Time optimal control of infinite dimensional linear systems

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Problem statement

Let X, U be Banach spaces and let $F : X \times U \to U$. Let $\mathcal{U} \subset L^\infty([0,\infty), U)$ be a closed bounded set.
Consider the control system

$$\dot{z} = F(z, u)$$

and assume that $z_0, z_1 \in X$ are such that there exist $u \in \mathcal{U}$ and $T > 0$ such that

$$z(0) = z_0, \qquad z(T) = z_1.$$

Time optimal control:

- Show that there exist $T^* > 0$ and $u^* \in \mathcal{U}$ such that T^* is the minimal T as above. Study the regularity of $z_0 \mapsto T^*(z_0, z_1)$.
- Derive optimality conditions (Pontryagin's maximum principle).
- Prove the bang-bang property $u^* \in \partial\mathcal{U}$.

Outline

- The finite dimensional case: linear and bilinear
- A class of exactly controllable systems
- Systems which are L^∞ null controllable over measurable sets
- Comments and open questions

The finite dimensional case : linear systems

Denote $X = \mathbb{R}^n$, $U = \mathbb{R}^m$ and let $\dot{z} = Az + Bu$.

Theorem 1 (Maximum Principle, Bellman et al. (1956))

Let $u^*(t)$ be the time optimal control, defined on $[0, \tau^*]$. Then there exists $z \in X$, $z \neq 0$ such that

$$\langle B^* \mathbb{T}^*_{\tau^*-t} z, u^*(t) \rangle = \max_{\|\mathrm{u}\| \leq 1} \langle B^* \mathbb{T}^*_{\tau^*-t} z, \mathrm{u} \rangle$$

Corollary 1

If (A, B) controllable then the time optimal control u^* is bang-bang, in the sense that

$$\|u^*(t)\| = 1 \qquad (t \in [0, \tau^*] \quad a.e.)$$

Moreover, the time optimal control is unique.

The finite dimensional case : a bilinear system (I)

Consider a simplified modeling self-propelling by radial defomations (Shapere and F. Wilczek (1989))

$$\dot{h} = M\alpha \cdot \beta,$$

$$\dot{\alpha} = \beta.$$

Given an integer $L \geqslant 2$, $M \in \mathcal{M}_L(\mathbb{R})$ with $M^* = -M$ and $h_1 \in \mathbb{R}^*$, our aim consists in determining the minimal time T^* for which there exists $\beta \in L^\infty\big((0, T^*), \mathbb{R}^L\big)$ such that

$$|\beta(t)|_2 \leqslant 1 \quad \text{(for a.e. } t \in (0, T^*)) \,,$$

and

$$h(0) = 0 \quad \text{and} \quad \alpha(0) = 0 \,.$$

satisfies

$$h(T^*) = h_1 \quad \text{and} \quad \alpha(T^*) = 0 \,.$$

The finite dimensional case : a bilinear system (II)

Proposition. (Lohéac, Scheid and M.T., 2011) The minimal time $T^\star$ is given by

$$T^\star = \sqrt{\frac{2\pi|h_1|}{\lambda^*}}\,,$$

where $\lambda^* = \max\{|\lambda|,\ \lambda \in \sigma(M)\} > 0$. Moreover, an optimal time control is

$$\beta^\star(t) = \exp\left(\operatorname{sign}(h_1)\sqrt{\frac{2\pi}{\lambda^*|h_1|}}\;t\,M\right)\beta_0 \qquad (t \in [0, T^*])\,, \quad (1)$$

where $|\beta_0|_2 = 1$ is chosen such that $\beta_0 \in \operatorname{Ker}\left(M^2 + |\lambda^*|^2\right)$

Notation and problem statement (I)

Notation:

- X (the state space) and U (the input space) are complex Hilbert spaces
- $\mathbb{T} = (\mathbb{T}_t)_{t \geqslant 0}$ is a strongly continuous semigroup on X generated by A.
- X_1 is $\mathcal{D}(A)$ equipped with the graph norm, while X_{-1} is the completion of X with respect to $\|z\|_{-1} := \|(\beta I - A)^{-1} z\|$.
- The semigroup $\mathbb{T}$ can be extended to X_{-1}, and then its generator is an extension of A, defined on X.
- $B \in \mathcal{L}(U; X_{-1})$ be a control operator and let $u \in L^2([0, \infty), U)$ be an input function.

