On the identifiability of a rigid body moving in a stationary viscous fluid

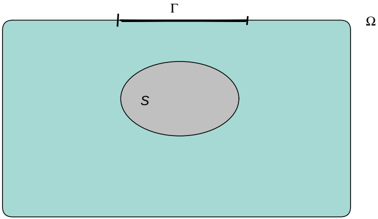
Erica L. Schwindt

Institut E. Cartan. Université de Lorraine

June 26^{th} 2012

Introduction

Suppose an unknown body $\mathcal{S}$ is immersed in a fluid, and the body and the fluid are contained in a fixed domain Ω. Assume Γ is a non empty open subset of $\partial\Omega$ where we can measure $\boldsymbol{\sigma}(\mathbf{u}, p)\, \mathbf{n}_{|\Gamma}$, at some time $t_0 > 0$.



Is it possible to recover $\mathcal{S}$?.

Formulation of the problem

We assume that the structure is a rigid body so that it can be described by its center of mass $\mathbf{a}(t) \in \mathbb{R}^3$ and by its orientation $\mathbf{Q}(t) \in SO_3(\mathbb{R})$:

$$\mathcal{S}(t) := \mathcal{S}(\mathbf{a}(t), \mathbf{Q}(t)),$$

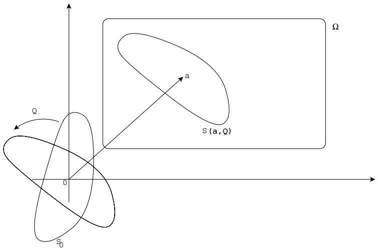
with

$$\mathcal{S}(\mathbf{a}, \mathbf{Q}) := \mathbf{Q}\mathcal{S}_0 + \mathbf{a}.$$

and

$$\mathcal{F}(\mathbf{a}, \mathbf{Q}) := \Omega \setminus \overline{\mathcal{S}(\mathbf{a}, \mathbf{Q})} = \mathcal{F}(\mathbf{a}(t), \mathbf{Q}(t))$$

is a smooth non empty domain.



Formulation of the problem

$$\begin{aligned} -\operatorname{div}(\boldsymbol{\sigma}(\mathbf{u}, p)) &= \mathbf{0} \quad \text{in } \mathcal{F}(t), \quad t \in (0, T), \\ \operatorname{div}(\mathbf{u}) &= 0 \quad \text{in } \mathcal{F}(t), \quad t \in (0, T), \end{aligned}$$

Formulation of the problem

$$\begin{aligned} -\operatorname{div}(\boldsymbol{\sigma}(\mathbf{u}, p)) &= \mathbf{0} \quad \text{in } \mathcal{F}(t), \quad t \in (0, T), \\ \operatorname{div}(\mathbf{u}) &= 0 \quad \text{in } \mathcal{F}(t), \quad t \in (0, T), \end{aligned}$$

$$\begin{aligned} \mathbf{u} &= \boldsymbol{\ell} + \boldsymbol{\omega} \times (\mathbf{x} - \mathbf{a}) \quad \text{on } \partial\mathcal{S}(t), \quad t \in (0, T), \\ \mathbf{u} &= \mathbf{u}_* \quad \text{on } \partial\Omega, \quad t \in (0, T), \end{aligned}$$

Formulation of the problem

$$
\begin{aligned}
-\operatorname{div}(\boldsymbol{\sigma}(\mathbf{u},p)) = \mathbf{0} \quad & \text{in } \mathcal{F}(t), \quad t \in (0,T), \\
\operatorname{div}(\mathbf{u}) = 0 \quad & \text{in } \mathcal{F}(t), \quad t \in (0,T),
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{u} = \boldsymbol{\ell} + \boldsymbol{\omega} \times (\mathbf{x} - \mathbf{a}) \quad & \text{on } \partial\mathcal{S}(t), \quad t \in (0,T), \\
\mathbf{u} = \mathbf{u}_* \quad & \text{on } \partial\Omega, \quad t \in (0,T),
\end{aligned}
$$

$$
\begin{aligned}
\int_{\partial\mathcal{S}(t)} \boldsymbol{\sigma}(\mathbf{u},p)\mathbf{n}\, d\gamma_{\mathbf{x}} = \mathbf{0} \quad & t \in (0,T), \\
\int_{\partial\mathcal{S}(t)} (\mathbf{x} - \mathbf{a}) \times \boldsymbol{\sigma}(\mathbf{u},p)\mathbf{n}\, d\gamma_{\mathbf{x}} = \mathbf{0} \quad & t \in (0,T),
\end{aligned}
$$

