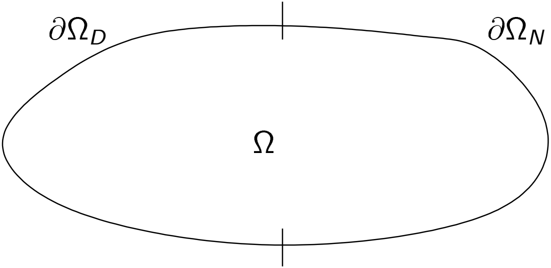
Carleman estimate for Zaremba boundary condition

Pierre Cornilleau and Luc Robbiano
Lycée du parc des Loges, Évry
and
Université de Versailles Saint-Quentin
LMV, UMR CNRS 8100

PROBLEM



$$\begin{cases} (\partial_t^2 - \Delta)u = 0 \text{ in } \Omega \\ \partial_n u + a(x)\partial_t u = 0 \text{ on } \partial\Omega_N \\ u = 0 \text{ on } \partial\Omega_D \end{cases}$$

where $a(x) \geq 0$, $a \not\equiv 0$, and a supported in $\partial\Omega_N$.

KNOWN RESULTS

$$\begin{cases} (\partial_t^2 - \Delta + a(x)\partial_t)u = 0 \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega. \end{cases}$$

(Lebeau 93)

$$\begin{cases} (\partial_t^2 - \Delta)u = 0 \text{ in } \Omega, \\ \partial_n u + a(x)\partial_t u = 0 \text{ on } \partial\Omega. \end{cases}$$

(Lebeau-Robbiano 95)

$$E(u(t)) = \frac{1}{2}\int_\Omega |\partial_t u(t)|^2 + |\partial_x u(t)|^2 dx.$$

We have $E(u(t)) \leq \dfrac{C}{\log^2(2+t)}$, (for "smooth" u).

Remind: $a(x) \geq 0$ and $a \not\equiv 0$.

RESULTS

Theorem

For Zaremba problem we have $E(u(t)) \leq \dfrac{C}{\log^2(2+t)}$.

Related result: Let v solution of

$$\begin{cases} (\Delta + \lambda^2)v = f_0 \text{ in } \Omega \\ (\partial_n + i\lambda a(x))v = f_1 \text{ on } \partial\Omega_N \\ v = 0 \text{ on } \partial\Omega_D \end{cases}$$

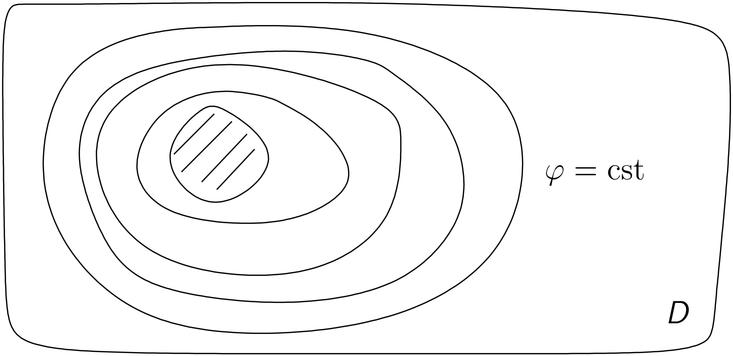
Theorem (Burq 98)

If there exists $C > 0$ *such that* $\|v\| \leq Ce^{C|\lambda|}(\|f_0\|_{L^2} + |f_1|_{L^2})$ *then we have* $E(u(t)) \leq \dfrac{C}{\log^2(2+t)}$.

This means estimate on resolvent $\Rightarrow$ stabilisation estimate.

WHAT IS CARLEMAN ESTIMATE?

φ a real function.