Notation and problem statement (II)

We consider the system $\dot{z}(t) = Az(t) + Bu(t) \qquad (t \geqslant 0)$.

$u \in \mathcal{U}_{ad} = \{u \in L^\infty([0,\infty), U) \;\mid\; \|u(t)\| \leqslant 1$ a. e. in $[0,\infty)\}$.

The state trajectory is $z(t) = \mathbb{T}_t z(0) + \Phi_t u$, where

$\Phi_t \in \mathcal{L}(L^2([0,\infty), U); X_{-1}), \quad \Phi_t u = \int_0^t \mathbb{T}_{t-\sigma} Bu(\sigma)\,\mathrm{d}\sigma.$

Assume that $z_0,\ z_1 \in X$ are s.t. there exists $u \in \mathcal{U}_{ad}$ and $\tau > 0$ s.t. $z_1 = \mathbb{T}_\tau z_0 + \Phi_\tau u$ (z_1 reachable from z_0.

Problem statement: Determine

$$\tau^*(z_0, z_1) = \min_{u \in \mathcal{U}_{ad}} \{\tau \;\mid\; z_1 = \mathbb{T}_\tau z_0 + \Phi_\tau u\},$$

and the corresponding control u^*.

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Exactly controllable systems(I)

Assume that $B \in \mathcal{L}(U, X)$ and that (A, B) is exactly controllable in any time $\tau > 0$.

Proposition 1 (Lohéac and M.T., 2011)

Let $u^*(t)$ be the time optimal control, defined on $[0, \tau^*]$. Then there exists $z \in X$, $z \neq 0$ such that

$$\langle B^* \mathbb{T}^*_{\tau^* - t} z, u^*(t) \rangle = \max_{\|\mathrm{u}\| \leq 1} \langle B^* \mathbb{T}^*_{\tau^* - t} z, \mathrm{u} \rangle$$

Corollary 2

Assume that (A, B) is approximatively controllable from sets of positive measure. Then

$$\|u^*(t)\| = 1 \qquad (t \in [0, \tau^*] \quad a.e.)$$

Moreover, the time optimal control is unique.

Exactly controllable systems(II): Idea of the proof

- For each $\tau > 0$, we endow X with the norm

$$|||z||| = \inf\{\|u\|_{L^\infty([0,\tau],U)} \ \mid \ \Phi_\tau u = z\}.$$

  Note that $|||\cdot|||$ is equivalent with the original norm $\|\cdot\|$.

- For $\tau > 0$ we set

$$B^\infty(\tau) = \{\Phi_\tau u \ \mid \ \|u\|_{L^\infty([0,\tau],U)} \leqslant 1\},$$

  and we show that if (τ^*, u^*) is an optimal pair then $\Phi_{\tau^*} u^* \in \partial B^\infty(\tau^*)$.

- Using the fact that $B^\infty(\tau^*)$ has a non empty interior, we apply a geometric version of the Hahn-Banach theorem to get the conclusion.

Exactly controllable systems(III): the Schrödinger equation

$\Omega \subset \mathbb{R}^m$ a rectangular domain;

$\mathcal{O}$ a non-empty open subset of Ω.

$$\frac{\partial z}{\partial t}(x,t) = i\Delta z(x,t) + u(x,t)\chi_{\mathcal{O}} \text{ for } (x,t) \in \Omega \times (0,\infty)$$

$$z(x,t) = 0 \quad \text{on} \quad \partial\Omega \times (0,\infty),$$

$$\|u(\cdot,t)\| \leqslant 1 \qquad a.e..$$

Proposition 2

The above system is approximatively controllable with controls supported in any set of positive measure $E \subset [0,\infty)$.

Corollary 3

Time optimal controls are bang-bang, i.e., $\|u(\cdot,t)\|_{L^2(\mathcal{O})} = 1$ a.e.