Formulation of the problem

$$
\begin{aligned}
-\operatorname{div}(\boldsymbol{\sigma}(\mathbf{u}, p)) = \mathbf{0} \quad & \text{in } \mathcal{F}(t), \quad t \in (0, T), \\
\operatorname{div}(\mathbf{u}) = 0 \quad & \text{in } \mathcal{F}(t), \quad t \in (0, T),
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{u} = \boldsymbol{\ell} + \boldsymbol{\omega} \times (\mathbf{x} - \mathbf{a}) \quad & \text{on } \partial\mathcal{S}(t), \quad t \in (0, T), \\
\mathbf{u} = \mathbf{u}_* \quad & \text{on } \partial\Omega, \quad t \in (0, T),
\end{aligned}
$$

$$
\begin{aligned}
\int_{\partial\mathcal{S}(t)} \boldsymbol{\sigma}(\mathbf{u}, p)\mathbf{n} \, d\gamma_{\mathbf{x}} = \mathbf{0} \quad & t \in (0, T), \\
\int_{\partial\mathcal{S}(t)} (\mathbf{x} - \mathbf{a}) \times \boldsymbol{\sigma}(\mathbf{u}, p)\mathbf{n} \, d\gamma_{\mathbf{x}} = \mathbf{0} \quad & t \in (0, T),
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{a}' = \boldsymbol{\ell} \quad & t \in (0, T), \\
\mathbf{Q}' = \mathbb{S}(\boldsymbol{\omega})\mathbf{Q} \quad & t \in (0, T), \quad (\mathbb{S}(\boldsymbol{\omega})\mathbf{z} = \boldsymbol{\omega} \times \mathbf{z}, \ \forall \mathbf{z} \in \mathbb{R}^3) \\
\mathbf{a}(0) = \mathbf{a}_0, & \\
\mathbf{Q}(0) = \mathbf{Q}_0. &
\end{aligned}
$$

Formulation of the problem

We assume

$$\mathbf{u} = \mathbf{u}_* \quad \text{on } \partial\Omega$$

where $\mathbf{u}_*$ is a given velocity satisfying the compatibility condition

$$\int_{\partial\Omega} \mathbf{u}_* \cdot \mathbf{n} \; d\boldsymbol{\gamma}_{\mathbf{x}} = 0.$$

Now, we can introduce the following operator called *Poincaré–Steklov operator*

$$\boldsymbol{\Lambda}_{\mathcal{S}}(\mathbf{u}_*) := \boldsymbol{\sigma}(\mathbf{u}, p)n \quad \text{on } \Gamma.$$

Our goal is to prove the injectivity of this operator, in the sense,

$$\boldsymbol{\sigma}\left(\mathbf{u}^{(1)}, p^{(1)}\right)\mathbf{n}_{|\Gamma} = \boldsymbol{\sigma}\left(\mathbf{u}^{(2)}, p^{(2)}\right)\mathbf{n}_{|\Gamma} \quad \Rightarrow \quad \mathcal{S}^{(1)} = \mathcal{S}^{(2)}.$$

Main results

Theorem (well-posedness)

Assume $\mathbf{u}_* \in \mathbf{H}^{3/2}(\partial\Omega)$, $\mathcal{S}_0$ *is a smooth non empty domain and assume* $(\mathbf{a}_0, \mathbf{Q}_0) \in \mathbb{R}^3 \times SO_3(\mathbb{R})$ *is such that* $\overline{\mathcal{S}(\mathbf{a}_0, \mathbf{Q}_0)} \subset \Omega$ *and* $\mathcal{F}(\mathbf{a}_0, \mathbf{Q}_0)$ *is a smooth non empty domain. Then there exist a maximal time* $T_* > 0$ *and a unique solution*

$$(\mathbf{a}, \mathbf{Q}) \in \mathbf{C}^1([0, T_*); \mathbb{R}^3 \times SO_3(\mathbb{R})), \quad (\boldsymbol{\ell}, \boldsymbol{\omega}) \in \mathbf{C}([0, T_*); \mathbb{R}^3 \times \mathbb{R}^3),$$

$$(\mathbf{u}, p) \in \mathbf{C}\left([0, T_*); \mathbf{H}^2(\mathcal{F}(\mathbf{a}(t), \mathbf{Q}(t))) \times H^1(\mathcal{F}(\mathbf{a}(t), \mathbf{Q}(t)))/\mathbb{R}\right)$$

satisfying the previous system. Moreover one of the following alternatives holds:

- $T_* = +\infty$;
- $\lim_{t \to T_*} \operatorname{dist}\left(\mathcal{S}(\mathbf{a}(t), \mathbf{Q}(t)), \partial\Omega\right) = 0.$

