Coupling estimation and control for a two dimensional Burgers type equation - Numerical experiments

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C.F.-S.S.I.P. - Toulouse Workshop 2012

Contents

Problem Formulation

We consider the following problem:
Find a control u that locally stabilizes the system

$$\begin{cases} \frac{\partial y}{\partial t} - \nu\Delta y + (\partial_1 y + \partial_2 y)y = f & \text{in } (0,\infty)\times\Omega \\ \nu\frac{\partial y}{\partial n}|_{\Gamma_{ob}} = g_N \ \text{ and } y|_{\Gamma_D} = g_D & \text{in } (0,\infty) \\ y|_{\Gamma_c} = g_c + Mu \ \text{ in } (0,\infty) \\ y(0) = y_0 \quad \text{in } \Omega \end{cases}$$

around a unstable solution w of the stationary problem

$$\begin{cases} -\nu\Delta w + (\partial_1 w + \partial_2 w)w = f & \text{in } \Omega \\ \nu\frac{\partial w}{\partial n}|_{\Gamma_{ob}} = g_N, \quad w|_{\Gamma_D} = g_D \\ w|_{\Gamma_c} = g_c \end{cases}$$

Problem Formulation

where $\Omega = [0,1] \times [0,1]$, $\Gamma_c = \{1\} \times [0,1]$ corresponds to the controlled boundary, $\Gamma_{ob=}\{0\} \times [0,1]$ the observed one and $\Gamma_D = \Gamma \setminus (\Gamma_c \cup \Gamma_{ob})$ as represented in Figure 1.

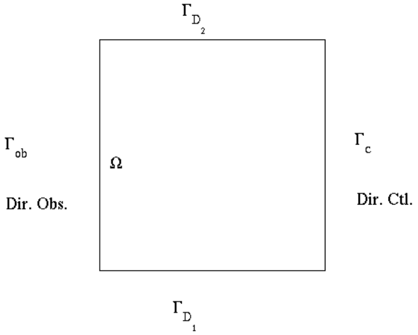

Figure: Domain Ω and boundary $\Gamma = \Gamma_D \cup \Gamma_c \cup \Gamma_{ob}$

Problem Formulation

In fact, the above problem is equivalent to consider the question of finding u such that the difference $z = y - w$ stabilizes to zero.

$$\begin{cases} \frac{\partial z}{\partial t} - \nu \Delta z + (\partial_1 w + \partial_2 w) z + (\partial_1 z + \partial_2 z) w + (\partial_1 z + \partial_2 z) z = 0 \quad \text{in } (0, \infty) \times \Omega \\ \nu \frac{\partial z}{\partial n}|_{\Gamma_{ob}} = 0 \ \text{ and } z|_{\Gamma_D} = 0 \quad \text{in } (0, \infty) \\ z|_{\Gamma_c} = Mu \ \text{ in } (0, \infty) \\ z(0) = z_0 \ (= y_0 - w) \quad \text{in } \Omega \end{cases}$$

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Available partial observations given by

$$Y_{ob} = Hz$$

$$Hz(t) = \left(\frac{1}{|\Gamma_1|} \int_{\Gamma_1} z(t), \cdots, \frac{1}{|\Gamma_{N_o}|} \int_{\Gamma_{N_o}} z(t) \right) \in \mathbb{R}^{N_o}$$

Semi-Discretization - Weak Formulation

$$\int_\Omega \dot{z}\phi_j dx = -\int_\Omega \nu \nabla z \cdot \nabla \phi_j dx + \int_\Gamma \nu \phi_j \frac{\partial z}{\partial n} ds - \int_\Omega (\partial_1 w + \partial_2 w) z \phi_j dx$$
$$- \int_\Omega (\partial_1 z + \partial_2 z) w \phi_j dx - \int_\Omega (\partial_1 z + \partial_2 z) z \phi_j dx \quad \text{for all } j \in \{1, n\}$$

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Robin approximation for Dirichlet and control conditions

$$\int_\Gamma \nu \phi_j \frac{\partial z}{\partial n} = \int_{\Gamma_{ob}} \nu \phi_j \times 0 + \int_{\Gamma \backslash \Gamma_{ob}} \phi_j \frac{1}{\varepsilon}(g - z)$$

$$g = \begin{cases} 0, \; x \in \Gamma_{D_1} \cup \Gamma_{D_1} \\ Mu, \; x \in \Gamma_c \end{cases}$$

so that

$$\int_\Gamma \nu \phi_j \frac{\partial z}{\partial n} = \int_{\Gamma_c} \frac{1}{\varepsilon} M u \phi_j - \int_{\Gamma \backslash \Gamma_{ob}} \frac{1}{\varepsilon} z \phi_j. \tag{1}$$

