Control and Stabilization of fluid-structure systems

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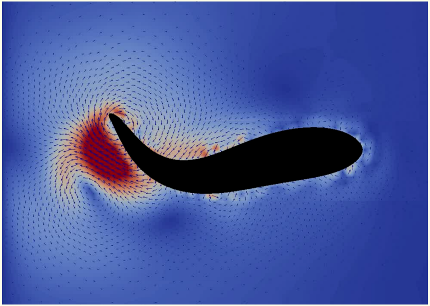
Outline

- Short Introduction on Fluid–Structure Systems
- Control as a swimming problem *in collaboration with Jorge San Martín (Chile) and Marius Tucsnak (Nancy)*
- Boundary Stabilization *in collaboration with Mehdi Badra (Toulouse)*

General Description



- Fluid:
  - inviscid and incompressible (particular case: potential fluid)
  - viscous and incompressible,
  - viscous and compressible, etc.
- Structure:
  - rigid body,
  - elastic solid,
  - aquatic animals, micro-organisms, etc.

Studied Problems

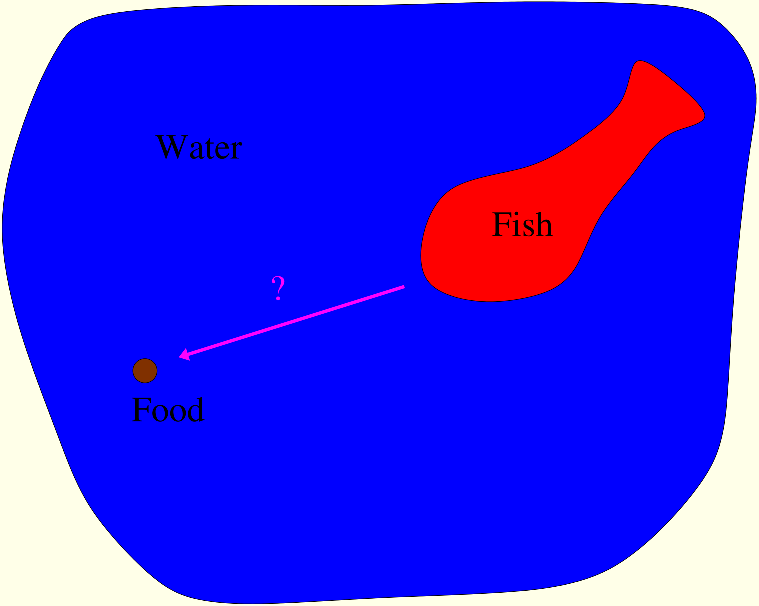
- Modeling (for instance: how to model a fish swimming ?)
- Theoretical Analysis (well-posedness, long time behavior, collisions, etc ...)
- Numerical Analysis (find numerical schemes, prove the convergence, numerical implementation)
- Control Problems (Control of both fluid and structure, self-propelled motions, etc ...)
- Inverse Problems (Detect a structure moving into a fluid, identify its shape, etc ...)

Difficulties

- The equations of the fluid are non linear and usually difficult to study even without structure (Navier–Stokes equations, Euler equations, etc.). The equations of the structure could also be difficult to solve.
- The equations of the fluid and the equations of the structure are strongly coupled.
- The fluid domain is variable.
- The fluid domain depends on the position of the structure which is part of the solution. The fluid domain is one of the unknowns of the problem!