L^∞ null controllability over measurable sets(I)

Take $U = L^2(\Gamma)$ where Γ is a compact manifold.

Definition 1

Let $\tau > 0$, $e \subset \Gamma \times [0, \tau]$ a set of positive measure. The pair (A, B) is said L^∞ *null controllable in time* τ *over* e if, for every $z_0 \in X$, there exists $u \in L^\infty(\Gamma \times [0, \tau])$ (null control) such that

$$\mathbb{T}_\tau z_0 + \int_0^\tau \mathbb{T}_{\tau - s} B \chi_e(s) u(s)\, ds = 0$$

where χ_e is the characteristic function of e.

If (A, B) is L^∞ null controllable in time τ over e then the quantity

$$C_{\tau,e} := \sup_{\|z_0\|=1} \inf \left\{ \|u\|_{L^\infty(e)} \mid u \text{ null control for } z_0 \right\} \qquad (2)$$

is called the *control cost in time* τ *over* e.

L^∞ null controllability over measurable sets(II): Duality

Proposition 3

Let $e \subset \Gamma \times [0,\tau]$ be a set of positive measure and $K_{\tau,e} > 0$. The following two properties are equivalent

1. *The inequality*

$$\|\mathbb{T}_\tau^* \varphi\| \leqslant K_{\tau,e} \int_0^\tau \|\chi_e B^* \mathbb{T}_\sigma^* \varphi\|_{L^1(\Gamma)}\, d\sigma \tag{3}$$

   holds for any $\varphi \in X$, where $e' = \{(x, \tau - t) \mid (x,t) \in e\}$.

2. *The pair (A, B) is L^∞ null controllable in time τ over e at cost not larger than $K_{\tau,e}$.*

L^∞ null controllability over measurable sets(III): main result

Theorem 2 (Mizel and Seidman (1997), G. Wang (2008), Micu, Roventa and M.T. (2011))

Assume that the pair (A, B) *is* L^∞ *null controllable in time* τ *over* e *for every* $\tau > 0$ *and for every set of positive measure* $e \subset \Gamma \times [0, \tau]$. *Then, for every* $z_0 \in X$ *and* $z_1 \in \mathcal{R}(z_0, \mathcal{U}_{ad})$, *the time optimal problem has a unique solution* u^* *which is bang-bang.*

Proof of Theorem 2:

- **Existence:** Consider a minimizing sequence $(z_n, \tau_n)_{n\geq 1}$ where $\tau_n \to \tau^*$ and z_n is a controlled solution and pass to the limit.

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- **Bang-bang property:** Suppose that $|u^*(x,t)| < 1 - \varepsilon$ for $(x,t) \in e \subset \Gamma \times [0,\tau^*]$ where e is a set of positive measure. From the L^∞ controllability over e property it follows that there exists an L^∞ null control $v \in L^\infty(\Gamma \times [0,\tau^*])$ such that
  - supp $(v) \subset e$ and $\|v\|_{L^\infty(e)} < \varepsilon$
  - v drives $z^*(\delta)$ to 0 in time $\tau^* - \delta$.

  It follows that $u(t) = u^*(t+\delta) + v(t+\delta)$ drives z_0 to z_1 in time $\tau^* - \delta$. Contradiction!

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  It follows that $u(t) = u^*(t+\delta) + v(t+\delta)$ drives z_0 to z_1 in time $\tau^* - \delta$. Contradiction!
- **Uniqueness:** If u^* and v^* are optimal time controls then $w^* = \frac{u^*+v^*}{2}$ is also an optimal time control.

$$|u^*(x,t)| = |v^*(x,t)| = |w^*(x,t)| = 1 \text{ a. e. in } \Gamma \times [0,\tau]$$

$$\Rightarrow u^*(x,t) = v^*(x,t) \text{ a. e. in } \Gamma \times [0,\tau]. \blacksquare$$

Bang-bang boundary controls for the heat equation (I)