Shifting the operator

For simplicity we consider $w = 0$ but we include in the computation of A the term

$$\alpha \int_{\Omega} \phi_i \phi_j,\ \alpha > 0.$$

The resulting matrix A corresponds to the discretization of the operator

$$\mathcal{A} = \Delta + \alpha Id, \quad D(\mathcal{A}) = \{z \in H^2 :\ z|_{\Gamma_D} = 0,\ \frac{\partial z}{\partial n}|_{\Gamma_{ob}} = 0\}$$

with eigenvalues

$$\lambda_{kl} = -\nu[\frac{(2k-1)^2}{4} + l^2]\pi^2 + \alpha \tag{2}$$

associated with the class of eigenfunctions

$$\Phi_{kl} = \gamma\, cos(\frac{2k-1}{2}\pi\, x_1)\sin(l\pi\, x_2), \quad k, l \in \mathbf{N},\ \delta \in \mathbf{R}.$$

Finite Dimensional Problem

The semi-discretized system can therefore be written as

$$\begin{cases} E(z^n)' = Az^n + Bu^n + F(z^n) + E\zeta^n \text{ in } (0, \infty) \\ z^n(0) = z_0^n \\ y_{ob}^n = Hz^n + \eta^n \end{cases} \tag{3}$$

The term $E\zeta^n$ is the discretization of the contribution of the error ζ in the weak form of the dynamics. As to matrix η^n it corresponds to the discretized observation noise η.

Coupled Finite Dimensional System - Linear Case

z^n will stabilize around zero if obtained from the coupled system

$$\begin{aligned} &E(z^n)' = Az^n + F(z^n) + BKz_e^n + E\zeta^n \quad \text{in } (0,\infty) \\ &E(z_e^n)' = LHz^n + (A + BK - LH)z_e^n + H\eta^n \\ &z^n(0) = z_0^n \\ &z_e^n(0) = z_{e0}^n = z_0^n + \varepsilon \end{aligned} \tag{4}$$

where the feedback gain K and the filter L

$$K = -R^{-1}B_0^T P \in M_{m^c \times n} \tag{5}$$

$$\tag{6}$$

$$L = P_e H^T R_d^{-1} \in M_{n \times m^{ob}} \tag{7}$$

are obtained from solutions P and P_e of the AREs

Algebraic Riccati Equations

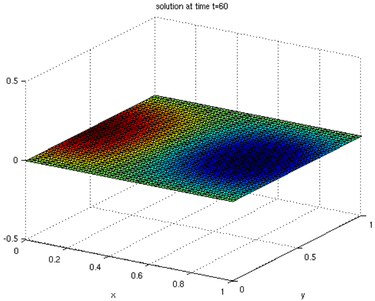
$$\begin{aligned} &P \in M_n,\ P^T = P,\ P \geq 0, \\ &A_0^T P + P A_0 - P B_0 R^{-1} B_0^T P + Q = 0 \end{aligned} \tag{8}$$

$$\begin{aligned} &P_e \in M_n,\ P_e^T = P_e,\ P_e \geq 0, \\ &A_0 P_e + P_e A_0^T - P_e C^T R_d^{-1} C P_e + Q_d = 0 \end{aligned} \tag{9}$$

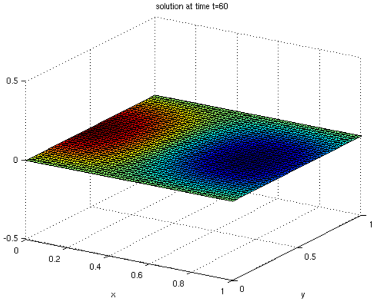
associated to two Linear Quadratic Optimal Control problems.

- $Q = E$ approximates $\int_\Omega z^2 ds$,
- $R = M_c$ approximates $\int_{\Gamma_c} u^2 ds$,
- R_d and Q_d correspond to the covariance fixed for the noise samples ζ and η possible weighted by some coefficients.