Control as a swimming problem

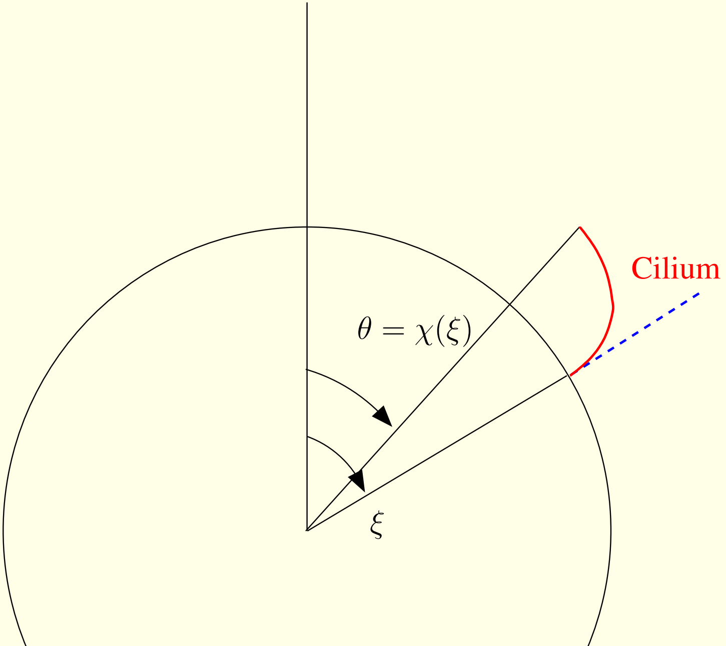
in collaboration with Jorge San Martín (Chile) and Marius Tucsnak (Nancy)


Water
Fish
?
Food

Swimming Control results

- San Martin-T.-Tucsnak (motion of ciliate micro-organism)
- Alouges-De Simone-Lefebvre (motion of the three-balls system in a Stokes fluid)
- Glass-Rosier (motion of a boat in an incompressible inviscid fluid),
- Chambrion-Munnier (deformable structure in a potential fluid)
- Lohéac-Scheid-Tucsnak (deformable ball in a Stokes fluid)

The Model



The Model

$$\left.\begin{aligned} -\nu\Delta v + \nabla p = 0 \\ \nabla \cdot v = 0 \end{aligned}\right\} \quad \text{in } \mathbb{R}^3 \setminus \mathcal{S}, \quad t > 0,$$

$$v = h' e_3 + u, \quad \text{on } \partial\mathcal{S}, \quad t > 0$$

$$\lim_{|x|\to\infty} v = 0, \quad t > 0$$

$$0 = -\int_{\partial\mathcal{S}} \sigma(v,p) n \; d\Gamma \cdot e_3, \quad t > 0$$

The Model

$$\theta = \chi(\xi, t) = \xi + \sum_{i=1}^{N} \alpha_i(t)\psi_i(\xi) \qquad (\xi \in [0, \pi],\ t \geqslant 0),$$

The "control" u is coming from the motion of the cilia:

$$u = u_\theta e_\theta$$

and

$$\partial_t \chi(t, \xi) = u_\theta(t, \chi(t, \xi)), \quad \chi(0, \xi) = \xi.$$

The Model

$$\dot{h}(t) = \sum_{i=1}^{N} \beta_i(t) F_i(\alpha_1(t), \ldots, \alpha_N(t)), \qquad (1)$$

$$\dot{\alpha}_i(t) = \beta_i(t) \qquad (i \in \{1, \ldots, N\}), \qquad (2)$$

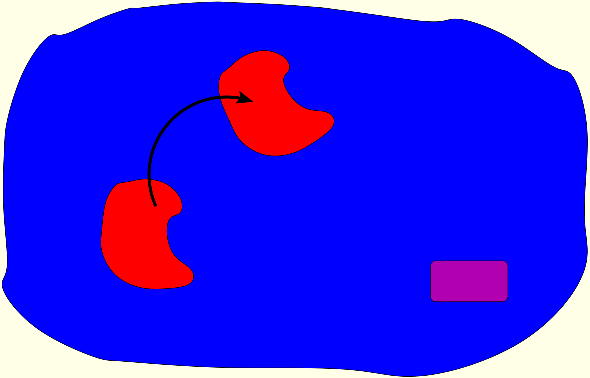
$$h(0) = 0, \qquad \alpha_i(0) = 0, \qquad (3)$$

Is it possible to have $h(T) = h_T$, and $\alpha_i(T) = 0$?
How to do it in an "efficient" way?