$\Omega \subset \mathbb{R}^m$ is an open and bounded set

Γ is a non-empty open subset of $\partial\Omega$

$$\frac{\partial z}{\partial t}(x,t) = \Delta z(x,t) \text{ for } (x,t) \in \Omega \times (0,\infty) \tag{4}$$

$$\left\{ \begin{array}{ll} z(x,t) = u(x,t) & \text{on } \Gamma \times (0,\infty) \\ z(x,t) = 0 & \text{on } (\partial\Omega \setminus \Gamma) \times (0,\infty) \end{array} \right. \tag{5}$$

$$z(x,0) = z_0(x) \text{ for } x \in \Omega \tag{6}$$

$$\mathcal{U}_{ad} = \{u \in L^\infty(\Gamma \times [0,\infty)) \;\;|\;\; |u(x,t)| \leqslant 1 \text{ a. e. in } \Gamma \times [0,\infty)\}.$$

$$\mathcal{R}(z_0, \mathcal{U}_{ad}) = \{z(\tau) \,|\, \tau > 0 \text{ and } z \text{ solution of (4)-(6) with } u \in \mathcal{U}_{ad}\}.$$

Given $z_0 \in H^{-1}(\Omega)$ and $z_1 \in \mathcal{R}(z_0, \mathcal{U}_{ad})$, *the time optimal control problem* for (4)-(6) consists in:

- determining $u^* \in \mathcal{U}_{ad}$ such that the corresponding solution z^* of (4)-(6) satisfies

$$z^*(\tau^*(z_0, z_1)) = z_1, \tag{7}$$

- where the control time $\tau^*(z_0, z_1)$ is

$$\tau^*(z_0, z_1) = \inf_{u \in \mathcal{U}_{ad}} \{\tau \;\;|\;\; z(\cdot, \tau) = z_1\}. \tag{8}$$

Bang-bang boundary controls for the heat equation (III): Main theorem

Theorem 3 (Micu, Roventa and M.T. (2011))

Let $m \geq 2$. Suppose that Ω is a rectangular domain in $\mathbb{R}^m$ and that Γ is a nonempty open set of $\partial\Omega$. Then, for every $z_0 \in H^{-1}(\Omega)$ and $z_1 \in \mathcal{R}(z_0, \mathcal{U}_{ad})$, there exits a unique solution u^* of the time optimal control problem (8). This solution u^* has the bang-bang property:

$$|u^*(x,t)| = 1 \quad \textit{a. e. in} \quad \Gamma \times [0, \tau^*(z_0, z_1)]. \tag{9}$$

Bang-bang boundary controls for the heat equation (IV): Main steps of the proof

Let A (respectively $\mathbb{T}$) be the Dirichlet Laplacian (respectively the heat semigroup) in $X = L^2(\Omega)$.

We know that there exists an orthonormal basis of eigenvectors $\{\varphi_k\}_{k\geqslant 1}$ of A and corresponding family of eigenvalues $\{-\lambda_k\}_{k\geqslant 1}$, where the sequence $\{\lambda_k\}$ is positive, non decreasing and satisfies $\lambda_k \to \infty$ as k tends to infinity.

For $\eta > 0$ we denote by

$$V_\eta = \text{Span}\,\{\varphi_k \ \mid \ \lambda_k^{\frac{1}{2}} \leqslant \eta\}.$$

Bang-bang boundary controls for the heat equation (V): A version of the Lebeau-Robbiano method

Proposition 4

Let $\tau > 0$ and let $e \subset \Gamma \times [0, \tau]$ be a set of positive measure. Assume $B \in \mathcal{L}(U, X_{-1/2})$. Moreover, assume that there exist positive constants d_0, d_1 and d_2 such that for every $\eta > 0$ and $[s, t] \subset (0, 1)$ we have that, for any $\varphi \in V_\eta$,

$$\|\mathbb{T}_\tau^* \varphi\| \leqslant d_0 e^{d_1 \eta \ln\left(\frac{1}{\mu(\mathcal{E})}\right) + \frac{d_2}{\mu(\mathcal{E})}} \int_0^\tau \|\chi_{\mathcal{E}'} B^* \mathbb{T}_s^* \varphi\|_{L^1(\Gamma)} \, \mathrm{d}s, \qquad (10)$$

where $\mathcal{E} = (e \cap \Gamma) \times [s, t]$ and $\mathcal{E}' = \{(x, \tau - t) \mid (x, t) \in \mathcal{E}\}$. Then the pair (A, B) is L^∞ null controllable in time τ over e.