Main results

Let us take two smooth non empty domains $\mathcal{S}_0^{(1)}$, $\mathcal{S}_0^{(2)}$ and we consider $\left(\mathbf{a}_0^{(1)}, \mathbf{Q}_0^{(1)}\right)$, $\left(\mathbf{a}_0^{(2)}, \mathbf{Q}_0^{(2)}\right) \in \mathbb{R}^3 \times SO_3(\mathbb{R})$ such that

$$\overline{\mathcal{S}^{(1)}\left(\mathbf{a}_0^{(1)}, \mathbf{Q}_0^{(1)}\right)} \subset \Omega \quad \text{and} \quad \overline{\mathcal{S}^{(2)}\left(\mathbf{a}_0^{(2)}, \mathbf{Q}_0^{(2)}\right)} \subset \Omega.$$

Applying the previous theorem, we deduce that for any $\mathbf{u}_* \in \mathbf{H}^{3/2}(\partial\Omega)$, there exist $T_*^{(1)} > 0$ $\left(\text{respectively } T_*^{(2)} > 0\right)$ and a unique solution $\left(\mathbf{a}^{(1)}, \mathbf{Q}^{(1)}, \ell^{(1)}, \boldsymbol{\omega}^{(1)}, \mathbf{u}^{(1)}, p^{(1)}\right)$ $\left(\text{respectively } \left(\mathbf{a}^{(2)}, \mathbf{Q}^{(2)}, \ell^{(2)}, \boldsymbol{\omega}^{(2)}, \mathbf{u}^{(2)}, p^{(2)}\right)\right)$ of our system in $\left[0, T_*^{(1)}\right)$ $\left(\text{respectively in } \left[0, T_*^{(2)}\right)\right)$.

Main results

Theorem (identifiability)

Assume that $\mathbf{u}_* \in \mathbf{H}^{3/2}(\partial\Omega)$ *and that* $\mathbf{u}_*$ *is not the trace of a rigid velocity on* Γ*. Assume also that* $\mathcal{S}_0^{(1)}$, $\mathcal{S}_0^{(2)}$ *are convex. If there exists* $0 < t_0 < \min\left(T_*^{(1)}, T_*^{(2)}\right)$ *such that*

$$\boldsymbol{\sigma}\left(\mathbf{u}^{(1)}(t_0), p^{(1)}(t_0)\right) \mathbf{n}_{|\Gamma} = \boldsymbol{\sigma}\left(\mathbf{u}^{(2)}(t_0), p^{(2)}(t_0)\right) \mathbf{n}_{|\Gamma}$$

then there exists $\mathbf{R} \in SO_3(\mathbb{R})$ *such that*

$$\mathbf{R}\mathcal{S}_0^{(1)} = \mathcal{S}_0^{(2)}$$

and

$$\mathbf{a}_0^{(1)} = \mathbf{a}_0^{(2)}, \qquad \mathbf{Q}_0^{(1)} = \mathbf{Q}_0^{(2)}\mathbf{R}.$$

In particular, $T_*^{(1)} = T_*^{(2)}$ *and* $\mathcal{S}^{(1)}(t) = \mathcal{S}^{(2)}(t) \quad \left(t \in \left[0, T_*^{(1)}\right)\right)$.

Main results

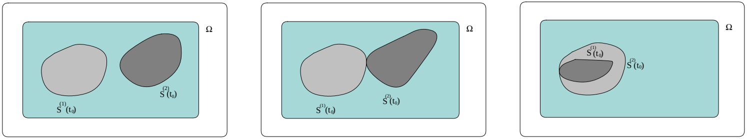
Idea of proof.
In order to prove Theorem of identifiability, we reason by contradiction and we assume that there exists $0 < t_0 < \min\left(T_*^{(1)}, T_*^{(2)}\right)$, such that

$$\boldsymbol{\sigma}\left(\mathbf{u}^{(1)}(t_0), p^{(1)}(t_0)\right)\ \mathbf{n}_{|\Gamma} = \boldsymbol{\sigma}\left(\mathbf{u}^{(2)}(t_0), p^{(2)}(t_0)\right)\ \mathbf{n}_{|\Gamma}$$

and $\mathcal{S}^{(1)}(t_0) \neq \mathcal{S}^{(2)}(t_0)$.
In that case, since $\mathcal{S}^{(1)}(t_0)$ and $\mathcal{S}^{(2)}(t_0)$ are convex sets, we have

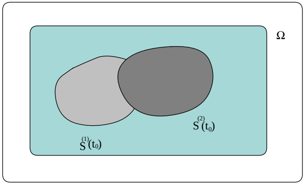
- $\partial\mathcal{S}^{(1)}(t_0) \cap \partial\mathcal{S}^{(2)}(t_0)$ is included in a line

This case can be split into the 3 following subcases



Main results

- $\partial\mathcal{S}^{(1)}(t_0) \cap \partial\mathcal{S}^{(2)}(t_0)$ contains 3 noncollinear points.



By using the unique continuation result for the Stokes equations due to Fabre and Lebeau, we prove that neither of the above case may be possible. □

Thanks for your attention !