Feedback stabilization of a 1d fluid-structure system

in collaboration with Mehdi Badra (Toulouse)

Controllability results



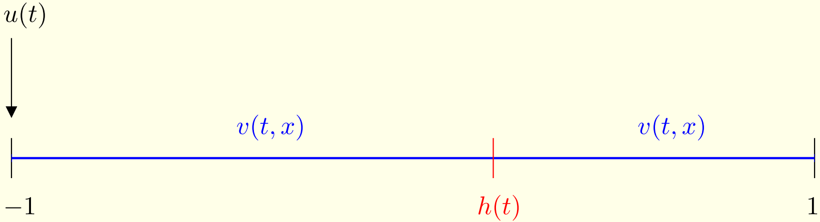
- 1d: Doubova and Fernandez-Cara, 2005
  Liu, T. and Tucsnak, 2012
- 2d: Imanuvilov and T., 2007, case of rigid ball
- 2d: Boulakia and Osses, 2008, rigid body with geometrical properties
- 3d: Boulakia and Guerrero, 2011

Stabilization results

- Raymond, 2009
- Lequeurre

Feedback Stabilization of a Fluid-Structure system (damped Euler-Bernoulli beam)

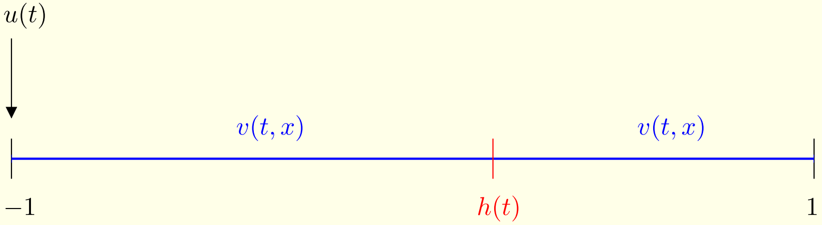
1d Stabilization Problem



- $h = h(t)$ trajectory of the particle
- $v = v(t,x)$ fluid velocity, defined in the fluid domain $[-1,1] \setminus \{h(t)\}$.
- $u = u(t)$ control of the system, velocity of the fluid at $x = -1$.

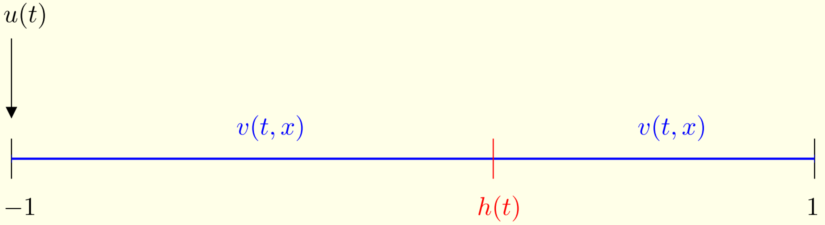
$$[f](x) = f(x^+) - f(x^-).$$

Feedback stabilization



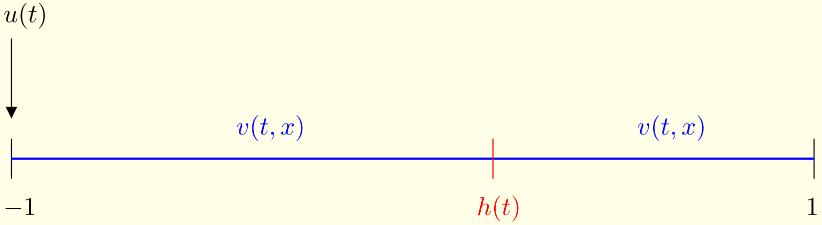
$$
\begin{cases}
v_t - v_{xx} + v v_x = f^S \qquad (t \geqslant 0, \quad x \in [-1,1] \setminus \{h(t)\}), \\
\\
v(t,-1) = a^S + u(t), \quad v(t,1) = b^S \qquad (t \geqslant 0), \\
\\
\end{cases}
$$