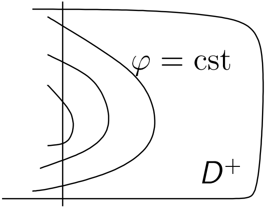
Carleman estimate has the following form,

$\exists h_0 > 0,\ C > 0$, $\forall u$ supported in D, $\forall h \in (0, h_0]$

$$h^{1/2} \| e^{\varphi/h} u \|_{L^2(D)} + h^{3/2} \| e^{\varphi/h} \nabla u \|_{L^2(D)} \leq C \| e^{\varphi/h} h^2 \Delta u \|_{L^2(D)}.$$

CARLEMAN ESTIMATES AT THE BOUNDARY (I)

First without boundary conditions,

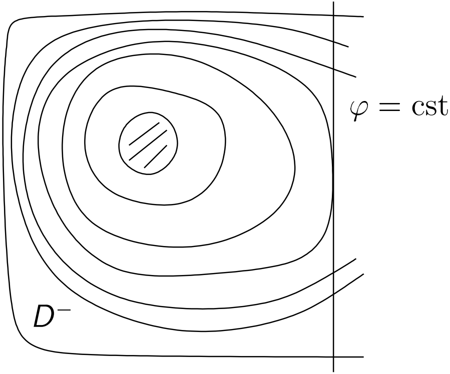


$\exists h_0 > 0,\ C > 0,\ \forall u$ supported in D, $\forall h \in (0, h_0]$

$$h^{1/2}\|e^{\varphi/h}u\|_{L^2(D^+)} + h^{3/2}\|e^{\varphi/h}\nabla u\|_{L^2(D^+)}$$
$$\leq C\|e^{\varphi/h}h^2\Delta u\|_{L^2(D^+)} + Ch^{1/2}|e^{\varphi/h}u|_{L^2(\partial D^+)} + Ch^{3/2}|e^{\varphi/h}\nabla u|_{L^2(\partial D^+)}.$$

CARLEMAN ESTIMATES AT THE BOUNDARY (II)

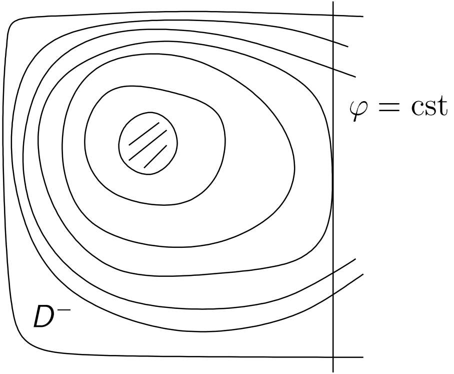
With boundary conditions,



$$h^{1/2}\|e^{\varphi/h}u\|_{L^2(D^-)} + h^{3/2}\|e^{\varphi/h}\nabla u\|_{L^2(D^-)} + Ch^{1/2}|e^{\varphi/h}u|_{L^2(\partial D^-)}$$
$$+Ch^{3/2}|e^{\varphi/h}\nabla u|_{L^2(\partial D^-)} \leq C\|e^{\varphi/h}h^2\Delta u\|_{L^2(D^-)} + C|e^{\varphi/h}Bu|_{L^2(\partial D^-)},$$

where B is a boundary condition, for instance, $Bu = u_{|\partial D^-}$ or $Bu = \partial_n u_{|\partial D^-}$.

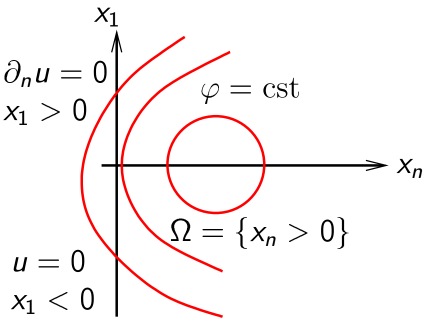
CARLEMAN FOR ZAREMBA BOUNDARY CONDITION



$$h^{1/2}\|e^{\varphi/h}u\|_{L^2(D^-)} + h^{3/2}\|e^{\varphi/h}\nabla u\|_{L^2(D^-)} + Ch^{1/2}|e^{\varphi/h}u|_{H^{1/2}(\partial D^-)}$$
$$+ Ch^{3/2}|e^{\varphi/h}\partial_{x_n}u|_{H^{-1/2}(\partial D^-)} \leq C\|e^{\varphi/h}h^2\Delta u\|_{L^2(D^-)} + C|e^{\varphi/h}Bu|_H,$$

where B is the Zaremba boundary condition and H adapted Hilbert space.

