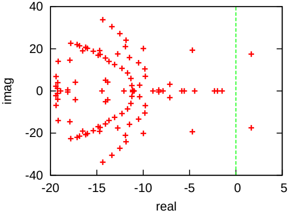
Outline of approach:

- With Dirichlet boundary conditions on $\Gamma_D \subset \partial\Omega$, we have $\mathbf{z} \in Z$ where

$$Z = \{z \in L^2(\Omega, \mathbb{R}^2) \ | \ \nabla \cdot \mathbf{z} = 0, \quad \mathbf{z} \cdot \mathbf{n} = 0 \ \text{ on } \ \Gamma_D\}$$

- The spectrum of the Oseen operator A, $\sigma(A)$, is contained in a sector. The eigenvalues are real or pairwise conjugate, of finite multiplicity.

Example: Numerical computation of eigenvalues at $Re_{cyl} = 80$



$\longrightarrow$ A is not stable, $\sigma(A) \cap \mathbb{R}^{+*} \neq \emptyset$

Computing the control

Outline of the approach (continued):

- For $\omega \geqslant 0$, we order the eigenvalues of A:

$$\mathrm{Re}(\lambda_1) \geqslant \cdots \geqslant \mathrm{Re}(\lambda_{N_{\omega,u}}) > -\omega > \mathrm{Re}(\lambda_{N_{\omega,u}+1}) \geqslant \cdots$$

- Z can be decomposed into an *unstable* subspace (unstable for $A + \omega I$) and a *stable* subspace.

$$Z = Z_{\omega,u} \oplus Z_{\omega,s} \text{ and } Z^* = Z^*_{\omega,u} \oplus Z^*_{\omega,s}$$

  The projector $\pi_{\omega,u}$ onto $Z_{\omega,u}$ along $Z_{\omega,s}$ can easily be defined by determining a basis of $Z_{\omega,u}$ and $Z^*_{\omega,u}$

- The control system is projected onto the *unstable* subspace
  → we obtain a low dimensional system
  → we can apply classical control techniques

Computing the control

Outline of the approach (continued):

- Computation of proportional feedback controls by this method then implies solving very small Riccati equations (dimension 2 for the cylinder !)
- Computations can be repeated for different actuator positions.

$$\dot{\mathbf{z}} = A\mathbf{z} + B_\theta \mathbf{u}, \qquad \mathbf{z}(0) = \mathbf{z}_0$$

- The norm of the control is directly linked to the solution of the Riccati equation.

Test cases:

- $Re_{cyl=80}$: Computation of a stabilizing control. Comparison of a classical actuator placement ($\theta = 70^o$) and the optimal placement determined by our algorithm ($\theta = 92^o$).
- $Re_{cyl=150}$: Computation of a control that reduces drag.

Best location of actuators for the stabilization of the Navier-Stokes equations

J. Weller-Calvo[1] , J.-M. Buchot[2], M. Fournié[2], J.-P. Raymond[2] et C. Airiau[1]

[1]Institut de Mécanique des Fluides de Toulouse
[2]Institut de Mathématiques de Toulouse

Control of Fluid-Structure Systems
and Inverse Problems
June 25, 2012