Feedback stabilization



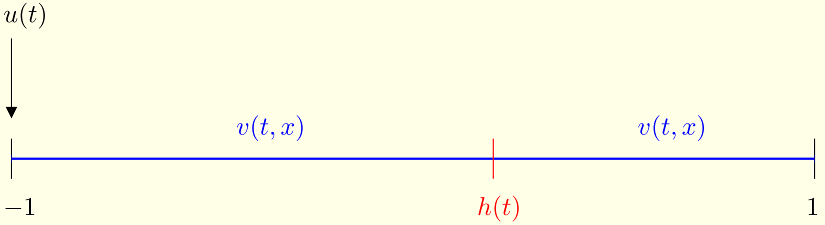
$$\begin{cases} v_t - v_{xx} + v v_x = f^S \qquad (t \geqslant 0, \quad x \in [-1,1] \setminus \{h(t)\}), \\ v(t,h(t)) = \dot{h}(t) \qquad (t \geqslant 0), \\ \\ v(t,-1) = a^S + u(t), \quad v(t,1) = b^S \qquad (t \geqslant 0), \\ \\ \end{cases}$$

Feedback stabilization



$$
\begin{cases}
v_t - v_{xx} + v v_x = f^S \qquad (t \geqslant 0, \quad x \in [-1,1] \setminus \{h(t)\}), \\
v(t,h(t)) = \dot{h}(t) \qquad (t \geqslant 0), \\
m\ddot{h}(t) = [v_x](t,h(t)) + m\ell^S \qquad (t \geqslant 0), \\
v(t,-1) = a^S + u(t), \quad v(t,1) = b^S \qquad (t \geqslant 0),
\end{cases}
$$

Feedback stabilization



$$\begin{cases} v_t - v_{xx} + vv_x = f^S \qquad (t \geqslant 0, \quad x \in [-1,1] \setminus \{h(t)\}), \\ v(t,h(t)) = \dot{h}(t) \qquad (t \geqslant 0), \\ m\ddot{h}(t) = [v_x](t,h(t)) + m\ell^S \qquad (t \geqslant 0), \\ v(t,-1) = a^S + u(t), \quad v(t,1) = b^S \qquad (t \geqslant 0), \\ h(0) = h_0, \quad \dot{h}(0) = h_1, \\ v(0,x) = v_0(x) \quad x \in [-1,1] \setminus \{h_0\}. \end{cases} \tag{4}$$

Stationary states

For f^S, ℓ^S, a^S and b^S given, find V^S and H^S such that

$$\left\{\begin{array}{c} -V^S_{yy} + V^S V^S_y = f^S \qquad (y \in (-1,1) \setminus \{H^s\}), \\ 0 = [V^S_y](H^S) + m\ell^S, \\ V^S(-1) = a^S, \quad V^S(1) = b^S, \\ V^S(H^S) = 0. \end{array}\right. \tag{5}$$

Feedback Stabilization result

Theorem

If

$$\|v_0 - V^S\|_{L^2((-1,1))} + |h_1| + |h_0 - H^S| \leqslant \mu$$

then there exists a solution (v, h) *with*

$$v(t,-1) = a^S + K\left(v(t,X) - V^S(y), \dot{h}(t), h(t)\right)$$

and

$$\|v(t) - V^S\|_{L^2(-1,1)} + |\dot{h}(t)| + |h(t) - H^S| \leqslant Ce^{-\sigma t}\left(\|v_0 - V^S\|_{L^2((-1,1))} + |h_1| + |h_0 - H^S|\right).$$

Change of variables

$$X(t,\cdot):(-1,1)\to(-1,1), \quad with \quad X(t,H^s)=h(t)$$

and

$$V(t,y) := v(t,X(t,y)).$$

This leads to the stabilization of the nonlinear system

$$\mathbf{Z}' = A\mathbf{Z} + Bu + F(\mathbf{Z}), \quad \mathbf{Z}(0)=\mathbf{Z}^0. \tag{6}$$

where

- $A:\mathcal{D}(A)\subset\mathcal{H}\to\mathcal{H}$ compact resolvent in the real Hilbert space $\mathcal{H}$ and generator of an analytic semigroup in $\mathcal{H}$;
- $B:\mathbb{R}\to[\mathcal{D}(A^*)]'$ is strictly relatively bounded.