Bang-bang boundary controls for the heat equation (VI): Proof of Theorem 3

From Proposition 4, a sufficient condition for existence, uniqueness and bang-bang property of time optimal controls is the inequality:

$$\|\mathbb{T}_\tau^*\varphi\| \leqslant d_0 e^{d_1\eta\,\ln\left(\frac{1}{\mu(\mathcal{E})}\right)+\frac{d_2}{\mu(\mathcal{E})}} \int_0^\tau \|\chi_{\mathcal{E}'} B^* \mathbb{T}_s^*\varphi\|_{L^1(\Gamma)} ds,$$

which, in out particular case, can be written equivalently as

$$\left(\sum_{n^2+m^2\leq\eta^2} |a_{nm}|^2 e^{-2\tau(n^2+m^2)}\right)^{\frac{1}{2}} \leq d_0 e^{d_1\eta\,\ln\left(\frac{1}{t-s}\right)+\frac{d_2}{t-s}} \int_{F\cap[s,t]}\int_{e_\sigma}\left|\sum_{n=1}^{\sqrt{\eta^2-1}}\left(\sum_{m=1}^{\sqrt{\eta^2-n^2}} a_{nm}e^{-(m^2+n^2)\sigma}\right)\sin(nx)\right| dx\, d\sigma,$$

$F\subset[0,\tau]$ verifies $\mu(F)\geqslant\frac{\mu(\mathcal{E})}{4\mu(\Gamma)}$ and $\mu(e_\sigma)\geqslant\frac{\mu(\mathcal{E})}{4\tau}$, $\forall\sigma\in F$.

Bang-bang boundary controls for the heat equation (V): Proof of Theorem 3

Theorem 4 (Nazarov, 1993)

Let $N \in \mathbb{N}$ be a nonnegative integer and $p(x) = \sum_{|k| \leqslant N} a_k e^{i\nu_k x}$ ($a_k \in \mathbb{C}$, $\nu_k \in \mathbb{R}$) be an exponential polynomial. Let $I \subset \mathbb{R}$ be a finite interval and E a measurable subset of I of positive measure. Then

$$\sup_{x \in I} |p(x)| \leqslant \left(\frac{C\mu(I)}{\mu(E)} \right)^{2N} \sup_{x \in E} |p(x)|, \tag{11}$$

where $C > 0$ is an absolute constant.

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where $C > 0$ is an absolute constant.

- Turán (1948): For every subinterval $E \subset I = [-\pi, \pi]$ of length $\mu(E) = 4\pi e L < 2\pi$ and $f(x) := \sum_{|n| \leq N} a_n \, e^{inx}$, we have

$$\sup_{x \in I} |f(x)| \leqslant \frac{1}{L^{2N}} \sup_{x \in E} |f(x)|.$$

Bang-bang boundary controls for the heat equation (VI): Proof of Theorem 3

- Nazarov (1993): The interval E is replaced by a measurable set of positive measure.
- Lebeau and Robbiano (1995): Inequality of similar type in which $e^{i\nu_k x}$ are replaced by eigenfunctions of an elliptic operator.

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Corollary 2

The following inequality holds for any sequence $(a_k)_{|k|\leqslant N} \subset \mathbb{C}$

$$\left(\sum_{|k|\leqslant N} |a_k|^2\right)^{\frac{1}{2}} \leqslant \frac{C^{2N}}{\mu(I)} \left(\frac{2\mu(I)}{\mu(E)}\right)^{2N+1} \int_E \left|\sum_{|k|\leqslant N} a_k e^{i\nu_k x}\right| dx, \quad (12)$$

where $C > 0$ *is the constant from* (11).

