Introduction: control problem and its stochastic counterpart
Two-dimensional Navier–Stokes system
Main result and some ideas of its proof

Control and mixing for 2D Navier–Stokes equations with space-time localised force

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Control of Fluid–Structure Systems and Inverse Problem
Toulouse, 25–28 June 2012

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Abstract setting

Let H be a Hilbert space let $E \subset H$ be a closed subspace.
Consider the equaton

$$\partial_t u = F(u) + \eta(t), \quad u(t) \in H, \quad \eta(t) \in E, \quad (1)$$

supplemented with the initial condition

$$u(0) = u_0 \in H. \quad (2)$$

We assume that problem (1), (2) is well posed and denote
by $S_t(u_0, \eta)$ its resolving operator.

Control problem. Assuming that η is a control with range in E,
investigate various controllability properties, such as exact or
approximate controllability, stabilisation of a given solution, etc.

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Random dynamical system. Let λ be a probability measure on $L^2(0, 1; E)$ and let $\{\eta_k\}$ be a sequence of i.i.d. random variables in $L^2(0, 1; E)$ such that $\mathcal{D}(\eta_k) = \lambda$, where $\mathcal{D}(\xi)$ denotes the law of a random variable ξ.

Define the mapping

$$\varphi_k : u_0 \mapsto u(k), \quad k \geq 0,$$

where $u(t) = S_t(u_0, \eta)$ is the solution of (1), (2) with η such that

$$\eta(t) = \eta_k(t - k + 1) \quad \text{for } t \in (k - 1, k), \quad k \geq 1.$$

Then $\{\varphi_k : H \to H, k \geq 0\}$ form an RDS in H, and the problem is to study the large-time asymptotics of its trajectories.

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General principle

We shall use the example of 2D Navier–Stokes equations to justify the following general principle:

"Right" controllability properties imply ergodicity

Ergodicity means the existence of a probability measure μ on H such that

$$\mathcal{D}(\varphi_k(u_0)) \rightharpoonup \mu \quad \text{as } k \to \infty. \tag{3}$$

"Right" controllability properties are the following ones:

- **Approximate controllability in infinite time;**
- **Squeezing:** *for any $v, v' \in H$ there is a finite-dimensional map $\Psi_{v,v'}$ acting in $L^2(0, 1; E)$ such that*

$$\left\| S_1(v, \zeta) - S_1(v', \zeta + \Psi_{v,v'}(\zeta)) \right\| \le \frac{1}{2} \|v - v'\|. \tag{4}$$

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Initial-boundary value problem
Problem of ergodicity

Equations and phase space

Let $D \subset \mathbb{R}^2$ be a domain with a smooth boundary ∂D. Consider the problem

$$\partial_t u + \langle u, \nabla \rangle u - \nu \Delta u + \nabla p = f(t, x), \quad \operatorname{div} u = 0, \quad x \in D, \quad (5)$$

$$u(0, x) = u_0(x), \quad u\big|_{\partial D} = 0. \quad (6)$$

Here $u = (u_1, u_2)$ and p are unknown velocity field and pressure, $\nu > 0$ is the viscosity, f is an external force, and

$$\langle u, \nabla \rangle = u_1(t, x) \partial_1 + u_2(t, x) \partial_2.$$

Problem (5), (6) is well posed in the space

$$H = \{u \in L^2(D, \mathbb{R}^2) : \operatorname{div} u = 0 \text{ in } D, \langle u, \boldsymbol{n} \rangle = 0 \text{ on } \partial D\},$$

where $\boldsymbol{n}$ stands for the unit normal to the boundary ∂D.

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Random force

Let $Q \subset (0,1) \times D$ be an open subset and let $\{\eta_k\}$ be a sequence of i.i.d. random variables in the space $H_0^1(Q, \mathbb{R}^2)$ regarded as a subspace in $L^2((0,1) \times D, \mathbb{R}^2)$. Assume that

$$f = h + \eta, \quad h \in H, \quad \eta = \sum_{k=1}^{\infty} I_{(k-1,k)}(t)\,\eta_k(t-k+1, x),$$

and define $\varphi_k : H \to H$ by the relation

$$\varphi_k(u_0) = u(k, \cdot), \quad u(t,x) = S_t(u_0, \eta) \text{ is the solution of (5), (6)}.$$

The independence of η_k ensures that the trajectories of the RDS φ_k are Markov processes in H, while the k-independence of their law implies that those Markov processes are homogeneous in time.

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Metric in the space of measures and ergodicity

Define the Kantorovich–Wasserstein distance (metrising the weak convergence) on the space $\mathcal{P}(H)$ of probabilities on H:

$$\|\mu_1 - \mu_2\|_L^* = \sup_{\|g\|_L \leq 1} \left| \int_H g \, d\mu_1 - \int_H g \, d\mu_2 \right|, \quad \mu_1, \mu_2 \in \mathcal{P}(H),$$

where for a function $g : H \to \mathbb{R}$ we set

$$\|g\|_L = \sup_{u \in H} |g(u)| + \sup_{0 < \|u-v\|_H \leq 1} \frac{|g(u) - g(v)|}{\|u - v\|_H}.$$

Problem of ergodicity. Find a measure $\mu \in \mathcal{P}(H)$ such that

$$\|\mathcal{D}(\varphi_k(u_0)) - \mu\|_L^* \leq C \, (1 + \|u_0\|_H) e^{-\gamma k}, \quad k \geq 0, \qquad (7)$$

where C and γ are positive constants not depending on u_0.