Feedback stabilization of parabolic systems

Theorem

Assume the following unique continuation property

$$\forall \varepsilon \in \mathcal{D}(A^*), \quad \forall \lambda \in \mathbb{C}$$
$$A^*\varepsilon = \lambda\varepsilon \quad \text{and} \quad B^*\varepsilon = 0 \implies \varepsilon = 0.$$

Then there exists $K \in \mathcal{L}(\mathcal{H}; \mathbb{R})$ *such that the solution* $\mathbf{Z}$ *of* (6) *with* $u = K\mathbf{Z}$ *satisfies for* $\|\mathbf{Z}^0\|$ *small enough*

$$\|\mathbf{Z}(t)\| \leqslant \|\mathbf{Z}^0\| e^{-\sigma t}.$$

Application to our case

Is it possible to find a non null solution to the following problem?

$$\begin{cases} V_y^S(0)\varphi - \varphi_{yy} - V^S\varphi_y = 0, \quad y \in (0,1), \\ \varphi(0) = \varphi(1) = 0. \end{cases} \qquad (\star)$$

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If we multiply by φ, we obtain

$$V_y^S(0)\int_0^1 |\varphi|^2 \mathrm{dy} + \int_0^1 |\varphi_y|^2 \mathrm{dy} + \frac{1}{2}\int_0^1 V_y^S |\varphi|^2 \mathrm{dy} = 0.$$

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Using Poincaré inequality, we deduce

$$\lambda = V_y^S(0) \leqslant \frac{\|[F^S]^+\|_{L^\infty} - 2\pi^2}{3}, \quad F^S(y) = \int_0^y f^S(s)\mathrm{ds}.$$

Conclusion

- We obtain the feedback stabilizability of a fluid-structure problem in 1d, with only one boundary control, provided σ is not too large or there does not exist a solution to $(\star)$.

$$\begin{cases} V^S_y(0)\varphi - \varphi_{yy} - V^S\varphi_y = 0, \quad y \in (0,1), \\ \varphi(0) = \varphi(1) = 0. \end{cases} \qquad (\star)$$

- We are working on the same problem in several dimensions.

Fluid-Rigid body in 2d

$$\left.\begin{array}{l}\dfrac{\partial v}{\partial t} + (v \cdot \nabla)v - \nu \Delta v + \nabla p = f^S \\ \nabla \cdot v = 0\end{array}\right\} \quad \text{in } \mathcal{F}(a,\theta), \quad t > 0,$$

$$v = \ell + \omega (x-a)^{\perp}, \quad \text{on } \partial\mathcal{S}(a,\theta),$$
$$v = b^S + u, \quad \text{on } \partial\Omega.$$

$$M\ell' = -\int_{\partial\mathcal{S}(a,\theta)} \sigma(v,p)n \; d\Gamma + M\ell^S, \quad t > 0$$

$$I\omega' = -\int_{\partial\mathcal{S}(a,\theta)} (x-a)^{\perp} \cdot \sigma(v,p)n \; d\Gamma + Ik^S, \quad t > 0,$$

$$a' = \ell, \quad \theta' = \omega, \quad t > 0.$$

The result

Theorem
Assume (f^S, ℓ^S, k^S, b^S) *is associated to a stationary solution. If*

$$\|v^0 - V^S\|_{L^2(\Omega)} + |a^0 - a^S| + |\theta^0 - \theta^S| \leqslant \mu, \tag{7}$$