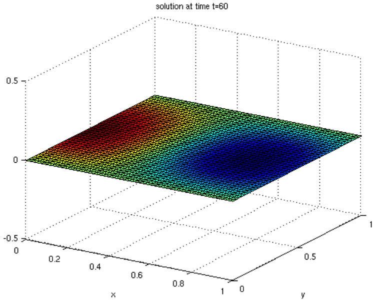
Uncontrolled solution for the linear problem with $\delta = 8$



Evolution of z^n solution for the linear problem with $\delta = 8$



Evolution of z^n solution for the linear problem with both noises and $\delta = 8$



Nonlinear Case

And what about the nonlinear case?

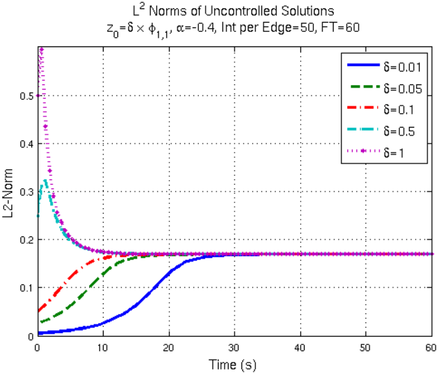


Figure: L^2 norm of uncontrolled solution for different initial values

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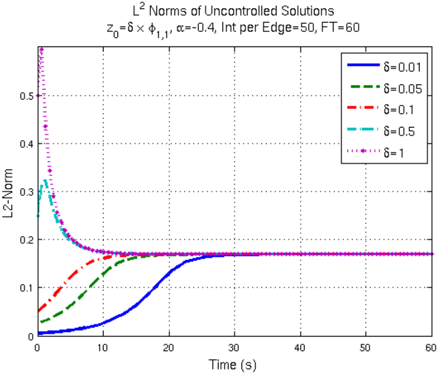

Figure: L^2 norm of uncontrolled solution for different initial values

Some answers can be found in the Hall. Welcome to the poster session!

Appendix - Nonlinear Case without n.

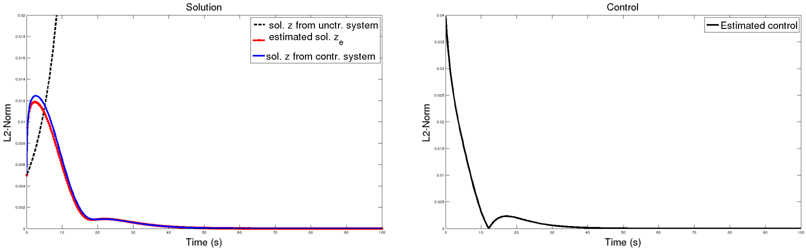

Figure: Evolution of the L^2 norm of the uncontrolled, estimated, controlled solution and control function for $\delta = 0.01$.

Appendix - Nonlinear Case Ob Noise

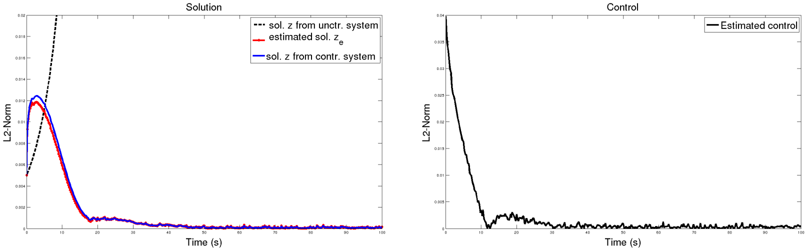

Figure: Evolution of the L^2 norm of the uncontrolled, estimated, controlled solution and control function for $\delta = 0.01$ when observation noise is present

Appendix - Nonlinear Case Ob Noise

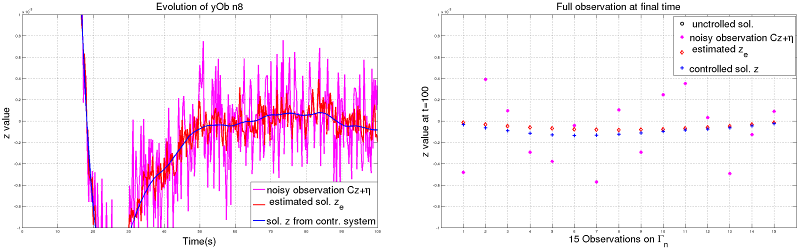

Figure: Evolution of observation y_8^n and value of the full observation at final time for $\delta = 0.01$ and η^N non-null