CONJUGAISON BY THE WEIGHT



We set $v = e^{\varphi/h} u$.
We have

$$h^2 e^{\varphi/h} \Delta u = h^2 e^{\varphi/h} \Delta(e^{-\varphi/h} v) = P_\varphi v,$$

where the semi-classical symbol of P_φ is given by

$$p_\varphi(x, \xi) = \xi_n^2 + 2i\varphi'_{x_n} \xi_n + q_2(x, \xi') + 2iq_1(x, \xi') \text{ where}$$

$$q_2(x, \xi') = |\xi'|^2 - |\partial_x \varphi(x)|^2$$

$$q_1(x, \xi') = \xi' \partial_{x'} \varphi(x)$$

THE PROPERTIES OF ROOTS

Informations on the roots: important for boundary value problems.
$p_\varphi(x,\xi) = \xi_n^2 + 2i\varphi'_{x_n}\xi_n + q_2(x,\xi') + 2iq_1(x,\xi')$.
There exists a real root if and only if

$\xi_n^2 + q_2(x,\xi') = 0$ and $\varphi'_{x_n}\xi_n + q_1(x,\xi') = 0$, thus

$$\mu(x,\xi') = q_2(x,\xi') + \frac{q_1^2(x,\xi')}{\varphi'^2_{x_n}(x)} = 0.$$

The roots are simple if

$\mu(x,\xi') > -(\varphi'_{x_n}(x))^2$ and in this case,

$\operatorname{Im}\rho_1(x,\xi') > -\varphi'_{x_n}(x) > \operatorname{Im}\rho_2(x,\xi')$,

where ρ_j are the roots.

We distinguish two cases, $\mu < 0$ and $\mu > -(\varphi'_{x_n})^2$.

CASE $\mu < 0$

We have $\operatorname{Im} \rho_j < 0$ if $\varphi'_{x_n}(x) > 0$ (it is our choice of φ).

$P_\varphi v = f$ in $x_n > 0$.

$$\text{Denote } \underline{v} = \begin{cases} v \text{ if } x_n > 0 \\ 0 \text{ if } x_n < 0 \end{cases}$$

$P_\varphi \underline{v} = \underline{f} + \gamma_0 \delta'_{x_n=0} + \gamma_1 \delta_{x_n=0}$,
where γ_j depend on $v_{|x_n=0}$ and $D_{x_n} v_{|x_n=0}$.

Choose Q a parametrix of $P \longrightarrow QP = Id + R$ where R smoothing.

$\underline{v} = Q\underline{f} + C_0\gamma_0 + C_1\gamma_1 + R\underline{f}$ where C_j are operators on traces.

On $x_n = 0$, (i.e. $x_n = 0^+$)

$v_{|x_n=0} = Q\underline{f}_{|x_n=0} + C'_0\gamma_0 + C'_1\gamma_1 + R\underline{f}_{|x_n=0}$,
and similar relation for $\partial_{x_n} v_{|x_n=0}$.

CALDERON OPERATORS

$v_{|x_n=0} = Q\underline{f}_{|x_n=0} + C_0'\gamma_0 + C_1'\gamma_1 + R\underline{f}_{|x_n=0}$.
C_0, C_1 have the following form (principal symbol)

$$\int e^{ix'\xi'}\hat{\gamma}(\xi')\int e^{ix_n\xi_n}\frac{\xi_n^\nu}{p_\varphi(x,\xi)}d\xi_n d\xi' = 0$$

WHY? $\longrightarrow$ By residues formula and $\operatorname{Im}\rho_j < 0$.

ξ_n^ν comes from Fourier transform of $\delta^{(\nu)}_{x_n=0}$.