and if

$$u(t,x) = \sum_{j=1}^{K} u_j(t) v_j(x)$$

$$\textit{with} \quad u_j = \Pi_j((v \circ X - V^S, \ell, \omega, a - a^S, \theta - \theta^S))$$

then there exists a weak solution $(v, p, \ell, \omega, a, \theta)$ *satisfying*

$$\|v(t) - V^S\|_{L^2(\Omega)} + |a(t) - a^S| + |\theta(t) - \theta^S|$$
$$\leqslant Ce^{-\varsigma t}\left(\|v^0 - V^S\|_{L^2(\Omega)} + |a^0 - a^S| + |\theta^0 - \theta^S|\right).$$

Stationnay states

$$\left.\begin{array}{l}(V^S \cdot \nabla) V^S - \nu \Delta V^S + \nabla P^S = f^S \\ \nabla \cdot V^S = 0\end{array}\right\} \quad \text{in } \mathcal{F}(a^S, \theta^S), \tag{8}$$

$$V^S(x) = 0, \quad x \in \partial\mathcal{S}(a^S, \theta^S), \tag{9}$$

$$V^S(x) = b^S, \quad x \in \partial\Omega. \tag{10}$$

$$0 = -\int_{\partial\mathcal{S}(a^S,\theta^S)} \sigma(V^S, P^S) n \; d\Gamma + M\ell^S, \tag{11}$$

$$0 = -\int_{\partial\mathcal{S}(a^S,\theta^S)} (y - a^S)^{\perp} \cdot \sigma(V^S, P^S) n \; d\Gamma + Ik^S. \tag{12}$$

Change of variables

$$X(t,y) := y+\eta(y)\left[a(t) + (R_{\theta(t)} - I_2)y\right] \quad \text{and} \quad Y(t,\cdot) := X(t,\cdot)^{-1}.$$

We define

$$\begin{gathered}\tilde{v}(t,y) := \mathrm{Cof}(\nabla X(t,y))^* v(t, X(t,y)), \quad \tilde{p}(t,y) = p(t, X(t,y)), \\ \tilde{\ell}(t) := R_{-\theta(t)}\ell(t), \quad \tilde{\omega}(t) := \omega(t) \\ \tilde{a}(t) := \int_0^t R_{-\theta(s)}\ell(s)\, ds, \quad \tilde{\theta}(t) := \theta(t).\end{gathered} \tag{13}$$

The system after change of variables

$$
\begin{gathered}
[\mathbf{K}\partial_t \tilde{v}] - \nu[\mathbf{L}\tilde{v}] + [\mathbf{M}\tilde{v}] + [\mathbf{N}\tilde{v}] + [\mathbf{G}\tilde{p}] = \tilde{f}^S \quad \text{in } (0,T)\times\mathcal{F}, \\
\nabla\cdot\tilde{v} = 0 \quad \text{in } (0,T)\times\mathcal{F}, \\
\tilde{v} = \tilde{\ell} + \tilde{\omega} y^{\perp} \quad \text{on } (0,T)\times\partial\mathcal{S}, \\
\tilde{v} = b^S + u \quad \text{in } (0,T)\times\Omega, \\
M[R_\theta \tilde{\ell}]' = -\int_{\partial\mathcal{S}} \sigma(\tilde{v},\tilde{p})n \; d\Gamma, \quad t>0, \\
I\tilde{\omega}'(t) = -\int_{\partial\mathcal{S}} y^{\perp}\cdot\sigma(\tilde{v},\tilde{p})n \; d\Gamma, \quad t>0, \\
\tilde{a}' = \tilde{\ell}, \quad t>0, \\
\tilde{\theta} = \tilde{\omega}, \quad t>0.
\end{gathered}
$$