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Formulation
"Right" controllability properties

Formulation of the main result

We first specify the structure of random variables η_k. Let $\chi \in C_0^\infty(Q)$ be a nonzero function and let $\{\varphi_j\} \subset H^1(Q, \mathbb{R}^2)$ be a orthonormal basis in $L^2(Q, \mathbb{R}^2)$. We assume that

$$\eta_k(t, x) = \sum_{j=1}^\infty b_j \xi_{jk} \psi_j(t, x), \quad \psi_j = \chi \varphi_j, \tag{8}$$

where $b_j \in \mathbb{R}$ go to zero sufficiently fast and ξ_{jk} are independent random variables such that

$$\mathcal{D}(\xi_{jk}) = \rho_j(r) dr, \quad \rho_j \in C_0^1([-1, 1]).$$

Main Theorem

Let $h \in H^1(D, \mathbb{R}^2)$. Then there are $\varepsilon > 0$ and $N \geq 1$ depending on $\nu > 0$ such that the RDS $\{\varphi_k\}$ is ergodic, provided that

$$\|h\|_H \leq \varepsilon, \qquad b_j \neq 0 \quad \text{for} \quad 1 \leq j \leq N. \tag{9}$$

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Sufficient condition for ergodicity

The existence and uniqueness of a measure $\mu \in \mathcal{P}(H)$ and its stability (without the rate of convergence) are consequences of the following two properties in which $\hat{u} \in H$ is a given function:

- **Recurrence.** For any $d > 0$ and $u_0 \in H$, the random time

$$\tau(u_0) = \min\{k \geq 0 : \|\varphi_k(u_0) - \hat{u}\|_H \leq d\}$$

  is almost surely finite.

- **Stability.** There is a continuous function $\varkappa(d)$ defined for $d \in [0, 1]$ and going to zero with d such that

$$\sup_{\|u_0 - \hat{u}\|_H \leq d} \left\| \mathcal{D}(\varphi_k(u_0)) - \mathcal{D}(\varphi_k(\hat{u})) \right\|_L^* \leq \varkappa(d). \tag{10}$$

The first property follows from the *approximate controllability to $\hat{u}$ in infinite time*, while the second can be established by combining the squeezing property with a general result on the transformation of measures by smooth mappings.

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Approximate controllability in infinite time

Let us denote by $\mathcal{K}$ the support of the law for η_k. Thus, $\mathcal{K}$ is a compact subspace in $H_0^1(Q, \mathbb{R}^2)$. Suppose there is $\hat{u} \in H$ for which the following properties hold.

Approximate controllability. *For any $R, d > 0$ there is an integer $l \geq 1$ such that, given $u_0 \in B_H(R)$, we can find controls $\zeta_1, \ldots, \zeta_l \in \mathcal{K}$ satisfying the inequality*

$$\|S_l(u_0; \zeta_1, \ldots, \zeta_l) - \hat{u}\|_H \leq d, \tag{11}$$

where $S_l(u_0; \zeta_1, \ldots, \zeta_k)$ stands for the restriction at $t = l$ of the solution for the Navier–Stokes system with the right-hand side

$$f(t, x) = h(x) + \sum_{k=1}^{l} I_{(k-1,k)}(t)\, \zeta_k(t - k + 1, x).$$

This property is well understood due to the works of *Coron* (1995-1996), *Fursikov–Imanuvilov* (1996-2000), and others. It is trivially satisfied when $\|h\| \ll 1$.

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Squeezing

Let $\mathcal{E}_m \subset H_0^1(Q, \mathbb{R}^2)$ be the vector span of $\psi_j(t, x)$, $j = 1, \ldots, m$, and $B_X(R)$ be the closed ball in X of radius R centred at zero.

Squeezing. $\forall\, R > 0$ there is $d_0 > 0$, $m \geq 1$, and a mapping

$$\Phi : B_H(R) \times B_{H_0^1(Q,\mathbb{R}^2)}(R) \to \mathcal{L}(H, \mathcal{E}_m)$$

(where $\mathcal{L}(H, \mathcal{E}_m)$ is the space of continuous linear operators from H to $\mathcal{E}_m$) with the following properties:

- **Contraction.** For any $u_0, u_0' \in B_H(R)$ with $\|u_0 - u_0'\| \leq d_0$ and any $\zeta \in B_{H_0^1(Q,\mathbb{R}^2)}(R)$ we have

$$\left\| S_1(u_0, \zeta) - S_1\big(u_0', \zeta + \Phi(u_0, \zeta)(u_0' - u_0)\big) \right\| \leq \frac{1}{2}\|u_0 - u_0'\|. \tag{12}$$

- **Regularity.** The mapping $\Phi(u_0, \zeta)$ is regular and uniformly Lipschitz with respect to its arguments.