We obtain,

$v_{|x_n=0} = Q\underline{f}_{|x_n=0} + R\underline{f}_{|x_n=0} + \tilde{R}(\gamma_0,\gamma_1)$, (where $\tilde{R}$ is smoothing).

Thus we can estimates the two traces $v_{|x_n=0}$ and $\partial_{x_n}v_{|x_n=0}$ microlocally in $\mu < 0$.

Important: in this region we do not need boundary conditions.

CASE $\mu > -(\varphi'_{x_n})^2$

We have
$p_\varphi(x,\xi) = (\xi_n - \rho_2(x,\xi'))\,(\xi_n - \rho_1(x,\xi'))$.
Modulo some remainder terms, we have
$P_\varphi v = (D_n - \mathrm{op}(\rho_2))\,(D_n - \mathrm{op}(\rho_1))\,v$.
We quantify symbols in semi-classical sense, i.e. $\mathrm{op}(\xi) = D = \dfrac{h}{i}\partial$.
Let $z = (D_n - \mathrm{op}(\rho_1))\,v$, we have,
$(D_n - \mathrm{op}(\rho_2))\,\underline{z} = \underline{f} + \frac{h}{i} z_{|x_n=0}\delta_{x_n=0}$.
The root of $\xi_n - \rho_2(x,\xi')$ is in $\{\mathrm{Im} < 0\}$ we can perform as in previous case and we can estimate
$z_{|x_n=0} = D_n v_{|x_n=0} - \mathrm{op}(\rho_1) v_{|x_n=0} = g$, where g depends on f.

EQUATION ON TRACES (I)

$D_n v_{|x_n=0} - \mathrm{op}(\rho_1) v_{|x_n=0} = g$

We recall $v = e^{\varphi/h} u$

$u_{|x_n=0} = 0$ on $x_1 < 0$ and $D_{x_n} u_{|x_n=0} = 0$ on $x_1 > 0$, then

$D_{x_n} u = D_{x_n}\left(e^{-\varphi/h} v\right) = e^{-\varphi/h}\left(D_{x_n} v + i\varphi'_{x_n} v\right)$.

Let $v_0 = v_{|x_n=0}$ and $v_1 = \left(D_{x_n} v + i\varphi'_{x_n} v\right)_{|x_n=0}$.

We obtain on v_0 and v_1 the equation

$$\begin{aligned} &v_1 - \mathrm{op}(\rho_1 + i\varphi'_{x_n}) v_0 = g \text{ on } x_n = 0, \text{ with} \\ &v_0 = 0 \text{ on } x_1 < 0, \\ &v_1 = 0 \text{ on } x_1 > 0. \end{aligned}$$

EQUATION ON TRACES (II)

$$v_1 - \mathrm{op}(\rho_1 + i\varphi'_{x_n})v_0 = g \text{ on } x_n = 0, \text{ with}$$
$$v_0 = 0 \text{ on } x_1 < 0,$$
$$v_1 = 0 \text{ on } x_1 > 0.$$

Let $r_{x_1>0}$ the restriction to $x_1 > 0$. We can write

$$r_{x_1>0}\,\mathrm{op}(\rho_1 + i\varphi'_{x_n})v_0 = -r_{x_1>0}g \text{ on } x_n = 0.$$

We have a pseudo-differential boundary problem, studied by Eskin, Boutet de Monvel, Rempel-Schulze, Grubb, Harutjunjan-Schulze ...
Problem: $\rho_1 + i\varphi'_{x_n}$ does not satisfy the transmission condition.
Before taking the restriction we must transform the equation on the traces.