The system after of change of variables

We write

$$\tilde{v} = \tilde{w} + V^S, \quad \tilde{p} = \tilde{q} + P^S,$$

so that

$$[\mathbf{K}\partial_t \tilde{w}] - \nu[\mathbf{L}\tilde{w}] - \nu[\mathbf{L}V^S] + [\mathbf{M}\tilde{w}] + [\mathbf{M}V^S] \\ + [\mathbf{N}(\tilde{w} + V^S)] + [\mathbf{G}\tilde{q}] + [\mathbf{G}P^S] = \tilde{f}^S.$$

We can rewrite the above equation as

$$\partial_t \tilde{w} - \nu\Delta\tilde{w} - \nu[\mathbf{L}V^S] + [\mathbf{M}V^S] + [\mathbf{N}(\tilde{w} + V^S)] + \nabla\tilde{q} + [\mathbf{G}P^S] \\ = \tilde{f}^S + [(\mathrm{Id} - \mathbf{K})\partial_t \tilde{w}] + \nu[(\mathbf{L} - \Delta)\tilde{w}] - [\mathbf{M}\tilde{w}] + [(\nabla - \mathbf{G})\tilde{q}].$$

The system after of change of variables

After some calculation, we deduce

$$\partial_t \tilde{w} - \nu \Delta \tilde{w} + \Gamma[\tilde{a}\ \tilde{\theta}\ \tilde{\ell}\ \tilde{\omega}]^* + \tilde{w} \cdot \nabla V^S + V^S \cdot \nabla \tilde{w} + \nabla \tilde{q} = F(\mathbf{Z}) \quad \text{in } (0,\infty) \times \mathcal{F}, \tag{14}$$

$$\nabla \cdot \tilde{w} = 0 \quad \text{in } (0,\infty) \times \mathcal{F}, \tag{15}$$

$$\tilde{w} = \tilde{\ell} + \tilde{\omega} y^\perp \quad \text{on } (0,\infty) \times \partial\mathcal{S}, \tag{16}$$

$$\tilde{w} = u \quad \text{in } (0,\infty) \times \Omega, \tag{17}$$

$$M\tilde{\ell}'(t) = -\int_{\partial\mathcal{S}} \sigma(\tilde{w},\tilde{q})n \, d\Gamma + M\varepsilon_\ell(\mathbf{Z}), \quad t > 0, \tag{18}$$

$$I\tilde{\omega}'(t) = -\int_{\partial\mathcal{S}} y^\perp \cdot \sigma(\tilde{w},\tilde{q})n \, d\Gamma, \quad t > 0, \tag{19}$$

$$\tilde{a}' = \tilde{\ell}, \quad t > 0, \tag{20}$$

$$\tilde{\theta}' = \tilde{\omega}, \quad t > 0. \tag{21}$$

In the above system, we have set

$$\mathbf{Z} = [\tilde{w},\ \tilde{q},\ \tilde{\ell},\ \tilde{\omega},\ \tilde{a},\ \tilde{\theta}]^*,$$

and

$$\Gamma[\tilde{a}\ \tilde{\theta}\ \tilde{\ell}\ \tilde{\omega}]^* = \tilde{a}_1\Gamma_1 + \tilde{a}_2\Gamma_2 + \tilde{\theta}\Gamma_3 + \tilde{\ell}_1\Gamma_4 + \tilde{\ell}_2\Gamma_5 + \tilde{\omega}\Gamma_6.$$

Feedback stabilization of parabolic systems

$$\mathbf{X}' = A\mathbf{X} + Bu \quad \text{in} \;\; [\mathcal{D}(A^*)]', \quad \mathbf{X}(0) = \mathbf{X}^0, \tag{22}$$

where

- $A : \mathcal{D}(A) \subset \mathcal{H} \to \mathcal{H}$ compact resolvent in the real Hilbert space $\mathcal{H}$ and generator of an analytic semigroup in $\mathcal{H}$;
- $B : U \to [\mathcal{D}(A^*)]'$ is strictly relatively bounded.

There is a finite number of "unstable" modes: for any prescribed $\sigma > 0$, there are only N eigenvalues of A with real part strictly greater than $-\sigma$: λ_k, $k = 1, \ldots, N$ (N depending on σ).