Bang-bang boundary controls for the heat equation (VII): Proof of Theorem 3

Theorem 5 (Borwein and Erdelyi, 1997, 1998)

Let $\nu_k := k^\theta$, $k \in \{1, 2, \dots\}$, $\theta > 1$. *Let* $\rho \in (0,1)$, $\varepsilon \in (0, 1-\rho)$ *and* $\varepsilon \leqslant 1/2$. *Then there exists a constant* $c_\theta > 0$ *such that*

$$\sup_{t \in [0,\rho]} |p(t)| \leqslant \exp\left(c_\theta \varepsilon^{1/(1-\theta)}\right) \sup_{t \in E} |p(t)|,$$

for every $p \in \mathcal{S}_{[0,1]} := \operatorname{Span}\{t^{\nu_1}, t^{\nu_2}, \dots\}$ *and for every set* $E \subset [\rho, 1]$ *of Lebesgue measure at least* $\varepsilon > 0$.

Bang-bang boundary controls for the heat equation (VII): Proof of Theorem 3

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for every $p \in \mathcal{S}_{[0,1]} := \operatorname{Span}\{t^{\nu_1}, t^{\nu_2}, \dots\}$ *and for every set* $E \subset [\rho, 1]$ *of Lebesgue measure at least* $\varepsilon > 0$.

- Müntz (1914): $\mathcal{S}_{[0,1]}$ is a proper subspace of $L^2(0,1)$ if and only if $\sum_{k=1}^{\infty} \frac{1}{\nu_k} < \infty$.
- L. Schwarz (1943): The restriction $R_\rho : \overline{\mathcal{S}_{[0,1]}}^{L^2} \to \overline{\mathcal{S}_{[\rho,1]}}^{L^2}$ is invertible. The proof is by contradiction (no explicit constant).

Bang-bang boundary controls for the heat equation (VIII): Proof of Theorem 3

Proof of the Erdelyi and Borwein Theorem:

- The norm $\sup_{t\in[0,1]} |p(t)|$ is bounded by the norm $\sup_{t\in[\rho,1]} |p(t)|$, with explicit constant $c := e^{\frac{\kappa}{1-\rho}}$.
  - Seidman (2008), Miller (2009), Tenenbaum and Tucsnak (2011): $\|R_\rho^{-1}\| \leq Ce^{\frac{\kappa}{1-\rho}}$ (results on the controllability's cost in small time for the heat equation).
- The interval $[1,\rho]$ is replaced by a measurable set E of positive measure by using the Chebyshev-type polynomials $T_{E,\nu_0,\nu_1,\dots,\nu_n}$ corresponding to the set E and exponents $\nu_0 = 0, \nu_1 \dots, \nu_n$:

$$|T_{E,\nu_0,\dots,\nu_n}(s)| \leq |T_{E,\nu_1,\dots,\nu_n}(0)| \leq c, \quad s \in (0, \inf(E))$$

$$|p(s)| \leq |T_{E,\nu_0,\dots,\nu_n}(s)| \sup_{t\in E} |p(t)|, \quad s \in (0, \inf(E)).$$

The above result of Erdelyi and Borwein has the following consequence:

Corollary 3

For every $\tau > 0$ there exist constants $C,\ \kappa > 0$ such that for every $F \subset [0,\tau]$ of positive measure the following inequality holds

$$Ce^{\kappa/\mu(F)} \int_F \left| \sum_{k\geqslant 1} a_k e^{-k^2 t} \right| \mathrm{d}t \geqslant \left[\sum_{k\geqslant 1} |a_k|^2 e^{-k^2\tau} \right]^{\frac{1}{2}} \quad ((a_k) \in \ell^2(\mathbb{C})).$$

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By combining the space and time estimates (Corollaries 2 and 3) we get the inequality (10) and the proof of the main theorem is finished.■

Comments

- We need the separation of variables x and y. Our proof can be generalized only for special geometries.
- Interior controllability results can be also obtained.

Open questions

- L^∞ boundary controls for the Schrödinger equation (even in one space dimension)
- Arbitrary shapes for the boundary control of the heat equation (APRAIZ, ESCAURIAZA, WANG, ZHANG (2012))
- Nonlinear problems