EQUATION ON TRACES (III)

$$v_1 - \text{op}(\rho_1 + i\varphi'_{x_n})v_0 = g \text{ on } x_n = 0, \text{ with}$$
$$v_0 = 0 \text{ on } x_1 < 0,$$
$$v_1 = 0 \text{ on } x_1 > 0.$$

Let $\lambda_-^s(\xi') = \left(\xi_1 + i\sqrt{|\xi''|^2 + \varepsilon^2}\right)^s$,
$\lambda_+^s(\xi') = \left(\xi_1 - i\sqrt{|\xi''|^2 + \varepsilon^2}\right)^s$.
The kernel of $\text{op}(\lambda_-^s)$ is supported in $x_1 \leq 0$. Let
$z_0 = \text{op}(\lambda_+^{1/2})v_0$ is supported in $x_1 \geq 0$,
$z_1 = \text{op}(\lambda_-^{-1/2})v_1$ is supported in $x_1 \leq 0$.
$z_1 - \text{op}(\lambda_-^{-1/2})\,\text{op}(\rho_1 + i\varphi'_{x_n})\,\text{op}(\lambda_+^{-1/2})z_0 = g_0$ where g_0 is known.
We obtain

$$-r_{x_1>0}\,\text{op}(\lambda_-^{-1/2})\,\text{op}(\rho_1 + i\varphi'_{x_n})\,\text{op}(\lambda_+^{-1/2})z_0 = r_{x_1>0}g_0.$$

EQUATION ON TRACES (IV)

$$-ir_{x_1>0}\,\mathrm{op}(\lambda_-^{-1/2})\,\mathrm{op}(\rho_1+i\varphi'_{x_n})\,\mathrm{op}(\lambda_+^{-1/2})z_0 = r_{x_1>0}g_0.$$

Computation of the symbol when $x=0$. The remainder are treated as errors terms.

First what is ρ_1?

$p_\varphi(x,\xi)=\xi_n^2+2i\varphi'_{x_n}\xi_n+q_2(x,\xi')+2iq_1(x,\xi')$.

For simplicity take $\partial_{x'}\varphi(0)=0$.

Thus $q_2(0,\xi')=|\xi'|^2-|\varphi'_{x_n}(0)|^2$ and $q_1(0,\xi')=0$. We obtain,

$p_\varphi(x,\xi)=\xi_n^2+2i\varphi'_{x_n}\xi_n+|\xi'|^2-|\varphi'_{x_n}(0)|^2=\left(\xi_n+i\varphi'_{x_n}\right)^2+|\xi'|^2$,

thus $\rho_1(0,\xi')=-i\varphi'_{x_n}(0)+i|\xi'|$.

Symbol of $\mathrm{op}(\lambda_-^{-1/2})\,\mathrm{op}(\rho_1+i\varphi'_{x_n})\,\mathrm{op}(\lambda_+^{-1/2})$ is (on $x=0$)

$$\lambda_-^{-1/2}i|\xi'|\lambda_+^{-1/2}=\left(\xi_1+i\sqrt{|\xi''|^2+\varepsilon^2}\right)^{-1/2}i|\xi'|\left(\xi_1-i\sqrt{|\xi''|^2+\varepsilon^2}\right)^{-1/2}$$

EQUATION ON TRACES (V)

$$-r_{x_1>0}\,\mathrm{op}(\lambda_-^{-1/2})\,\mathrm{op}(\rho_1 + i\varphi'_{x_n})\,\mathrm{op}(\lambda_+^{-1/2})z_0 = r_{x_1>0}g_0.$$

$$\lambda_-^{-1/2} i|\xi'|\lambda_+^{-1/2} = \left(\xi_1 + i\sqrt{|\xi''|^2+\varepsilon^2}\right)^{-1/2} i|\xi'| \left(\xi_1 - i\sqrt{|\xi''|^2+\varepsilon^2}\right)^{-1/2}$$
$$= \frac{i|\xi'|}{\sqrt{|\xi'|^2+\varepsilon^2}} = i + O(\varepsilon).$$

Then we obtain,

$$-ir_{x_1>0}z_0 = r_{x_1>0}g_0 + O(\varepsilon)(z_0) + O(x')(z_0)$$

and we can estimate z_0 (supported in $\{x_1 \geq 0\}$) by g_0 in suitable norm.
From traces we can estimate v in interior by more classical way.