General Theorem for Feedback stabilization

Theorem

Assume the following unique continuation property

$$\forall \varepsilon \in \mathcal{D}(A^*), \quad \forall k \in \{1, \dots, N\}$$
$$A^* \varepsilon = \bar{\lambda}_k \varepsilon \quad \textit{and} \quad B^* \varepsilon = 0 \implies \varepsilon = 0.$$

Then there exists $\Pi \in \mathcal{L}(\mathcal{H}; \mathbb{R}^K)$ *such that* $A + BB^*\Pi$ *is exponentially stable of order* $-\sigma$*: the solution* $\mathbf{X}$ *of* (22) *with* $u = K\mathbf{X}$ *satisfies*

$$\|\mathbf{X}(t)\| \leqslant \|\mathbf{X}^0\| e^{-\sigma t}.$$

K *can be chosen as the the maximum of the geometric multiplicities of the* λ_k, $k \in \{1, \dots, N\}$.

Unique continuation

$$\left\{\begin{array}{l} \lambda\varphi - \nu\Delta\varphi + \nabla\pi + (\nabla V^S)^*\varphi - V^S\cdot\nabla\varphi = 0, \\ m\lambda\xi + \displaystyle\int_{\partial\mathcal{S}} \sigma(\varphi,\pi)n\; d\Gamma + \begin{bmatrix} \displaystyle\int_{\mathcal{F}} \Gamma_1\cdot\varphi\; dy \\ \displaystyle\int_{\mathcal{F}} \Gamma_2\cdot\varphi\; dy \end{bmatrix} - b = 0, \\ I_0\lambda\zeta + \displaystyle\int_{\partial\mathcal{S}} y^\perp\cdot\sigma(\varphi,\pi)n\; d\Gamma + \int_{\mathcal{F}} \Gamma_3\cdot\varphi\; dy - \kappa = 0, \\ \lambda b + \displaystyle\int_{\mathcal{F}} \Gamma_4\cdot\varphi\; dy = 0, \\ \lambda\kappa + \displaystyle\int_{\mathcal{F}} \Gamma_5\cdot\varphi\; dy = 0. \end{array}\right. \tag{23}$$

and

$$\sigma(\varphi,\pi)n = 0 \quad \text{on } \partial\Omega. \tag{24}$$

Fixed Point Argument

$$\mathbf{F} \mapsto \mathbf{X}$$

where

$$\mathbb{P}\mathbf{X}' = A_\Pi \mathbb{P}\mathbf{X} + \mathbb{P}\mathbf{F},$$
$$(I - \mathbb{P})\mathbf{X} = (I - \mathbb{P})DB^*\Pi(\mathbf{X}),$$
$$\mathbf{X}(0) = \mathbf{X}^0.$$

$$\mathbf{X} = [\tilde{w}\ \tilde{\ell}\ \tilde{\omega}\ \tilde{a}\ \tilde{\ell}].$$

$$F(\mathbf{Z}) = [(\mathrm{Id} - K)\partial_t \tilde{w}] + \nu[(L - \Delta)\tilde{w}] + [(\nabla - G)\tilde{q}] + \ldots$$

Fixed point in

$$L^2(D(A_\Pi^{1/2})) \cap H^1(D((A^{1/2})^*)') \cap BC(H), \qquad (25)$$

and in

$$L^2_\chi(D(A_\Pi)) \cap H^1_\chi(H) \cap BC_\chi(D(A_\Pi^{1/2})). \qquad (26)$$

Conclusion

- We obtain the feedback stabilizability of a fluid-structure problem in 1d, with only one boundary control, provided σ is not too large or there does not exist a solution to $(\star)$.

- We obtain the feedback stabilizability of a fluid-structure problem in 2d for "mild solutions" provided the initial position and the final position of the rigid body is the same.

- We obtain the feedback stabilizability of a fluid-structure problem in 2d and 3d for "strong solutions" with dynamical controllers.